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Obtaining $K \to \pi\pi$ from Off-Shell $K \to \pi$ Amplitudes.

Johan Bijnens[a], **Elisabetta Pallante**[b], **and Joaquim Prades**[c]

[a] Department of Theoretical Physics 2, University of Lund,
Sölvegatan 14A, S 22362 Lund, Sweden

[b] Institut für Theoretische Physik, Universität Bern,
Sidlerstrasse 5, CH-3012 Bern, Switzerland

[c] Departamento de Física Teórica y del Cosmos, Universidad de Granada,
Campus de Fuente Nueva, E-18002 Granada, Spain

Abstract

We properly define off-shell $K \to \pi$ transition amplitudes and use them to extract information for on-shell $K \to \pi\pi$ amplitudes within Chiral Perturbation Theory. At order p^2 in the chiral expansion all three parameters of weak interaction can be determined. At order p^4 we are able to fix eleven additional constants out of thirteen contributing to off-shell $K \to \pi$ transitions, which leaves four undetermined constants in the on-shell $K \to \pi\pi$ amplitudes. All $\mathcal{O}(p^4)$ contributions have been exactly derived with $m_\pi^2 \neq 0$. We finally discuss the weak mass term issue and find contributions to on-shell $\Delta S = \pm 1$ Kaon decays, in particular to transitions like $K_L \to \gamma\gamma$, $K_L \to \mu^+\mu^-$ and $K_S \to \pi^0\gamma\gamma$ at the lowest non-zero order.

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1 Introduction

The explanation of the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decays remains one of the challenges in Kaon physics and in our understanding of strong interactions. Various non-leptonic Kaon decays are also used to put limits on CP-violation and several other quantities of the Standard Model and extensions of it [1]. The short distance part of the relevant operators can be treated using renormalization group within perturbative QCD, while the computation of matrix elements of the relevant operators between meson states is a pure non perturbative problem.

In the long term lattice QCD should be able to perform a direct computation of weak matrix elements. It is however much easier on the lattice[1], and often also in analytical attempts to reproduce the weak matrix elements, to calculate correlators involving fewer external legs. As a first step, current algebra can be used to relate $K \to 2\pi$ to $K \to \pi$ amplitudes where it involves off-shell $K \to \pi$ transitions. Chiral Perturbation Theory (CHPT) [3] is however the more modern tool to exploit the consequences of current algebra. At lowest order in the chiral expansion this problem was first worked out in [4] and subsequently the non-analytic parts of the loop contributions to $K \to \pi\pi$ and $K \to \pi$ were calculated in [5]. It was also discussed in [6] and in the context of Wilson Fermions on the lattice in [7]. In this paper we extend the previous study in two ways:
1) We systematically go to next-to-leading order (i.e. order p^4) in CHPT and
2) Instead of the vague notion of off-shell Kaon and pion fields, we use pseudo-scalar current correlators which are well defined quantities[2].

The use of this type of correlators to extract information on non-leptonic Kaon matrix elements is quite common in lattice studies (see for instance [7]), though in those cases an on-shell extrapolation is usually performed. In addition, this extrapolation is done at lowest order p^2 in the chiral expansion, i.e. using pure current algebra relations. However, due to the large Kaon mass, one expects non-negligible higher order CHPT corrections to Kaon weak matrix elements and in general to the pseudo-scalar current correlators involved. The use of the off-shell behaviour of this type of correlators to obtain additional information on the relevant matrix element has been advocated in [8], where it was used to unravel the quark mass dependence of B_K, and in [9] to disentangle the structure of the electromagnetic mass differences.

That chiral corrections are important in non-leptonic Kaon decays is already known since a long time [5, 10] and has been fully worked out in CHPT by [11, 12, 13]. Here we present results for the octet and the 27-plet contributions to $K \to \pi\pi$ transitions both at order p^2 and order p^4 and without neglecting m_π^2/m_K^2 suppressed contributions. (We have a small disagreement here with

[1]Computing the $K \to \pi\pi$ amplitudes directly is quite difficult because of the Maiani-Testa argument [2].

[2]We have verified that in the cases discussed here the use of other two-point correlators does not yield additional information.

respect to previous literature for the 27-plet [13].) While for the physical case neglecting m_π^2/m_K^2 terms is a reasonable approximation it will not be the case for foreseeable lattice calculations. We keep $\overline{m} \equiv m_u = m_d \neq 0$ throughout the derivation. Another issue to be clarified is the weak mass term contribution. It is well known that the weak mass term, which gives rise to the tadpole contributions in the original formulation of the weak effective Lagrangian [11], does not enter the $K \to \pi\pi$ on-shell matrix elements at order p^2 [14, 15]. It does however contribute in a well defined way to off-shell quantities at order p^2 and higher. For this reason we shall discuss the precise role of the weak mass term up to order p^4 in Kaon transition amplitudes in Sect. 6.

It turns out that, while at order p^2 all the weak parameters can be determined from our two-point correlators[3], this is no longer true at order p^4. There are in total nineteen parameters entering the weak effective Lagrangian up to that order as discussed in Sect. 2.1. Of these, we can obtain fourteen using our procedure. Five of them can be obtained in more than one place, thus providing as many CHPT relations.

Several relations are also implied between different two-point correlators so that the same quantities can also be used to check how well calculations within CHPT obey the chiral symmetry predictions to order p^4. In particular, we can obtain several coefficients of terms which involve quark masses at higher order. In the purely strong sector these are the most difficult ones to predict from models and/or dispersive constraints. Determining some of them through our procedure will provide a good check on models used in this context (see e.g. [16]).

We study in CHPT the pseudo-scalar current correlators

$$\Pi^{ij}(q^2) \equiv i \int \mathrm{d}^4x \, e^{iq\cdot x} \langle 0| T \left(P^{i\dagger}(0) P^j(x) \, e^{i\Gamma_{\Delta S=a}} \right) |0\rangle \tag{1.1}$$

in the presence of strong interactions. Above, $a = \pm 1, \pm 2$ stands for $|\Delta S| = 1, 2$ transitions and i, j are light quarks combinations corresponding to the octet of light pseudo-scalar mesons:

$$\begin{aligned} P^{\pi^0}(x) &\equiv \frac{1}{\sqrt{2}} \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d \right); \quad P^{\pi^+}(x) \equiv \overline{d} i\gamma_5 u; \quad P^{K^0}(x) \equiv \overline{s} i\gamma_5 d; \\ P^{K^+}(x) &\equiv \overline{s} i\gamma_5 u; \quad P^{\eta_8}(x) \equiv \frac{1}{\sqrt{6}} \left(\overline{u} i\gamma_5 u + \overline{d} i\gamma_5 d - 2\overline{s} i\gamma_5 s \right) . \end{aligned} \tag{1.2}$$

The effective action of weak interactions $\Gamma_{\Delta S=a}$ describes strangeness changing processes in one and two units. Within the Standard Model it can be written as follows

$$\Gamma_{\Delta S=a} \equiv -C_{\Delta S=a} \, G_F \int \mathrm{d}^4\mathrm{y} \, \mathcal{O}_{\Delta \mathrm{S=a}}(\mathrm{y}) \, , \tag{1.3}$$

where $\mathcal{O}_{\Delta S=a}$ is a sum over the effective operators arising after integrating out the heavy bosons, i.e. W, Z, and the Higgs boson, and heavy fermions, i.e. top,

[3] These are called three-point correlators in lattice QCD because of the extra weak vertex.

bottom, and charm quark (see e.g. [1, 17, 18] for the actual derivation). The constant $C_{\Delta S=a}$ collects Clebsch-Gordan factors and G_F is the Fermi constant.

In (1.1), the first term in the expansion of $\exp[i\Gamma_{\Delta S=a}]$ describes strangeness zero changing transitions, the second term describes strangeness one and two changing processes, while the third term includes $(\Delta S=\pm 1)^2$ transitions. The $|\Delta S|=2$ case relevant to the B_K factor which parameterizes the $\overline{K^0}-K^0$ mixing was already studied in [8]; in Section 3, we will just repeat the relevant expressions for completeness. The plan of the paper is as follows. In Section 2 we construct the weak effective Lagrangian up to order p^4 and relevant to our analysis. The $1/N_c$ counting of the weak constants is also done in subsection 2.2. In Section 3 the fully renormalized two-point current correlators up to order p^4 are derived for $|\Delta S|=0,1,2$ cases. The non-analytic contributions to one loop are collected in Appendix A. In Section 4 the $K\to\pi\pi$ on-shell amplitudes are derived in CHPT up to order p^4. In Appendix B are the non-analytic contributions to one loop. Section 5 is devoted to the connection between off-shell $K\to\pi$ transition amplitudes and on-shell $K\to 2\pi$ amplitudes. Resonance saturation also for the 27-plet sector is used here and derived in Appendix C. Finally, in Section 6 we clarify the role of the weak mass term in Kaon decays up to order p^4 and in Section 7 we state our conclusions.

2 CHPT Lagrangian and $1/N_c$-Discussion

At lowest order in CHPT (i.e. $\mathcal{O}(p^2)$) the strangeness changing interactions up to two units amongst the pseudo-Goldstone bosons and external scalar, pseudo-scalar, vector and axial-vector sources (neglecting virtual photon interactions) are described by the following effective Lagrangian:

$$\mathcal{L}^{(2)}_{\rm eff} \;=\; \mathcal{L}^{(2)}_{\Delta S=0}+\mathcal{L}^{(2)}_{\Delta S=1}+\mathcal{L}^{(2)}_{\Delta S=2}\,. \tag{2.1}$$

The first term is the strong interaction Lagrangian

$$\mathcal{L}^{(2)}_{\Delta S=0} \;=\; \frac{F_0^2}{4}\left[\,\mathrm{tr}\,(u^\mu u_\mu)+\,\mathrm{tr}\,(\chi_+)\right]\,, \tag{2.2}$$

where $\mathrm{tr}(A)$ is the flavour trace of A,

$$u_\mu \equiv i u^\dagger\,(D_\mu U)\,u^\dagger = u_\mu^\dagger \tag{2.3}$$

and $U\equiv u^2=\exp(i\sqrt{2}\Phi/F_0)$ is the exponential representation incorporating the SU(3) matrix of the octet of light pseudo-scalar mesons

$$\Phi(x)=\frac{\vec{\lambda}\cdot\vec{\phi}}{\sqrt{2}}=\begin{pmatrix} \frac{\pi^0}{\sqrt{2}}+\frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}}+\frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}. \tag{2.4}$$

$D_\mu U$ denotes the covariant derivative on the U field

$$D_\mu U \equiv \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu)\,, \tag{2.5}$$

where $v_\mu(x)$ and $a_\mu(x)$ are external SU(3) vector and axial-vector matrices (no singlet component will be included in the present analysis). The matrix χ_+ in Eq. (2.2) and its pseudoscalar counterpart χ_- are defined as follows

$$\chi_\pm \;\equiv\; u^\dagger \chi u^\dagger \pm u\chi^\dagger u\,, \tag{2.6}$$

where $\chi \equiv 2B_0(\mathcal{M} + s(x) + ip(x))$, $s(x)$ and $p(x)$ are external scalar and pseudoscalar SU(3) sources and $\mathcal{M} \equiv \mathrm{diag}(m_u, m_d, m_s)$ is the light quarks mass matrix. The constant B_0 is related to the vacuum expectation value of the scalar quark density

$$\langle 0|\overline{q}q|0\rangle\,|_{q=u,d,s} \equiv -F_0^2 B_0\,(1 + \mathcal{O}(\mathcal{M}))\;. \tag{2.7}$$

With this normalization, F_0 is the chiral limit value of the pion decay constant $F_\pi \simeq 92.4$ MeV. In the absence of the $\mathrm{U}(1)_A$ anomaly (i.e. in the large N_c limit) [19], the U(3) singlet field η_1 becomes the ninth Goldstone boson which is incorporated in the $\Phi(x)$ field as

$$\Phi(x) = \frac{\vec{\lambda}\cdot\vec{\phi}}{\sqrt{2}} + \frac{\eta_1}{\sqrt{3}}\mathbf{1}\,. \tag{2.8}$$

In this work we limit ourselves to the octet symmetry case, meaning that we assume the singlet degree of freedom η_1 as heavy and integrated out. This is enough for our purpose of showing how to relate off-shell $K \to \pi$ and $K \to \eta_8$ transitions to on-shell $K \to \pi\pi$ amplitudes. In octet symmetry, i.e. $\det(u) = 1$ and $\mathrm{tr}(u_\mu) = 0$, the $\Delta S = \pm 1$ contribution to the l.h.s. of (1.3) is given by

$$\begin{aligned}\mathcal{L}^{(2)}_{\Delta S=1} \;=\;& C\,F_0^4\,\Big[G_8\ \mathrm{tr}\,(\Delta_{32}u_\mu u^\mu) + G_8'\ \mathrm{tr}\,(\Delta_{32}\chi_+) \\ &+ G_{27} t^{ij,kl}\ \mathrm{tr}\,(\Delta_{ij}u_\mu)\ \mathrm{tr}\,(\Delta_{kl}u^\mu)\Big] + \ \mathrm{h.c.}\,,\end{aligned} \tag{2.9}$$

where $i,j,k,l = 1,2,3$ correspond to the light flavour indices u,d,s and the tensor $t^{ij,kl}$ has

$$\begin{aligned} t^{21,13} &= t^{13,21} = \frac{1}{3}\,;\ t^{22,23} = t^{23,22} = -\frac{1}{6}\,; \\ t^{23,33} &= t^{33,23} = -\frac{1}{6}\,;\ t^{23,11} = t^{11,23} = \frac{1}{3} \end{aligned} \tag{2.10}$$

and zero otherwise. The matrix Δ_{ij} is defined as

$$\begin{aligned} \Delta_{ij} &\equiv u\lambda_{ij}u^\dagger \\ (\lambda_{ij})_{ab} &\equiv \delta_{ia}\,\delta_{jb}\,, \end{aligned} \tag{2.11}$$

with δ_{ia} the Kronecker delta acting on the SU(3) light flavour space. The constant $C = C_{\Delta S=1}$ in (1.3) includes normalization factors and the Cabibbo-Kobayashi-Maskawa matrix elements

$$C = -\frac{3}{5}\frac{G_F}{\sqrt{2}} V_{ud}\, V_{us}^{*} \,. \tag{2.12}$$

The couplings G_8 and G_8' modulate octet operators under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, while G_{27} modulates a 27-plet operator. For on-shell $K \to \pi\pi$ transitions at order p^2 one can set $G_8' = 0$ [4, 14, 15]. At order p^4 this question was studied in [11, 20]; the result is that one can always use a basis for the order p^4 counterterms where the effects of the weak mass term (G_8') on the on-shell $K \to \pi\pi$ amplitudes are fully reabsorbed at this order. Clearly, the use of the shifted basis implies a redefinition of the order p^4 couplings in order to absorb the weak mass term contributions. This was not done in [16]. Since G_8' does always appear in off-shell $K \to \pi$ transitions, we keep the unshifted basis in our analysis, where $K \to \pi\pi$ amplitudes explicitly contain order p^4 contributions proportional to G_8'.

The $|\Delta S| = 2$ term in Eq. (1.3) can be written as follows

$$\mathcal{L}^{(2)}_{\Delta S=2} \;=\; C_{\Delta S=2}\, F_0^4\, G_{27}\; \mathrm{tr}\,(\Delta_{32} u_\mu)\; \mathrm{tr}\,(\Delta_{32} u^\mu) + \text{ h.c.}\,, \tag{2.13}$$

with

$$C_{\Delta S=2} = -\frac{G_F}{4}\mathcal{F}(m_t^2, m_c^2, M_W^2, \mathbf{V}_{\mathrm{CKM}}) \tag{2.14}$$

and $\mathcal{F}(m_t^2, m_c^2, M_W^2, \mathbf{V}_{\mathrm{CKM}})$ being a known function of the heavy fermions and bosons masses, and the Cabibbo-Kobayashi-Maskawa matrix elements [1].

The weak couplings G_8, G_8' and G_{27} in Eqs. (2.9) and (2.13) are dimensionless and they are related to those used in [11] as follows

$$C\, F_0^4\, G_8 = c_2 \quad C\, F_0^4\, G_{27} = 3c_3\,; \quad C\, F_0^4\, G_8' = c_5\,. \tag{2.15}$$

2.1 The order p^4

At next-to-leading order in the chiral expansion (i.e. at order p^4) the complete list of counterterms in the strong interaction sector and in the octet symmetry case has been given by Gasser and Leutwyler in [21]. In the SU(3) flavour case and for on-shell Green's functions there appear ten counterterms denoted with L_i, $i = 1, \ldots, 10$, while in the off-shell case there are two extra contact terms H_1 and H_2 involving external sources only.

The complete basis of counterterms in the weak interaction sector at order p^4 and describing transitions with strangeness changing in one and two units was first derived by Kambor, Missimer and Wyler in [11] and in [22]. This basis was confirmed and reduced to various minimal sets by Esposito-Farèse for the octet and 27-plet operators [23], and by Ecker, Kambor and Wyler for the octet operators [24].

For the analysis of the $\Delta S = \pm 1$ transitions we use a minimal set of operators which differs from the one in [23] in one octet operator, but has the advantage of producing shorter expressions. Our octet subset coincides with that of Ecker et al. in [24] up to two operators. We also give below the translation from the counterterms we are using to those in [24].

In the octet symmetry case, a minimal set of counterterms contributing to the $K \to \pi$ and $K \to \pi\pi$ transitions at order p^4 in CHPT is given by:

$$\begin{aligned}
\mathcal{L}^{(4)}_{\Delta S=1} &= C\, F_0^2\, G_8 \Big[E_1 \mathcal{O}_1^8 + E_2 \mathcal{O}_2^8 + E_3 \mathcal{O}_3^8 + E_4 \mathcal{O}_4^8 + E_5 \mathcal{O}_5^8 \\
&+ E_{10} \mathcal{O}_{10}^8 + E_{11} \mathcal{O}_{11}^8 + E_{12} \mathcal{O}_{12}^8 + E_{13} \mathcal{O}_{13}^8 + E_{15} \mathcal{O}_{15}^8 \Big] \\
&+ C\, F_0^2\, G_{27} \Big[D_1 \mathcal{O}_1^{27} + D_2 \mathcal{O}_2^{27} + D_4 \mathcal{O}_4^{27} + D_5 \mathcal{O}_5^{27} \\
&+ D_6 \mathcal{O}_6^{27} + D_7 \mathcal{O}_7^{27} \Big] + \text{h.c.}\,.
\end{aligned} \tag{2.16}$$

The octet operators above are

$$\begin{aligned}
\mathcal{O}_1^8 &= \mathrm{tr}\left(\Delta_{32} \chi_+ \chi_+\right)\,; \\
\mathcal{O}_2^8 &= \mathrm{tr}\left(\Delta_{32} \chi_+\right)\ \mathrm{tr}\left(\chi_+\right)\,; \\
\mathcal{O}_3^8 &= \mathrm{tr}\left(\Delta_{32} \chi_- \chi_-\right)\,; \\
\mathcal{O}_4^8 &= \mathrm{tr}\left(\Delta_{32} \chi_-\right)\ \mathrm{tr}\left(\chi_-\right)\,; \\
\mathcal{O}_5^8 &= \mathrm{tr}\left(\Delta_{32} \left[\chi_+, \chi_-\right]\right)\,; \\
\mathcal{O}_{10}^8 &= \mathrm{tr}\left(\Delta_{32} \left\{\chi_+, u^\mu u_\mu\right\}\right)\,; \\
\mathcal{O}_{11}^8 &= \mathrm{tr}\left(\Delta_{32} u_\mu \chi_+ u^\mu\right)\,; \\
\mathcal{O}_{12}^8 &= \mathrm{tr}\left(\Delta_{32} u_\mu\right)\ \mathrm{tr}\left(\left\{u^\mu, \chi_+\right\}\right)\,; \\
\mathcal{O}_{13}^8 &= \mathrm{tr}\left(\Delta_{32} \chi_+\right)\ \mathrm{tr}\left(u^\mu u_\mu\right)\,; \\
\mathcal{O}_{15}^8 &= \mathrm{tr}\left(\Delta_{32} \left[\chi_-, u^\mu u_\mu\right]\right)\,.
\end{aligned} \tag{2.17}$$

The 27-plet operators are

$$\begin{aligned}
\mathcal{O}_1^{27} &= t^{ij,kl}\ \mathrm{tr}\left(\Delta_{ij} \chi_+\right)\ \mathrm{tr}\left(\Delta_{kl} \chi_+\right)\,; \\
\mathcal{O}_2^{27} &= t^{ij,kl}\ \mathrm{tr}\left(\Delta_{ij} \chi_-\right)\ \mathrm{tr}\left(\Delta_{kl} \chi_-\right)\,; \\
\mathcal{O}_4^{27} &= t^{ij,kl}\ \mathrm{tr}\left(\Delta_{ij} u_\mu\right)\ \mathrm{tr}\left(\Delta_{kl} \left\{u^\mu, \chi_+\right\}\right)\,; \\
\mathcal{O}_5^{27} &= t^{ij,kl}\ \mathrm{tr}\left(\Delta_{ij} u_\mu\right)\ \mathrm{tr}\left(\Delta_{kl} \left[u^\mu, \chi_-\right]\right)\,; \\
\mathcal{O}_6^{27} &= t^{ij,kl}\ \mathrm{tr}\left(\Delta_{ij} \chi_+\right)\ \mathrm{tr}\left(\Delta_{kl} u_\mu u^\mu\right)\,; \\
\mathcal{O}_7^{27} &= t^{ij,kl}\ \mathrm{tr}\left(\Delta_{ij} u_\mu\right)\ \mathrm{tr}\left(\Delta_{kl} u^\mu\right)\ \mathrm{tr}\left(\chi_+\right)\,,
\end{aligned} \tag{2.18}$$

where the tensor $t^{ij,kl}$ is defined in (2.10).

The translation from the octet counterterms in (2.17) to the ones used by Ecker et al. [24] is as follows

$$N_5 = E_{10} - E_{11}\,; \qquad N_6 = E_{11} + 2E_{12}\,;$$

$$
\begin{aligned}
N_7 &= \frac{1}{2}E_{11} + E_{13}\,; \quad N_8 = E_{11}\,; \\
N_9 &= E_{15}\,; \qquad\quad N_{10} = E_1 - E_5\,; \\
N_{11} &= E_2\,; \qquad\quad N_{12} = -E_3 + E_5\,; \\
N_{13} &= -E_4\,; \qquad\, N_{36} = E_5\,. \qquad (2.19)
\end{aligned}
$$

In octet symmetry and at the same order in CHPT, a minimal set of counterterms contributing to the $\Delta S = 2$ component of the $K^0 - \overline{K^0}$ mixing is:

$$
\begin{aligned}
\mathcal{L}^{(4)}_{\Delta S=2} &= C\,F_0^2\,G_{27}\left[D_1\mathcal{O}_1^{\Delta S=2} + D_2\mathcal{O}_2^{\Delta S=2} + D_4\mathcal{O}_4^{\Delta S=2} + D_5\mathcal{O}_5^{\Delta S=2}\right. \\
&+ \left. D_6\mathcal{O}_6^{\Delta S=2} + D_7\mathcal{O}_7^{\Delta S=2}\right] + \mathrm{h.c.}\,. \qquad (2.20)
\end{aligned}
$$

Notice that this basis is not exactly the one used in [8]; the one in (2.20) is a minimal set. The $\Delta S = 2$ operators are given by

$$
\begin{aligned}
\mathcal{O}_1^{\Delta S=2} &= \mathrm{tr}\left(\Delta_{32}\chi_+\right)\ \mathrm{tr}\left(\Delta_{32}\chi_+\right)\,; \\
\mathcal{O}_2^{\Delta S=2} &= \mathrm{tr}\left(\Delta_{32}\chi_-\right)\ \mathrm{tr}\left(\Delta_{32}\chi_-\right)\,; \\
\mathcal{O}_4^{\Delta S=2} &= \mathrm{tr}\left(\Delta_{32}u_\mu\right)\ \mathrm{tr}\left(\Delta_{32}\left\{u^\mu, \chi_+\right\}\right)\,; \\
\mathcal{O}_5^{\Delta S=2} &= \mathrm{tr}\left(\Delta_{32}u_\mu\right)\ \mathrm{tr}\left(\Delta_{32}\left[u^\mu, \chi_-\right]\right)\,; \\
\mathcal{O}_6^{\Delta S=2} &= \mathrm{tr}\left(\Delta_{32}\chi_+\right)\ \mathrm{tr}\left(\Delta_{32}u_\mu u^\mu\right)\,; \\
\mathcal{O}_7^{\Delta S=2} &= \mathrm{tr}\left(\Delta_{32}u_\mu\right)\ \mathrm{tr}\left(\Delta_{32}u^\mu\right)\ \mathrm{tr}\left(\chi_+\right)\,. \qquad (2.21)
\end{aligned}
$$

Since the 27-plet operators with $\Delta S = 1$ in (2.16) and the $\Delta S = 2$ ones above are components of the same irreducible tensor under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, the D_i couplings in both Lagrangians have to be the same.

The divergences associated with the minimal set of counterterms in (2.16) and (2.20) can be extracted from Kambor et al. [11] with the use of the strong equation of motion, partial integration and the Cayley-Hamilton theorem. We explicitly verified the results in [11] and differ in the 27-plet sector by an overall sign[4]. The subtraction procedure is defined in the usual manner by

$$
E_i \equiv E_i^r + \frac{\nu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right]\right\}\left[\varepsilon_i + \frac{G_8'}{G_8}\varepsilon_i'\right] \qquad (2.22)
$$

and

$$
D_i \equiv D_i^r + \frac{\nu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right]\right\}\gamma_i\,. \qquad (2.23)
$$

In the strong sector we need the counterterms

$$
L_i \equiv L_i^r + \frac{\nu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right]\right\}\Gamma_i\,, \qquad (2.24)
$$

[4]EP has independently redone the Generating Functional calculation of the infinities and agrees with [24, 23] and [11] modulo the 27-plet overall sign.

with $i = 4, 5, 6, 7, 8$, and the one associated with the contact term

$$H_j \equiv H_j^r + \frac{\nu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right]\right\}\delta_j\,, \qquad (2.25)$$

with $j = 2$. The coefficients of the divergent parts are fixed to be

$$\Gamma_4 = \frac{1}{8},\quad \Gamma_5 = \frac{3}{8},\quad \Gamma_6 = \frac{11}{144},\quad \Gamma_7 = 0,\quad \Gamma_8 = \frac{5}{48},\quad \text{and}\quad \delta_2 = \frac{5}{24}\,, \qquad (2.26)$$

and those of Table 1.

E_i	ε_i	ε_i'	N_c	D_i	γ_i	N_c
1	$1/4$	$5/6$	1	1	$-1/6$	1
2	$-13/18$	$11/18$	$-2/3E_1 + \mathcal{O}(1/N_c)$	2	0	1
3	0	0	1	4	3	N_c
4	0	0	1	5	1	1
5	$-5/12$	$5/12$	1	6	$-3/2$	1
10	1	$3/4$	N_c	7	1	1
11	$-1/2$	0	N_c	–	–	–
12	$1/8$	0	N_c	–	–	–
13	$-7/8$	$1/2$	1	–	–	–
15	$3/4$	$-3/4$	1	–	–	–

Table 1: The divergences and the leading in $1/N_c$ behaviour of the weak $\mathcal{O}(p^4)$ octet counterterms E_i and 27-plet counterterms D_i. Notice that F_0^2 (i.e. an N_c factor) is factored out.

2.2 $1/N_c$ Counting

It is also useful to know the $1/N_c$ counting of the different weak couplings in the Lagrangians (2.16) and (2.20). We remind that in this counting F_0^2 is order N_c, while B_0 is order 1 [21]. The effective operators in (1.3) are four-quark operators and the leading contributions to them are of order N_c^2 [19]. In particular the leading large N_c contribution (i.e. in the absence of gluonic corrections) to the $|\Delta S| = 1$ operator comes from the one W-exchange diagram, while the Box diagram leads the leading contribution to the $|\Delta S| = 2$ operator. In both cases only one effective four-quark operator arises in the large N_c limit:

$$\mathcal{O}_{\Delta S=1}(x) = Q_2(x) \equiv 4\left(\overline{s}_L\gamma^\mu u_L\right)(x)\left(\overline{u}_L\gamma_\mu d_L\right)(x)\,, \qquad (2.27)$$

for $|\Delta S| = 1$ transitions and

$$\mathcal{O}_{\Delta S=2}(x) = Q_{\Delta S=2}(x) \equiv 4\left(\overline{s}_L\gamma^\mu d_L\right)(x)\left(\overline{s}_L\gamma_\mu d_L\right)(x)\,, \tag{2.28}$$

for $|\Delta S| = 2$ transitions. Above we defined $q_L(x) \equiv [(1-\gamma_5)/2]\, q(x)$ and summation over colour indices is understood inside each bracket. For $N_c \to \infty$ both currents bosonize independently. So the terms in (2.16) and (2.20) with two flavour traces, each of the current type, may receive contributions of order N_c^2. However, to assign the correct $1/N_c$ counting to single and double flavour traces in the weak effective Lagrangian the operators have to be traceless. This can be best seen at the quark level. Light four-quark operators are described by $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ tensors

$$\hat{t}^{ij,kl}(\bar{q}_i\Gamma q_k)(\bar{q}_j\Gamma' q_l) \tag{2.29}$$

with $\Gamma^{(\prime)}$ the adequate Dirac structure (pseudo-scalar, scalar, axial-vector and vector)[5]. The coupling modulating the traceless part of $\hat{t}$, namely $\hat{t}^{ij,kl} - (2/3)\,\hat{t}^{mj,ml}$ is leading in the $1/N_c$ counting, i.e. of order N_c^2. The rest of $\hat{t}$ is of order N_c since it has an additional flavour trace which is $1/N_c$ suppressed. Already at order p^2 the weak octet Lagrangian, if not written in terms of flavour traceless operators, does contain double trace terms which are of leading order N_c^2. At order p^4, the basis of 27-plet operators in (2.18), the one in (2.21) and the one used in Kambor, Missimer, and Wyler in [11] are written in terms of traceless operators so that the $1/N_c$ counting is correct in those cases.

Neither the octet basis we use in (2.17) nor the one in [11] are written in terms of traceless operators so that one has to proceed in two steps to do the correct $1/N_c$ counting. First, writing the octet Lagrangian in terms of flavour traceless operators. The second step eventually needed is the reduction to a minimal basis, which involves the use of the eqs. of motion, integration by parts and Cayley-Hamilton relations. Again, some of these relations can spoil the correct $1/N_c$ counting. In this respect the counting given in [11] for the octet sector is not fully correct, while some of the Cayley-Hamilton relations used in [23, 24] for the reduction to a minimal basis were not appropriate for a correct large N_c counting.

A way of deducing the leading in $1/N_c$ contributions to the weak couplings at order p^4 is the use of the strong factorization assumption (See [24] and references therein) where all the weak parameters in (2.16) are known to all orders in CHPT, in terms of the parameters of the strong effective Lagrangian. All the large N_c contributions to the weak parameters are contained in their factorizable part. Up to order p^4 (and assuming the fudge factor of naive factorization $k_f = 1$) the weak couplings receive the following large N_c contributions

$$G_8 \;=\; 1\,, \quad G_{27} = 1\,,$$

[5]The tensor $\hat{t}$ can always be decomposed in a symmetric part $\hat{t}^{ij,kl} = \hat{t}^{ji,lk}$ and an antisymmetric part $\hat{t}^{ij,kl} = -\hat{t}^{ji,lk}$. We assume $\hat{t}$ is symmetric. If $\hat{t}$ is antisymmetric then $\hat{t}$ is flavour traceless and the couplings modulating it are order N_c^2.

$$
\begin{aligned}
E_{10} &= 8L_4 + 2L_5\,, \quad E_{11} = 8L_4 + 4L_5\,, \quad E_{12} = -4L_4 - \frac{4}{3}L_5\,, \quad E_{13} = -4L_4\,,\\
D_4 &= 4L_5\,, \quad D_7 = 8L_4
\end{aligned}
\tag{2.30}
$$

and all the others in (2.16) being zero. The strict large N_c limit would also imply $L_4 = 0$. The factorization assumption actually corresponds to keeping L_4 non-zero in (2.30) and leaving G_8 and G_{27} as free parameters.

At next-to-leading order in $1/N_c$ (i.e. in the presence of gluonic corrections) all the weak parameters receive corrections both non-factorizable and factorizable of short- and long-distance origin. $G_8 \neq G_{27}$ at this order. Analogously all the p^4 couplings in (2.30) and the rest of the counterterms in (2.16) which are zero in the large N_c limit do get unknown contributions at next-to-leading order. Their estimate is one of the main challenges of low-energy physics. Above we have included the factorizable next-to-leading in $1/N_c$ contributions too – the L_4 parts. As shown in Table 1 the octet counterterms E_{10}, E_{11}, E_{12} and the 27-plet counterterm D_4 get leading order contributions in $1/N_c$. To arrive at the basis we are using in (2.16) and (2.20) we made use of the equations of motion, partial integrations and Cayley-Hamilton relations. The latter have been used in such a way that the $1/N_c$ counting is not broken. The use of the equations of motion does not break the counting either. However some integrations by parts do, e.g. one can remove the apparent current–current structure of some operators in this way. Therefore we have to do the $1/N_c$ counting of the other counterterms in Table 1 before the integrations by parts are done. Afterwards, the counting is translated to the basis in (2.16)[6].

We summarize the $1/N_c$ counting of the weak couplings we are using. The $\mathcal{O}(p^2)$ couplings G_8 and G_{27} are order 1 while G_8' is order $1/N_c$ (notice that F_0^4 has been factored out). The $\mathcal{O}(p^4)$ couplings E_{10}, E_{11}, E_{12}, and D_4 are order N_c (F_0^2 has been factored out in (2.16)). The couplings E_1, E_3, E_4, E_5, E_{13}, E_{15}, D_1, D_2, D_5, D_6, and D_7 are order 1. The combination of couplings $E_2 + 2E_1/3$ is order $1/N_c$.

3 Two-Point Functions

In this section we give the two-point functions in (1.1) at order p^4 for all the relevant ij combinations.

[6]In the strict $N_c \to \infty$ limit the singlet η_1 degree of freedom needs to be included. This one should afterwards be integrated out, leading to counterintuitive N_c-counting as is the case for L_7 in the strong sector [21]. Since the numerical value of L_7 is such that this counting seems inappropriate for our real world we neglect this issue. It should however be included in modeling approaches since there can be sizeable contributions from it as e.g. seen in B_K [8].

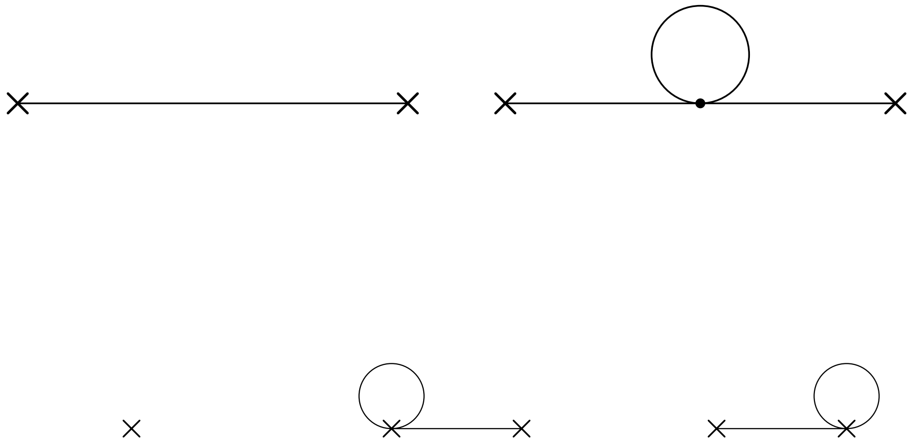

Figure 1: The diagrams contributing to the $\Delta S = 0$ two-point functions. A line is a pseudoscalar meson propagator, a dot a strong vertex with only meson legs and a cross a vertex from the strong Lagrangian with one or more insertions of the external pseudoscalar currents.

3.1 Strangeness Zero

Here we give the two-point Green's functions in (1.1) with $i = j$ to order p^4. Notice that those conserving strangeness with $i \neq j$ vanish since $m_u = m_d$. The poles of these two-point functions define the masses of the corresponding mesons and set the renormalization factors Z_i for the pseudoscalar sources P^i needed for the reduction procedure.

$$\Pi_{ii}(q^2) \equiv -\left[\frac{Z_i}{q^2 - m_i^2} + Z_i'\right] . \tag{3.1}$$

To order p^4, using $m_u = m_d = \overline{m}$ and neglecting electromagnetic corrections, we get from the diagrams in Fig. 1

$$\begin{aligned}
Z_{\pi^0} = Z_{\pi^+} &= 2B_0^2F_0^2\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)(4L_6 - L_4) + \frac{8}{F_0^2}m_\pi^2(4L_8 - L_5)\right.\\
&- \left. 2\mu_\pi - 2\mu_K - \frac{2}{3}\mu_{\eta_8}\right] ;\\
Z_{K^0} = Z_{K^+} &= 2B_0^2F_0^2\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)(4L_6 - L_4) + \frac{8}{F_0^2}m_K^2(4L_8 - L_5)\right.\\
&- \left. \frac{3}{2}\mu_\pi - 3\mu_K - \frac{1}{6}\mu_{\eta_8}\right] ;\\
Z_{\eta_8} &= 2B_0^2F_0^2\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)(4L_6 - L_4) + \frac{8}{F_0^2}m_{\eta_8}^2(4L_8 - L_5)\right.
\end{aligned}$$

$$- \quad 2\mu_\pi - \frac{2}{3}\mu_K - 2\mu_{\eta_8}\Big] \,. \tag{3.2}$$

$$Z'_{\pi^0} = Z'_{\pi^+} = Z'_{K^0} = Z'_{K^+} = Z'_{\eta_8} = 8B_0^2(2L_8 - H_2)\,, \tag{3.3}$$

and

$$\begin{aligned}
m^2_{\pi^0} &= m^2_{\pi^+} = 2\overline{m}B_0\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)(2L_6 - L_4)\right.\\
&+ \left.\frac{8}{F_0^2}m_\pi^2(2L_8 - L_5) + \mu_\pi - \frac{1}{3}\mu_{\eta_8}\right]\,;\\
m^2_{K^0} &= m^2_{K^+} = (\overline{m} + m_s)B_0\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)(2L_6 - L_4)\right.\\
&+ \left.\frac{8}{F_0^2}m_K^2(2L_8 - L_5) + \frac{2}{3}\mu_{\eta_8}\right]\,;\\
m^2_{\eta_8} &= \frac{2}{3}(\overline{m} + 2m_s)B_0\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)(2L_6 - L_4)\right.\\
&+ \left.\frac{8}{F_0^2}m_{\eta_8}^2(2L_8 - L_5) + 2\mu_K - \frac{4}{3}\mu_{\eta_8}\right]\\
&+ 2\overline{m}B_0\left[-\mu_\pi + \frac{2}{3}\mu_K + \frac{1}{3}\mu_{\eta_8}\right] + B_0^2(m_s - \overline{m})^2\frac{128}{9\,F_0^2}\,(3L_7 + L_8)\,.
\end{aligned} \tag{3.4}$$

For completeness and later use we also quote the decay constants f_π, f_K, and f_{η_8}, to the same order:

$$\begin{aligned}
f^2_{\pi^0} = f^2_{\pi^+} &= F_0^2\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)L_4 + \frac{8}{F_0^2}m_\pi^2 L_5 - 4\mu_\pi - 2\mu_K\right]\,;\\
f^2_{K^0} = f^2_{K^+} &= F_0^2\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)L_4 + \frac{8}{F_0^2}m_K^2 L_5 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{3}{2}\mu_{\eta_8}\right]\,;\\
f^2_{\eta_8} &= F_0^2\left[1 + \frac{8}{F_0^2}(2m_K^2 + m_\pi^2)L_4 + \frac{8}{F_0^2}m_{\eta_8}^2 L_5 - 6\mu_K\right]\,;
\end{aligned} \tag{3.5}$$

We use the notation [21]

$$\mu_i \equiv \frac{m_i^2}{32\pi^2 F_0^2}\ln\left(\frac{m_i^2}{\nu^2}\right)\,. \tag{3.6}$$

3.2 Strangeness One

In this section we give the renormalized two-point Green's functions in (1.1) up to order p^4. We define them as

$$\Pi_{ij}(q^2) \equiv \Pi_{ij}(q^2)\,|_{\rm Count} + \Pi_{ij}(q^2)\,|_{\rm Logs} \tag{3.7}$$

Only the analytic contributions from the counterterm Lagrangian $\Pi_{ij}(q^2)\,|_{\rm Count}$ are written here, while we give the non-analytic contributions from the one loop integration $\Pi_{ij}(q^2)\,|_{\rm Logs}$ in Appendix A. The contributing diagrams are shown in Fig. 2 With the pseudo-scalar sources $P^i(x) = P^{K^0}(x)$ and $P^j(x) = P^{\eta_8}(x)$ the two-point Green's function is given by

$$\begin{aligned}
\Pi_{K^0\eta_8}(q^2)\,|_{\rm Count} &= -\frac{\sqrt{Z_{K^0}Z_{\eta_8}}}{(q^2-m_{K^0}^2)(q^2-m_{\eta_8}^2)}\,\frac{C}{\sqrt{6}} \\
&\times\; 2\,\frac{F_0^4}{f_{K^0}f_{\eta_8}}\left[q^2(G_{27}-G_8+\frac{f_{\eta_8}^2+f_K^2}{F_0^2}G_8') - m_{\eta_8}^2\,\frac{f_{\eta_8}^2}{F_0^2}\,G_8'\right. \\
&+\; \frac{2}{F_0^2}\left\{2q^4\left(G_8E_3-G_{27}D_2\right)\right. \\
&+\; q^2m_K^2\left[G_8\left(4E_1+4E_2+\frac{16}{3}E_3+8E_4+6E_5\right.\right. \\
&-\; 2E_{10}-4E_{11}-8E_{12}) \\
&-\; \left.\frac{G_{27}}{3}\left(4D_1-7D_4-D_6-6D_7\right)-8G_8'\left(4L_6+\frac{7}{3}L_8\right)\right] \\
&+\; q^2m_\pi^2\left[G_8\left(2E_2-\frac{16}{3}E_3-8E_4-6E_5+3E_{11}+8E_{12}\right)\right. \\
&+\; \left.\frac{G_{27}}{3}\left(4D_1-D_4-D_6+3D_7\right)-8G_8'\left(2L_6-\frac{1}{3}L_8\right)\right] \\
&-\; \frac{8}{3}m_K^4\left[G_8\left(E_1+E_2\right)-\frac{G_{27}}{3}D_1-8G_8'\left(L_6+\frac{2}{3}L_8\right)\right] \\
&+\; \frac{2}{3}m_K^2m_\pi^2\left[G_8\left(E_1-E_2\right)-5\frac{G_{27}}{3}D_1+8G_8'\left(L_6-\frac{4}{3}L_8\right)\right] \\
&+\; \left.\left.\frac{1}{3}m_\pi^4\left[G_8E_2+2\frac{G_{27}}{3}D_1-8G_8'\left(L_6-\frac{1}{3}L_8\right)\right]\right\}\right]\,.
\end{aligned} \tag{3.8}$$

Notice that the renormalized meson masses m_π, m_K, m_{η_8} are used everywhere. For the case of pseudo-scalar sources $P^i(x) = P^{K^0}(x)$ and $P^j(x) = P^{\pi^0}(x)$ the result is

$$\begin{aligned}
\Pi_{K^0\pi^0}(q^2)\,|_{\rm Count} &= -\frac{\sqrt{Z_{K^0}Z_{\pi^0}}}{(q^2-m_{K^0}^2)(q^2-m_{\pi^0}^2)}\,\frac{C}{\sqrt{2}} \\
&\times\; 2\frac{F_0^4}{f_{K^0}f_{\pi^0}}\left[q^2\left(G_{27}-G_8+\frac{f_\pi^2+f_K^2}{F_0^2}G_8'\right)-m_\pi^2\frac{f_\pi^2}{F_0^2}G_8'\right.
\end{aligned}$$

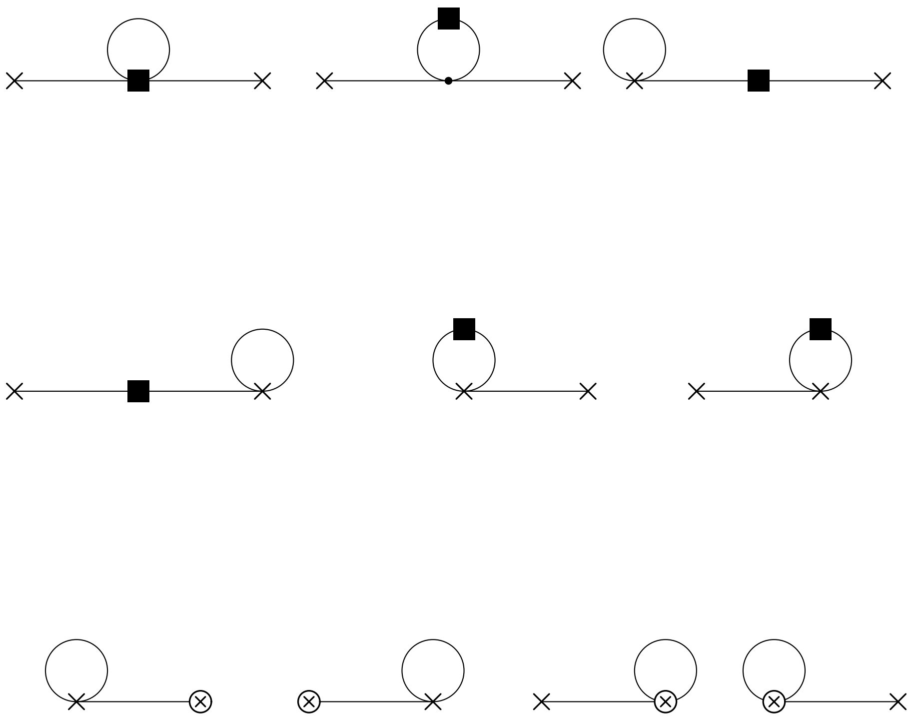

Figure 2: The diagrams contributing to the $|\Delta S| = 1$ and $|\Delta S| = 2$ two-point functions up to one loop. In addition to the symbols of Fig. 1, the full square is a weak vertex with only meson legs and the circled cross is a weak vertex with one or more insertions of the external pseudoscalar currents.

$$
\begin{aligned}
&+ \quad \frac{2}{F_0^2}\left\{2q^4\left(G_8E_3 - G_{27}D_2\right)\right.\\
&+ \quad q^2m_K^2\left[G_8\left(4E_1+4E_2-2E_5-2E_{10}\right)\right.\\
&- \quad \left.\frac{G_{27}}{3}\left(4D_1-3D_4-D_6-6D_7\right) - 8G_8'\left(4L_6+L_8\right)\right]\\
&+ \quad q^2m_\pi^2\left[G_8\left(2E_2+2E_5-E_{11}\right)\right.\\
&+ \quad \left.\frac{G_{27}}{3}\left(4D_1+3D_4-D_6+3D_7\right) - 8G_8'\left(2L_6+L_8\right)\right]\\
&- \quad 2m_K^2m_\pi^2\left[G_8\left(E_1+E_2\right) - \frac{G_{27}}{3}D_1 - 8G_8'L_6\right]\\
&- \quad \left.m_\pi^4\left[G_8E_2 + 2\frac{G_{27}}{3}D_1 - 8G_8'\left(L_6+L_8\right)\right]\right\}\Bigg] \,. \qquad (3.9)
\end{aligned}
$$

Finally, for $P^i(x) = P^{K^+}$ and $P^j(x) = P^{\pi^+}$ we obtain

$$
\begin{aligned}
\Pi_{K^+\pi^+}(q^2)\,|_{\mathrm{Count}} \quad &= \quad -\frac{\sqrt{Z_{K^+}Z_{\pi^+}}}{(q^2-m_{K^+}^2)(q^2-m_{\pi^+}^2)}\,C\\
&\times \quad 2\frac{F_0^4}{f_{K^+}f_{\pi^+}}\left[q^2\left(G_8 + 2\frac{G_{27}}{3} - \frac{f_\pi^2+f_K^2}{F_0^2}G_8'\right) + m_\pi^2\frac{f_\pi^2}{F_0^2}G_8'\right.\\
&+ \quad \frac{2}{F_0^2}\left\{-2q^4\left(G_8E_3 + 2\frac{G_{27}}{3}D_2\right)\right.\\
&- \quad q^2m_K^2\left[G_8\left(4E_1+4E_2-2E_5-2E_{10}\right)\right.\\
&+ \quad \left.\frac{G_{27}}{3}\left(-4D_1-2D_4+D_6-4D_7\right) - 8G_8'\left(4L_6+L_8\right)\right]\\
&- \quad q^2m_\pi^2\left[G_8\left(2E_2+2E_5-E_{11}\right)\right.\\
&+ \quad \left.\frac{G_{27}}{3}\left(4D_1-2D_4-D_6-2D_7\right) - 8G_8'\left(2L_6+L_8\right)\right]\\
&+ \quad 2m_K^2m_\pi^2\left[G_8\left(E_1+E_2\right) - \frac{G_{27}}{3}D_1 - 8G_8'L_6\right]\\
&+ \quad \left.\left.m_\pi^4\left[G_8E_2 + 2\frac{G_{27}}{3}D_1 - 8G_8'(L_6+L_8)\right]\right\}\right] \,. \qquad (3.10)
\end{aligned}
$$

3.3 Strangeness Two

The two-point Green's function for the $\Delta S = 2$ transition was already calculated in [8]. We include it here for sake of completeness. With the notation used in the present work we need the $\Delta S = 2$ part of the two-point Green's function in (1.1) with $P^i(x) = P^{K^0}$ and $P^j(x) = P^{\overline{K}^0}$. This gives

$$
\begin{aligned}
\Pi^{\Delta S=2}_{K^0\overline{K}^0}(q^2)\,|_{\mathrm{Count}} \quad &= \quad -\frac{Z_{K^0}}{(q^2-m_{K^0}^2)^2}\,C_{\Delta S=2}\,G_{27}\,4\frac{F_0^4}{f_{K^0}^2}\left[q^2 + \frac{1}{F_0^2}\left\{-4q^4D_2\right.\right.\\
&+ \quad \left.\left.2q^2\left(2D_4m_K^2 + D_7\left(2m_K^2+m_\pi^2\right)\right) - 4D_1\left(m_K^2-m_\pi^2\right)^2\right\}\right] \,.
\end{aligned}
$$

$$(3.11)$$

The non-analytic contributions are in Appendix A. There is another contribution $(\Delta S = \pm 1)^2$ to this two-point function which comes from expanding the exponential in (1.1) up to second order. They are the so-called long-distance contributions to $K^0 - \overline{K}^0$ mixing.

4 $K \to \pi\pi$ Amplitudes

We have the following decomposition into definite isospin quantum numbers invariant amplitudes $[A \equiv -iT]$,

$$\begin{aligned} A\left[K_S \to \pi^0\pi^0\right] &\equiv \sqrt{\frac{2}{3}}A_0 - \frac{2}{\sqrt{3}}A_2\,; \\ A\left[K_S \to \pi^+\pi^-\right] &\equiv \sqrt{\frac{2}{3}}A_0 + \frac{1}{\sqrt{3}}A_2\,; \\ A\left[K^+ \to \pi^+\pi^0\right] &\equiv \frac{\sqrt{3}}{2}\,A_2\,. \end{aligned} \tag{4.1}$$

Where $K_S \simeq K_1^0 + \varepsilon\, K_2^0$, $K^0_{1(2)} \equiv (K^0 - (+)\overline{K^0})/\sqrt{2}$, and CP $K^0_{1(2)} = +(-)K^0_{1(2)}$. Since CP violation is small we set $\varepsilon = 0$ and therefore $\Im m\, G_8 = 0$, $\Im m\, G_{27} = 0$, and $\Im m\, G_8' = 0$. We have also included the final state interaction phases into the amplitudes A_0 and A_2. For the isospin 1/2 amplitude we have

$$A_0 \equiv -i a_0\, e^{i\delta_0} \tag{4.2}$$

and for the isospin 3/2 amplitude we have

$$A_2 \equiv -i a_2\, e^{i\delta_2}\,. \tag{4.3}$$

To order p^2 we get

$$\begin{aligned} a_0 &\equiv a_0^8 + a_0^{27} = C\left[G_8 + \frac{1}{9}G_{27}\right]\sqrt{6}\,F_0\,(m_K^2 - m_\pi^2)\,, \\ a_2 &= C\,G_{27}\,\frac{10\sqrt{3}}{9}\,F_0\,(m_K^2 - m_\pi^2)\,, \end{aligned} \tag{4.4}$$

and

$$\delta_0 = \delta_2 = 0\,. \tag{4.5}$$

The order p^4 counterterms contributions to A_0^8, A_0^{27}, and A_2 (see Appendix B for the non-analytic contributions) are

$$\Im m\, A_0^8\,|_{\rm Count.} = -C\,G_8\sqrt{6}\,\frac{F_0^4}{f_K f_\pi^2}\,(m_K^2 - m_\pi^2)$$

$$
\begin{aligned}
&\times \quad \left[1+\frac{2}{F_0^2}\left[m_\pi^2\left(-2E_1-4E_2-2E_3+2E_{10}+E_{11}+4E_{13}\right)\right.\right.\\
&+ \quad \left.\left. m_K^2\left(E_{10}-2E_{13}+E_{15}\right)\right]\right]\\
&- \quad C\,G_8' 8\sqrt{6}\,\frac{F_0^2}{f_K f_\pi^2}\left(m_K^2-m_\pi^2\right)\\
&\times \quad \left[m_\pi^2\left(-4L_4-L_5+8L_6+4L_8\right)+2m_K^2 L_4\right]
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
\Im m\, A_0^{27}\,|_{\rm Count.} \quad &= \quad -C\,G_{27}\frac{\sqrt{6}}{9}\,\frac{F_0^4}{f_K f_\pi^2}\left(m_K^2-m_\pi^2\right)\\
&\times \quad \left[1+\frac{1}{F_0^2}\left[2m_\pi^2\left(-6D_1-2D_2+2D_4+6D_6+D_7\right)\right.\right.\\
&+ \quad \left.\left. m_K^2\left(D_4-D_5-9D_6+4D_7\right)\right]\right]
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
\Im m\, A_2\,|_{\rm Count.} \quad &= \quad -C\,G_{27}\frac{10\sqrt{3}}{9}\,\frac{F_0^4}{f_K f_\pi^2}\left(m_K^2-m_\pi^2\right)\\
&\times \quad \left[1+\frac{1}{F_0^2}\left[2m_\pi^2\left(-2D_2+2D_4+D_7\right)+m_K^2\left(D_4-D_5+4D_7\right)\right]\right]\,.
\end{aligned}
\tag{4.8}
$$

The diagrams are depicted in Fig. 3. In addition there are the corrections on the external legs and on the internal propagators of the tree level diagrams.

5 $K \to \pi\pi$ from $K \to \pi, \eta_8$ Amplitudes

We discuss here the information we can extract from the $K \to \pi, \eta_8$ two-point functions and make some remarks about the parameters needed for $K \to \pi\pi$ we cannot obtain. As discussed in Sect. 6, the weak mass term contribution to $K \to \pi\pi$ decays can be absorbed in a redefinition of the other coefficients [11], while this is not true in the case of the two-point functions. In typical approaches used in lattice QCD or effective models like the one proposed in [16], the weak mass term should be treated as an extra parameter in the determination of $K \to \pi\pi$ amplitudes at order p^4.

A few other remarks are needed here. The contribution from the 27_L and 8_L cannot be easily disentangled in general, since for $m_s \neq m_d = m_u$ the two components are mixed by higher order effects in $m_s - \overline{m}$. The $\Delta I = 1/2$ and

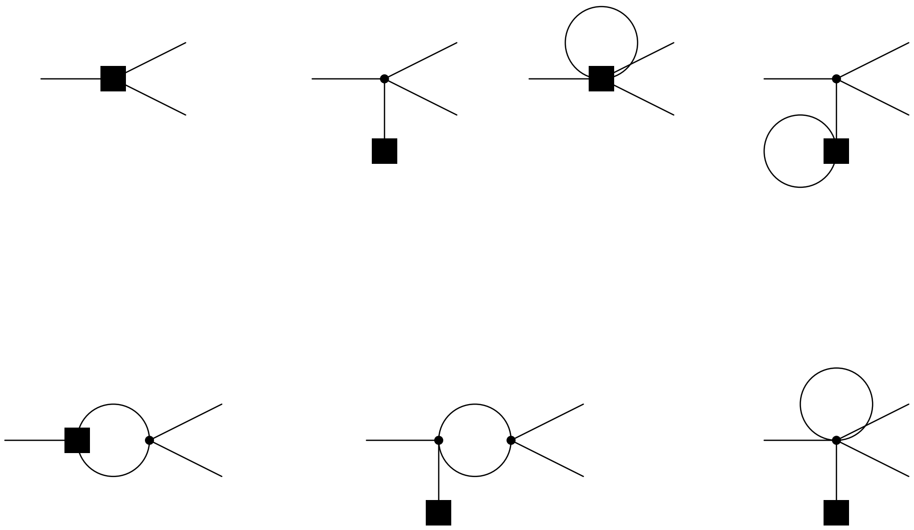

Figure 3: The diagrams contributing to on-shell $K \to \pi\pi$ amplitude up to one loop. Symbols as in Figs. 1 and 2. In addition, renormalization of external legs and internal propagators of the tree level diagrams have to be added.

$\Delta I = 3/2$ contributions are however separate. The two-point function $\Pi_{K^0\eta_8}(q^2)$ and the combination $(1/\sqrt{2})\Pi_{K^0\pi^0}(q^2) - \Pi_{K^+\pi^+}(q^2)$ are pure $\Delta I = 1/2$, while $\sqrt{2}\Pi_{K^0\pi^0}(q^2) + \Pi_{K^+\pi^+}(q^2)$ is pure $\Delta I = 3/2$.

The $K \to \pi$ and $K \to \eta_8$ two-point functions defined here are not measurable in experiments, so obtaining the counterterms from them implies that one has to calculate the relevant two-point functions either using lattice QCD [25, 26] or using other hadronic approaches [8, 9].

Recent work on $\Delta I = 3/2$ transitions in quenched CHPT [27] gives numerically consistent results with quenched results on B_K. The main uncertainty are the unknown quenched counterterms: G_{27} and D_i's. The value of several of them can be similarly extracted from two-point functions but this has not been done so far [28]. In the near future it might be possible to calculate the two-point functions we propose in (1.1) both in the quenched and unquenched case and the $K \to \pi\pi$ amplitude only quenched. The comparison of all the constants that can be calculated using the simpler two-point correlator would then be a useful tool to estimate quenching errors on the remainder.

We now discuss the expressions of Sect. 3 to check which constants are obtainable. The order p^4 counterterms E_{13}, E_{15}, and D_5 cannot be obtained from the $\Pi_{ij}(q^2)$ since they do not contribute to them. They do however contribute to $K \to \pi\pi$. Their value in the large N_c limit and with the factorization assumption is known, see Sect. 2.2.

In the chiral limit we can get G_8, G_{27}, E_3, and D_2. Away from the chiral limit, we can get G_8' from the terms quadratic in the meson masses. From the terms quartic in the meson masses we can get E_1, E_2, and D_1. The terms proportional to $q^2 m^2_{\pi,K}$ allow us to obtain the combination $2E_{10} + E_{11}$, D_4, D_6, and D_7. The latter also determine two more combinations of couplings, though not needed for $K \to \pi\pi$: $E_4 + E_{12}$ and $E_5 + E_{10}$. E_{10} is needed for $K \to \pi\pi$ but cannot be separately disentangled from the two-point functions $\Pi_{ij}(q^2)$.

We have determined three order p^2 couplings and eleven order p^4 ones. We have in addition four relations which are independent of the value of the couplings to test chiral symmetry at order p^4. This is the main result of this manuscript. We are left with four unknowns E_{10}, E_{13}, E_{15}, and D_5 in order to extrapolate to $K \to \pi\pi$.

What can be said about the missing coefficients. Three kind of arguments can be used:

1) Order of magnitude: We know the leading in $1/N_c$ contributions to all of them, Eq. (2.30). In particular, E_{13}, E_{15}, and D_5 only receive non-factorizable contributions. From the discussion above, we have seen that we can determine eleven next-to-leading in $1/N_c$ contributions to the couplings, this should give us some information on the ones we cannot get.

For instance, assuming that all the $1/N_c$ contributions are of the same order, since $2E_{10} + E_{11}$ is obtained from terms of the type $q^2 m^2_{\pi,K}$ and we know its factorizable contribution

$$(2E_{10} + E_{11})_{\text{Factorizable}} = 8L_5 + 24L_4 \,, \tag{5.1}$$

we can use (conservative choice) the next-to-leading result we have for $2E_{10} + E_{11}$ as a good estimate for E_{13}, E_{15} and the non-factorizable part of E_{10}. Analogously for D_5 we can use the result

$$(D_4)_{\text{Factorizable}} = 4L_5 \,, \quad (D_7)_{\text{Factorizable}} = 8L_4 \,, \tag{5.2}$$

for predicting the non-factorizable part of D_5.

2) Resonance Saturation: Another possibility is using estimates of higher order parameters coming from resonance exchange saturation, as done in [24]. It is well known and experimentally proven that vector and/or axial-vector dominance is not at work in the weak sector (which is instead the case in the strong one). Within the tensor formulation of vector (axial-vector) resonances used in [24] all the counterterms in (2.16) only receive contributions from scalar and/or pseudo-scalar resonances. In Table 2 we summarize the resonance contributions to the counterterms contained in (2.16). The notation is the one used in [24]. For the 27-plet the derivation is done in Appendix C. All of the unknown terms E_{10}, E_{13}, E_{15} and D_5 only receive scalar and/or pseudo-scalar contributions. This is consistent with the consequences of the factorization assumption as shown in (2.30), where the weak counterterms are all expressed in terms of the strong

	S	S_1	P
E_1	$2c_m g_S^1$	–	–
E_2	$c_m\left(g_S^2 - \frac{2}{3}g_S^1\right)$	$\tilde{c}_m\tilde{g}_S^1$	–
E_3	–	–	$2d_m g_P^1$
E_4	–	–	$d_m\left(-\frac{2g_P^1}{3} + g_P^2\right)$
E_5	$c_m g_S^3$	–	$-d_m g_P^3$
E_{10}	$c_d(g_S^1 + g_S^2) + \frac{c_m}{3}g_S^4$	$\tilde{c}_m\tilde{g}_S^2$	–
E_{11}	$c_d g_S^2 - \frac{2}{3}c_m g_S^4$	$\tilde{c}_m\tilde{g}_S^2$	–
E_{12}	$-\frac{c_d}{2}g_S^2 + c_m\left(\frac{g_S^4}{3} + \frac{g_S^5}{2}\right)$	$-\frac{\tilde{c}_m}{2}\tilde{g}_S^2$	–
E_{13}	$-c_d\left(\frac{2g_S^1}{3} + \frac{g_S^2}{2}\right) + c_m\left(g_S^6 + \frac{g_S^4}{3}\right)$	$\tilde{c}_d\tilde{g}_S^1 - \frac{\tilde{c}_m}{2}\tilde{g}_S^2$	–
E_{15}	$-c_d g_S^3$	–	$-d_m g_P^4$
D_1	$c_m\bar{g}_S^1$	–	–
D_2	–	–	$-d_m\bar{g}_P^1$
D_4	$c_m\bar{g}_S^2$	–	–
D_5	–	–	$-d_m\bar{g}_P^2$
D_6	$c_d\bar{g}_S^1 + c_m\bar{g}_S^3$	–	–
D_7	–	$\tilde{c}_m\tilde{\bar{g}}_S^1$	–

Table 2: The contributions to the octet E_i and 27-plet D_i counterterms in (2.16) from scalar octet (S), scalar singlet (S_1) and pseudo-scalar octet (P) resonance. The pseudo-scalar singlet (P_1) only contributes to E_4 with a term $\tilde{d}_m\tilde{g}_P^1$. A factor $1/M_R^2$ is pulled out. Vector and axial-vector resonances exchange do not contribute [24].

counterterms L_4 and L_5, which are in turn saturated by scalar exchange [24, 29]. A simplified resonance model is e.g. the one where pseudo-scalar resonances exchange is neglected [7]. Their contribution is small in the strong sector [29]. With this reduction several relations are valid. $E_3 = 0, E_4 = 0, D_2 = 0$ and $D_5 = 0$ are a test of scalar dominance, while

$$2E_{10}^r + E_{11}^r - \frac{c_d}{c_m}\left(2E_1^r - 2E_3^r + 3E_2^r\right) = 0 \tag{5.3}$$

[7] One can, of course, use other resonance models to make the analysis. We mention that in the vector formulation with vector fields used in [30], not antisymmetric tensor fields as in [24], E_{15} also receives contribution from vector resonance exchange. We do not address here the question of the equivalence of different resonance models in the weak sector.

tests the absence of singlet scalar resonance contributions. The couplings c_d and c_m are scalar couplings from the strong sector [24, 29]. Additional relations are $(c_m/c_d)\ E_{15}^r + E_5^r - E_3^r = 0$, valid in the absence of pseudo-scalar resonance contributions, and

$$E_{13}^r = \frac{1}{6}\left[-(2E_{10}^r + E_{11}^r) - \frac{c_d}{c_m}(E_1^r - E_3^r) + \frac{c_m}{c_d}(3N_1^r + 3N_2^r + 6N_3^r)\right], \quad (5.4)$$

valid in the absence of the singlet scalar resonance. The combination of N_i corresponds to K_2 which receives no vector/axial-vector contributions and is a combination of the p^4 constants that can be determined from $K \to 3\pi$ decays [24, 31]. Hence, introducing additional information about $K \to 3\pi$ decays we can use resonance arguments to determine E_{13}. One main observation is that the direct determination of most of the couplings from the analysis of the two-point functions offers already a powerful test of the validity of different resonance saturation assumptions.

3) Factorization: We can of course also adopt the strong factorization assumption as often used to get at the undetermined parameters. Again this procedure can be well tested by the fourteen parameters we can actually determine and comparing them with the predictions of Eq. (2.30) with G_8 and G_{27} as free parameters.

6 The Weak Mass Term Contributions

In the literature there are conflicting opinions about whether the weak mass term contributes to $K \to \pi$ and $K \to$ vacuum matrix elements. In [15] the claim is that they do not and in [4, 6, 7] they do. The underlying reason for the difference is that the meson fields used in those two references differ by a field redefinition. For *on-shell* matrix elements this makes no difference and as a consequence both analyses agree for the $K \to \pi\pi$ amplitudes. Since neither the $K \to \pi$ transition nor the $K \to$ vacuum transition can be allowed on-shell, if the masses are such that $K \to \pi\pi$ is possible on-shell, we first have to correctly *define* what we mean by off-shell matrix elements.

Green's functions defined by quark currents, as introduced in CHPT by Gasser and Leutwyler [21], are well defined for all values of momenta and thus provide a proper definition of off-shell quantities. We have shown here that for the pseudo-scalar currents as sources the two-point function that defines properly an off-shell $K \to \pi$ transition does depend on the weak mass term, or the coefficient G_8'.

Of course, we also find that for the on-shell transition $K \to \pi\pi$, the weak mass term does not depend on G_8' to order p^2 as shown before in [4, 14, 15]; but it does have contributions at higher order, see the discussion below.

The discussion of [11] and [20] indicates at which level the weak mass term can contribute to $K \to \pi\pi$ amplitudes. The basic argument can be phrased in terms of the strong equations of motion for the u field in (2.2). Terms that can be removed using the equations of motion can be described by changes in the other coefficients of the Lagrangian. The argument of Sonoda and Georgi is that the weak mass term can be written as a total derivative assuming $m_s \neq m_d$ (see also [20]) and as such does not contribute to physical amplitudes. This argument fails in the presence of external fields. It crucially requires $s+ip = \mathcal{M}$, otherwise other non-derivative terms involving $s+ip-\mathcal{M}$ remain after the use of the equations of motion. That is why the argument fails for the non-tadpole diagrams like the two-point functions considered here. These extra non-derivative terms give the G_8' contributions to our $\Pi_{ij}(q^2)$ two-point functions.

To see what happens in the case we add external vector and axial vector fields, let us show the argument of Sonoda and Georgi in more detail extended to include vector and axial-vector external fields. The basic underlying argument is that the effects of terms that vanish using the lowest order equation of motion can be described by changes in the other parameters of the effective Lagrangian. As we will show below this implies that the effects of the weak mass term can be absorbed in shifts of the other parameters for all processes involving on-shell pseudo-scalars and photons. As a consequence the contributions from the G_8' term vanish at order p^2 for these type of diagrams and can be absorbed in shifts of the other parameters at higher order, i.e. the E_i and D_i of [11] at order p^4. As stressed earlier, this does not mean that the contributions from the weak mass term are zero in this case, only that they can be described by shifts of the other parameters.

The equation of motion from the lowest order Lagrangian in (2.2) is

$$2D_\mu(U^\dagger D^\mu U) - U^\dagger\chi + \chi^\dagger U - \frac{1}{3}\,\mathrm{tr}(-U^\dagger\chi + \chi^\dagger U) = 0\,. \tag{6.1}$$

When the external scalar and pseudoscalar fields are zero this becomes for $i \neq j$

$$\begin{aligned} 2D_\mu(U^\dagger D^\mu U)_{ij} - 2B_0 U^\dagger_{ij} m_j + 2B_0 U_{ij} m_i &= 0 \\ -2D_\mu(U D^\mu U^\dagger)_{ij} - 2B_0 U^\dagger_{ij} m_i + 2B_0 U_{ij} m_j &= 0\,. \end{aligned} \tag{6.2}$$

The second equation can be derived by first multiplying Eq. (6.1) on the left by U and on the right by $U^\dagger$ and using the unitarity of U extensively. The weak mass term for $\Delta S = 1$ transitions is proportional to

$$\begin{aligned} (\chi^\dagger U + U^\dagger\chi)_{23} &= 2B_0\left(m_d U_{23} + m_s U^\dagger_{23}\right) \\ &= 2\frac{m_s^2+m_d^2}{m_s^2-m_d^2}(D_\mu(U^\dagger D^\mu U))_{23} + 4\frac{m_s m_d}{m_s^2-m_d^2}(D_\mu(U D^\mu U^\dagger))_{23}\,. \end{aligned} \tag{6.3}$$

For external vector and axial-vector fields zero or equal to the photon field the last line is a total derivative and thus does not contribute to the action. This proves the comments made above.

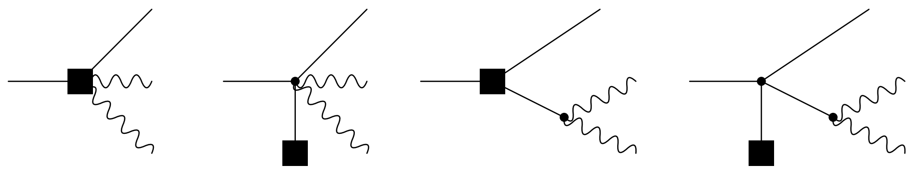

Figure 4: The Feynman diagrams for $K_S \to \pi^0\gamma\gamma$ at order p^4. Notation as in previous figures. The strong vertices can now also be the Wess-Zumino term.

Processes with on-shell pions and kaons and photons could in principle depend on G_8' already at their lowest non-zero order. We have verified that G_8' does contribute to $K_L \to \gamma\gamma$, and hence to $K_L \to \mu^+\mu^-$, at lowest non-zero order (i.e. at order p^6) similarly to the part from G_8[8]. For $K_S \to \pi^0\gamma\gamma$ there is already a contribution at order p^4. The relevant diagrams are depicted in Fig. 4. The contributions from the weak mass term are non-zero but local, the non-local parts containing $1/(q_{\gamma\gamma}^2 - m_M^2)$ cancel, and thus as proved in general above can be absorbed in shifts of the E_i.

These results are important since this contribution has never been included in the long distance estimates of the processes above. Notice also that the value of G_8' is unknown.

In addition, physical amplitudes like $K \to \pi\pi$ that do not depend on G_8' at lowest order can depend on it at next-to-leading order in CHPT. In this case the G_8' contribution can be reabsorbed in shifts of order p^4 and higher couplings. Nevertheless notice that in analytical predictions, the value of G_8' should be known to do this shift.

7 Conclusions

We have extended the p^2 analysis of [4] with the non-analytic contributions of [5] in two ways: first, we have included all the relevant p^4 couplings and the full one loop contributions to the two-point functions $K \to \pi, \eta_8$ and the $K \to \pi\pi$ amplitude and second, we have changed from the vague notion of an off-shell meson field to the well-defined notion of Green's functions of external fields in the presence of the weak non-leptonic interaction.

We have confirmed the results of [4] that the three relevant couplings at order p^2 can be fully determined and discussed the necessity of including the weak mass term in this analysis.

We concluded that the weak mass term can contribute in some physical amplitudes even at their lowest non-zero order and have given several examples

[8] See [32] for the standard discussion and the warning about SU(3) breaking parameters

where this happens. In general this can happen in the case of on-shell Green's functions which receive contributions from off-shell flavour changing two-point functions. As an example, the weak mass term gives lowest non-zero order unknown long-distance contributions to processes like $K_L \to \gamma\gamma$, $K_S \to \pi^0\gamma\gamma$ and $K_L \to \mu^+\mu^-$.

To order p^4 in the chiral expansion we find six more parameters in the 27-plet sector that could in principle contribute to $K \to \pi, \eta_8, \pi\pi$. Of these, we can directly determine five from the two-point functions, leaving one, D_7 as a free parameter. This parameter vanishes in the large N_c limit and is proportional to L_4 in the factorization model. If the predictions of this model turn out to be satisfied by the other five parameters we can take the factorization prediction for D_7 and obtain a value for $K \to \pi\pi$.

In the octet sector, there are ten more operators of which we can also directly determine six combinations. We can use them to test the predictions of various models like factorization, the weak deformation model [24], resonance models, etc. The weaker assumptions of resonance saturation by vector, axial-vector and scalar resonance exchange allows to determine one more combination of counterterms from the two-point functions and one more parameter (E_{13}) can be fixed if one of the slope parameters of $K \to 3\pi$, namely K_2 of Ref. [13], is known. To obtain the full set of counterterms at order p^4 we need to use factorization or another more restrictive model. Factorization and alternative models can be strongly constrained by the value of the parameter combinations that can be directly extracted from $K \to \pi, \eta_8$ two-point functions.

As a calculational tool, we have provided the complete one-loop formulas for the two-point functions of the octet symmetry case and for the on-shell $K \to \pi\pi$ amplitudes with quark masses $m_s \neq m_d = m_u$ all different from zero. In [13] and [24] the combinations suppressed by m_π^2/m_K^2 were neglected. This might be a good approximation for the real quark mass values [at the level of 10 % though], but will not necessarily be true on the lattice.

Since our two-point functions are much easier to determine on the lattice they also provide a better laboratory to study unquenching effects in the non-leptonic weak sector than the full $K \to \pi\pi$ amplitudes.

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A Order p^4 Loop Contributions to $\Pi_{ij}(q^2)$

Using

$$P(m^2) \equiv \frac{m_K^2}{16\pi^2 F_0^2} \frac{m^2}{m_K^2 - m^2} \ln\left(\frac{m_K^2}{m^2}\right) \tag{A.1}$$

we get

$$\begin{aligned}
\Pi_{K^0\eta_8}(q^2)\,|_{\text{Logs}} &= -\frac{\sqrt{Z_{K^0}Z_{\eta_8}}}{(q^2-m_{K^0}^2)(q^2-m_{\eta_8}^2)} \frac{C}{\sqrt{6}} \frac{F_0^4}{f_{K^0}f_{\eta_8}} \\
&\times \Big[G_8\Big\{q^2\left(\frac{20}{3}\mu_{\eta_8} + \frac{38}{3}\mu_K - \frac{3}{2}P(m_\pi^2) + \frac{11}{6}P(m_{\eta_8}^2)\right) \\
&- m_K^2\left(\frac{13}{9}\mu_{\eta_8} + \frac{28}{9}\mu_K + \mu_\pi + \frac{1}{2}P(m_\pi^2) + \frac{1}{18}P(m_{\eta_8}^2)\right) \\
&- m_\pi^2\left(-\frac{1}{9}\mu_{\eta_8} + \frac{20}{9}\mu_K - \mu_\pi + P(m_\pi^2) + \frac{1}{9}P(m_{\eta_8}^2)\right)\Big\} \\
&+ \frac{G_{27}}{3}\Big\{-q^2\left(20\mu_{\eta_8} + 78\mu_K + 10\mu_\pi + \frac{1}{2}P(m_\pi^2) + \frac{11}{2}P(m_{\eta_8}^2)\right) \\
&+ m_K^2\left(\frac{13}{3}\mu_{\eta_8} - 4\mu_K - \frac{1}{3}\mu_\pi - \frac{1}{6}P(m_\pi^2) + \frac{1}{6}P(m_{\eta_8}^2)\right) \\
&+ \frac{m_\pi^2}{3}\left(-\mu_{\eta_8} + \mu_\pi - P(m_\pi^2) + P(m_{\eta_8}^2)\right)\Big\} \\
&- G_8'\Big\{q^2\left(\frac{8}{3}\mu_8 + 6\mu_K - 8\mu_\pi - \frac{3}{2}P(m_\pi^2) + \frac{11}{6}P(m_{\eta_8}^2)\right) \\
&+ m_K^2\left(-\frac{5}{9}\mu_{\eta_8} - \frac{20}{9}\mu_K + \frac{5}{3}\mu_\pi - \frac{1}{2}P(m_\pi^2) - \frac{1}{18}P(m_{\eta_8}^2)\right) \\
&+ m_\pi^2\left(-\frac{1}{9}\mu_{\eta_8} - \frac{22}{9}\mu_K + \frac{1}{3}\mu_\pi - P(m_\pi^2) - \frac{1}{9}P(m_{\eta_8}^2)\right)\Big\}\Big]
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\Pi_{K^0\pi^0}(q^2)\,|_{\text{Logs}} &= -\frac{\sqrt{Z_{K^0}Z_{\pi^0}}}{(q^2-m_{K^0}^2)(q^2-m_{\pi^0}^2)} \frac{C}{\sqrt{2}} \frac{F_0^4}{f_{K^0}f_{\pi^0}} \\
&\times \Big[G_8\Big\{q^2\left(\frac{8}{3}\mu_{\eta_8} + \frac{26}{3}\mu_K + 8\mu_\pi + \frac{1}{2}P(m_\pi^2) - \frac{1}{6}P(m_{\eta_8}^2)\right) \\
&- m_K^2\left(-\frac{1}{3}\mu_{\eta_8} + \frac{4}{9}\mu_K + \frac{7}{3}\mu_\pi + \frac{7}{6}P(m_\pi^2) + \frac{1}{18}P(m_{\eta_8}^2)\right) \\
&- m_\pi^2\left(\frac{1}{3}\mu_{\eta_8} + \frac{8}{9}\mu_K + 3\mu_\pi + \frac{1}{3}P(m_\pi^2) + \frac{1}{9}P(m_{\eta_8}^2)\right)\Big\} \\
&+ \frac{G_{27}}{3}\Big\{-q^2\left(8\mu_{\eta_8} + 46\mu_K + 54\mu_\pi + \frac{13}{2}P(m_\pi^2) - \frac{1}{2}P(m_{\eta_8}^2)\right) \\
&+ m_K^2\left(-\mu_{\eta_8} + \frac{4}{3}\mu_K - 3\mu_\pi - \frac{3}{2}P(m_\pi^2) + \frac{1}{6}P(m_{\eta_8}^2)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad m_\pi^2 \left(\mu_{\eta_8} + \frac{8}{3}\mu_K - \mu_\pi + P(m_\pi^2) + \frac{1}{3} P(m_{\eta_8}^2) \right) \Big\} \\
&- \quad G_8' \left\{ q^2 \left(\frac{2}{3}\mu_K + \frac{1}{2} P(m_\pi^2) - \frac{1}{6} P(m_{\eta_8}^2) \right) \right. \\
&+ \quad m_K^2 \left(\frac{1}{3}\mu_{\eta_8} - \frac{4}{9}\mu_K - \frac{7}{3}\mu_\pi - \frac{7}{6} P(m_\pi^2) - \frac{1}{18} P(m_{\eta_8}^2) \right) \\
&+ \quad m_\pi^2 \left(-\mu_{\eta_8} - \frac{26}{9}\mu_K + 3\mu_\pi - \frac{1}{3} P(m_\pi^2) - \frac{1}{9} P(m_{\eta_8}^2) \right) \Big\} \Big] \quad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{K^+\pi^+}(q^2)\,|_{\rm Logs} \quad &= \quad - \frac{\sqrt{Z_{K^+} Z_{\pi^+}}}{(q^2 - m_{K^+}^2)(q^2 - m_{\pi^+}^2)}\, C \; \frac{F_0^4}{f_{K^+} f_{\pi^+}} \\
&\times \quad \Big[G_8 \left\{ -q^2 \left(\frac{8}{3}\mu_{\eta_8} + \frac{26}{3}\mu_K + 8\mu_\pi + \frac{1}{2} P(m_\pi^2) - \frac{1}{6} P(m_{\eta_8}^2) \right) \right. \\
&+ \quad m_K^2 \left(-\frac{1}{3}\mu_{\eta_8} + \frac{4}{9}\mu_K + \frac{7}{3}\mu_\pi + \frac{7}{6} P(m_\pi^2) + \frac{1}{18} P(m_{\eta_8}^2) \right) \\
&+ \quad m_\pi^2 \left(\frac{1}{3}\mu_{\eta_8} + \frac{8}{9}\mu_K + 3\mu_\pi + \frac{1}{3} P(m_\pi^2) + \frac{1}{9} P(m_{\eta_8}^2) \right) \Big\} \\
&+ \quad \frac{G_{27}}{3} \left\{ q^2 \left(-7\mu_{\eta_8} - 34\mu_K - 31\mu_\pi - \frac{7}{2} P(m_\pi^2) - \frac{1}{2} P(m_{\eta_8}^2) \right) \right. \\
&+ \quad m_K^2 \left(\mu_{\eta_8} - \frac{4}{3}\mu_K - \frac{1}{3}\mu_\pi - \frac{1}{6} P(m_\pi^2) - \frac{1}{6} P(m\eta_8{}^2) \right) \\
&+ \quad m_\pi^2 \left(-\mu_{\eta_8} + \frac{2}{3}\mu_K + \mu_\pi + \frac{2}{3} P(m_\pi^2) - \frac{1}{3} P(m_{\eta_8}^2) \right) \Big\} \\
&+ \quad G_8' \left\{ q^2 \left(\frac{2}{3}\mu_K + \frac{1}{2} P(m_\pi^2) - \frac{1}{6} P(m_{\eta_8}^2) \right) \right. \\
&+ \quad m_K^2 \left(\frac{1}{3}\mu_{\eta_8} - \frac{4}{9}\mu_K - \frac{7}{3}\mu_\pi - \frac{7}{6} P(m_\pi^2) - \frac{1}{18} P(m_{\eta_8}^2) \right) \\
&+ \quad m_\pi^2 \left(-\mu_{\eta_8} - \frac{26}{9}\mu_K + 3\mu_\pi - \frac{1}{3} P(m_\pi^2) - \frac{1}{9} P(m_{\eta_8}^2) \right) \Big\} \Big] \quad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{K^0\overline{K}^0}(q^2)\,|_{\rm Logs} \quad &= \quad - \frac{Z_{K^0}}{(q^2 - m_{K^0}^2)^2}\, C_{\Delta S=2}\, G_{27} \\
&\times \quad 4 \frac{F_0^4}{f_{K^0}^2} \left[-q^2 \left[\frac{m_K^2}{16\pi^2 F_0^2} + 10\mu_K + \frac{7}{2}\mu_\pi + \frac{9}{2}\mu_{\eta_8} \right] \right. \\
&+ \quad 2m_K^2 \mu_K - \frac{1}{2} m_\pi^2 \mu_\pi - \frac{3}{2} m_{\eta_8}^2 \mu_{\eta_8} \Big] \, . \qquad \text{(A.5)}
\end{aligned}
$$

B Order p^4 Loop Contributions to $K \to \pi\pi$ Amplitudes

In addition to the definitions used above we need now,

$$
\begin{aligned}
B(m_1^2, m_2^2, p^2) &= \frac{1}{16\pi^2 F_0^2}\left[-1 + \ln\left(\frac{m_2^2}{\nu^2}\right) + \frac{1}{2}\ln\left(\frac{m_1^2}{m_2^2}\right)\left(1 + \frac{m_1^2}{p^2} - \frac{m_2^2}{p^2}\right)\right. \\
&+ \left.\frac{1}{2}\lambda^{1/2}\left(1, \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right)\ln\left[\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)}\right]\right],
\end{aligned}
$$

for $p^2 > (m_1 + m_2)^2$ and $p^2 \le (m_1 - m_2)^2$ while

$$
\begin{aligned}
B(m_1^2, m_2^2, p^2) &= \frac{1}{16\pi^2 F_0^2}\left[-1 + \ln\left(\frac{m_2^2}{\nu^2}\right) + \frac{1}{2}\ln\left(\frac{m_1^2}{m_2^2}\right)\left(1 + \frac{m_1^2}{p^2} - \frac{m_2^2}{p^2}\right)\right. \\
&- \left.\frac{1}{2}\sqrt{-\lambda\left(1, \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right)}\arctan\left[\frac{(p^2 - m_1^2 - m_2^2)\sqrt{-\lambda(p^2, m_1^2, m_2^2)}}{(p^2 - m_1^2 - m_2^2)^2 - 2m_1^2 m_2^2}\right]\right]
\end{aligned}
$$

for $(m_1 - m_2)^2 < p^2 \le (m_1 + m_2)^2$ and $-\pi/2 < \arctan(x) < \pi/2$

$$
\lambda(x, y, z) = (x + y - z)^2 - 4xy\,. \tag{B.1}
$$

The non-analytic parts of A_0^8, A_0^{27}, and A_2 are

$$
\begin{aligned}
\Im m\, A_0^8|_{\rm Logs} &= -C\, G_8\sqrt{6}\,\frac{F_0^4}{f_K f_\pi^2}\left[m_\pi^2\left(2\mu_K - \mu_\pi - \mu_{\eta_8}\right)\right. \\
&+ (m_K^2 - m_\pi^2)\left\{\frac{m_K^2}{2m_\pi^2}\left(\mu_\pi - \mu_{\eta_8}\right) - \frac{5}{2}\mu_\pi - \frac{1}{2}\mu_{\eta_8} - 3\mu_K\right\} \\
&+ (m_K^2 - m_\pi^2)\left(\frac{1}{2}m_\pi^2\left[B(m_\pi^2, m_\pi^2, m_K^2) - \frac{1}{9}B(m_{\eta_8}^2, m_{\eta_8}^2, m_K^2)\right]\right. \\
&+ \frac{1}{4}\frac{m_K^4}{m_\pi^2}\left[B(m_K^2, m_\pi^2, m_\pi^2) + \frac{1}{3}B(m_K^2, m_{\eta_8}^2, m_\pi^2)\right] \\
&- \left.\left. m_K^2\left[B(m_\pi^2, m_\pi^2, m_K^2) + B(m_K^2, m_\pi^2, m_\pi^2)\right]\right)\right]\,;
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
\Im m\, A_0^{27}|_{\rm Logs} &= -C\, G_{27}\frac{\sqrt{6}}{9}\,\frac{F_0^4}{f_K f_\pi^2}\left[m_\pi^2\left(2\mu_K - \mu_\pi - \mu_{\eta_8}\right)\right. \\
&+ (m_K^2 - m_\pi^2)\left\{\frac{m_K^2}{2m_\pi^2}\left(\mu_\pi + 4\mu_{\eta_8} - 5\mu_K\right) - \frac{5}{2}\mu_\pi - \frac{11}{2}\mu_{\eta_8} - 8\mu_K\right\} \\
&+ (m_K^2 - m_\pi^2)\left(\frac{1}{2}m_\pi^2\left[B(m_\pi^2, m_\pi^2, m_K^2) + B(m_{\eta_8}^2, m_{\eta_8}^2, m_K^2)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{4}\frac{m_K^4}{m_\pi^2}\left[B(m_K^2,m_\pi^2,m_\pi^2)-\frac{4}{3}B(m_K^2,m_{\eta_8}^2,m_\pi^2)\right] \\
&- \quad m_K^2\left[B(m_\pi^2,m_\pi^2,m_K^2)+B(m_K^2,m_\pi^2,m_\pi^2)\right]\Big)\Big] \; ;
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\Im m\, A_2\,|_{\mathrm{Logs}} \;=\;& -C\,G_{27}\frac{10\sqrt{3}}{9}\,\frac{F_0^4}{f_K f_\pi^2}\Big[m_\pi^2\,(2\mu_\pi-\mu_{\eta_8}-\mu_K) \\
&+\; (m_K^2-m_\pi^2)\left\{\frac{m_K^2}{4m_\pi^2}\,(5\mu_\pi-\mu_{\eta_8}-4\mu_K)-10\mu_\pi-\mu_{\eta_8}-5\mu_K\right\} \\
&+\; (m_K^2-m_\pi^2)\Big(-m_\pi^2 B(m_\pi^2,m_\pi^2,m_K^2) \\
&+\; \frac{1}{8}\frac{m_K^4}{m_\pi^2}\left[5B(m_K^2,m_\pi^2,m_\pi^2)+\frac{1}{3}B(m_K^2,m_{\eta_8}^2,m_\pi^2)\right] \\
&+\; \frac{1}{2}m_K^2\left[B(m_\pi^2,m_\pi^2,m_K^2)-2B(m_K^2,m_\pi^2,m_\pi^2)\right]\Big)\Big] \; ;
\end{aligned}
\tag{B.4}
$$

$$
\Re e\, A_0^8 \;=\; -C\,G_8\sqrt{6}\,\frac{F_0^2}{f_K f_\pi^2}\,\frac{m_K^2-m_\pi^2}{64\pi}\,(m_\pi^2-2m_K^2)\sqrt{1-\frac{4m_\pi^2}{m_K^2}}\; ; \tag{B.5}
$$

$$
\Re e\, A_0^{27} \;=\; -C\,G_{27}\frac{\sqrt{6}}{9}\,\frac{F_0^2}{f_K f_\pi^2}\,\frac{m_K^2-m_\pi^2}{64\pi}\,(m_\pi^2-2m_K^2)\sqrt{1-\frac{4m_\pi^2}{m_K^2}}\; ; \tag{B.6}
$$

$$
\Re e\, A_2 \;=\; -C\,G_{27}\frac{10\sqrt{3}}{9}\,\frac{F_0^2}{f_K f_\pi^2}\,\frac{m_K^2-m_\pi^2}{64\pi}\,(m_K^2-2m_\pi^2)\sqrt{1-\frac{4m_\pi^2}{m_K^2}}\; ; \tag{B.7}
$$

$$
\delta_i \;=\; -\arctan\left(\frac{\Re e\, A_i}{\Im m\, A_i}\right) \qquad \text{for} \qquad i=0,2\,. \tag{B.8}
$$

C The 27-plet Weak Lagrangian from Resonance Exchange Saturation

We derive the weak effective Lagrangian at order p^4 in the 27-plet sector for $\Delta S=\pm1$ transitions by assuming resonance exchange saturation of the couplings.

We restrict the derivation to those terms listed in (2.18). The derivation of the octet sector can be found in [24]. We refer the reader to Refs. [24, 29] for details on the method. The weak 27-plet Lagrangian in (2.16) can only receive contributions from scalar (octet S and singlet S_1) and pseudo-scalar octet P resonances. The relevant weak couplings of the light meson fields to resonances can be written as follows:

$$\mathcal{L}_R^{27} = \sum_{i=1}^{3} \bar{g}_S^i K_i^S + \sum_{i=1}^{2} \bar{g}_P^i K_i^P + \tilde{\bar{g}}_S^1 K_1^{S_1} \; + \text{h.c.}\,, \tag{C.1}$$

where

$$\begin{aligned}
K_1^S &= t_{kl}^{ij}\ \text{tr}(\Delta_{ij} S)\ \text{tr}(\Delta_{kl}\chi_+)\,, \quad K_2^S = t_{kl}^{ij}\ \text{tr}(\Delta_{ij}\{S, u_\mu\})\ \text{tr}(\Delta_{kl} u^\mu)\,, \\
K_3^S &= t_{kl}^{ij}\ \text{tr}(\Delta_{ij} S)\ \text{tr}(\Delta_{kl} u_\mu u^\mu)\,, \\
K_1^P &= i t_{kl}^{ij}\ \text{tr}(\Delta_{ij} P)\ \text{tr}(\Delta_{kl}\chi_-)\,, \quad K_2^P = i t_{kl}^{ij}\ \text{tr}(\Delta_{ij}[u_\mu, P])\ \text{tr}(\Delta_{kl} u^\mu)\,, \\
K^{S_1} &= S_1\ \ \text{tr}(\Delta_{ij} u_\mu)\ \text{tr}(\Delta_{kl} u^\mu)\,.
\end{aligned} \tag{C.2}$$

Inserting the lowest order solution of the equations of motion for the resonance fields in (C.2), the 27-plet weak effective Lagrangian at order p^4 and order G_F is given by

$$\mathcal{L}_{27}^{(4)} = \sum_{R=S,P,S_1} \frac{1}{M_R^2}\ \text{tr}(J_s^R J_w^R)\,, \tag{C.3}$$

where J_s^R, J_w^R are the strong and weak currents respectively, coupled to the resonance R at lowest chiral order p^2. The weak currents are defined from (C.1) as follows

$$\mathcal{L}_R^{27} = \text{tr}(S J_w^S) + \text{tr}(P J_w^P) + S_1 J_w^{S_1}\,, \tag{C.4}$$

while the strong currents are given by [29]

$$\begin{aligned}
J_s^S &= c_d\ u_\mu u^\mu + c_m\ \chi_+ \qquad J_s^{S1} = \tilde{c}_d\ \text{tr}(u_\mu u^\mu) + \tilde{c}_m\ \text{tr}(\chi_+) \\
J_s^P &= i d_m\ \chi_-\,.
\end{aligned} \tag{C.5}$$

The contributions to the low energy 27-plet weak effective Lagrangian (C.3) are summarized in Table 2.

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