Short-distance constraints for the HLbL contribution to the muon anomalous magnetic moment

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Abstract

We derive short-distance constraints for the hadronic light-by-light contribution (HLbL) to the anomalous magnetic moment of the muon in the kinematic region where the three virtual momenta are all large. We include the external soft photon via an external field leading to a well-defined Operator Product Expansion. We establish that the perturbative quark loop gives the leading contribution in a well defined expansion. We compute the first nonzero power correction. It is related to the magnetic susceptibility of the QCD vacuum. The results can be used as model-independent short-distance constraints for the very many different approaches to the HLbL contribution. Numerically the power correction is found to be small.
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Abstract

We derive short-distance constraints for the hadronic light-by-light contribution (HLbL) to the anomalous magnetic moment of the muon in the kinematic region where the three virtual momenta are all large. We include the external soft photon via an external field leading to a well-defined Operator Product Expansion. We establish that the perturbative quark loop gives the leading contribution in a well-defined expansion. We compute the first nonzero power correction. It is related to the magnetic susceptibility of the QCD vacuum. The results can be used as model-independent short-distance constraints for the very many different approaches to the HLbL contribution. Numerically the power correction is found to be small.

1. Introduction

The anomalous magnetic moment of the muon is one of the most powerful low-energy probes of the Standard Model (SM). Its experimental value via $a_\mu = (g_\mu - 2)/2$,\cite{1,2},

$$a_\mu^{\text{exp}} = 116592091(63) \times 10^{-11},$$

is expected to be significantly improved\cite{3,4}. The present theoretical prediction is \cite{2}

$$a_\mu^{\text{SM}} = 116591823(43) \times 10^{-11}.$$\hspace{1cm}(2)

The tension between (1) and (2) might be a sign of physics beyond the SM. Both the theoretical prediction and the measured value thus need improvement. Reviews of the theory are \cite{5,6}.

A major contributor to the theoretical error is the hadronic light-by-light contribution (HLbL or $a_\mu^{\text{HLbL}}$) depicted in Figure 1. It involves the evaluation of the 4-point correlation function of electromagnetic quark currents

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x \, d^4y \, d^4z \, e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \times (0 \cdot T \left\{ J^\mu(x) J^\nu(y) J^\lambda(z) J^\sigma(0) \right\} |0\rangle,$$\hspace{1cm}(3)

the HLbL tensor. The currents are $J^\mu(x) = \bar{q} Q q^\mu q$ with the quark fields $q = (u, d, s)$ and charge matrix $Q_q = \text{diag}(2/3, -1/3, -1/3)$. The contribution from the heavy quarks, $c, b,$ and $t$, can be evaluated fully perturbatively\cite{7}. The evaluation of $a_\mu^{\text{HLbL}}$ involves an integration with the loop momenta, $q_1, q_2,$ and $q_3$, running over all possible values and the fourth, $q_4 = q_1 + q_2 + q_3$, in the static limit, i.e. $q_4 \to 0$. This class of diagrams thus contains a complex interplay of strong interactions at different scales. In the below we work in the Euclidean domain and use $Q_4^2 = -q_4^2$.

The first full calculations of HLbL were done in the 1990s\cite{8,9} using mainly models. A model independent approach using dispersive theory allows for a much more precise determination\cite{10,11} for individual intermediate states but the short-distance part contains very many. Perturbative short-distance constraints have been used in constraining individual contributions starting in \cite{10,12} as well as some matching with the quark loop\cite{8}. The part with $Q_4^2 \approx Q_2^2 \gg Q_3^2$ was treated in\cite{13}.

Our best theoretical understanding of $\Pi^{\mu\nu\lambda\sigma}$ lies in the kinematic regions where the four Euclidean momenta are large, $Q_1 \sim Q_2 \sim Q_3 \sim Q_4 \gg \Lambda_{QCD}$, where $\Lambda_{QCD}$ is the hadronic scale. This allows for a perturbative description in terms of quarks and gluons. In this regime, one may construct a well-defined Operator Product Expansion (OPE), where the leading contribution corresponds

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure1.png}
\caption{The HLbL contribution to the $g - 2$. The bottom line is the muon. The blob is filled with hadrons.}
\end{figure}
be produced not only by hard quark lines, but also by low-energy degrees of freedom via vacuum expectation values of operators. Note that not only operators with vacuum quantum numbers acquire non-zero values, but also those with the same quantum numbers as the external electromagnetic field $F_{\mu \nu}$, e.g., $\langle q \gamma^\mu q \rangle$.

In this letter we show how this formalism can be used to provide a model-independent and accurate description of the region where the three incoming loop momenta are large.

2. Some generalities about the HLbL tensor

We use the notation of [10, 11] to facilitate using our results together with theirs. This section summarizes what we need from there. The HLbL tensor satisfies the Ward identities

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} \Pi_{\mu \nu \lambda \sigma} (q_1, q_2, q_3) = 0. \quad (4)$$

Note that this implies that

$$\Pi_{\mu \nu \lambda \sigma} (q_1, q_2, q_3) = - q_{4\rho} \frac{\partial \Pi_{\mu \nu \lambda \sigma}}{\partial q_{4\rho}} (q_1, q_2, q_3). \quad (5)$$

The dependence on $q_4$ is via $q_4 = q_1 + q_2 + q_3$. Equation (5) allows to compute $\alpha_{\mu \nu \lambda \sigma}^{HLbL}$ directly from the derivative [17].

In [11], the HLbL tensor is decomposed in a basis with 54 Lorentz scalar functions $\hat{\Pi}$, free of kinematic singularities as

$$\Pi_{\mu \nu \lambda \sigma} (q_1, q_2, q_3) = \sum_{i=1}^{54} \hat{T}_i^{\mu \nu \lambda \sigma} \hat{\Pi}_i (q_1, q_2, q_3). \quad (6)$$

The $\hat{T}_i^{\mu \nu \lambda \sigma}$ satisfy Ward identities equivalent to (4) and thus, in the static limit $q_4 \to 0$,

$$\frac{\partial \Pi_{\mu \nu \lambda \sigma} (q_1, q_2, q_3)}{\partial q_{4\sigma}} = \sum_{i=1}^{54} \frac{\partial \hat{T}_i^{\mu \nu \lambda \sigma} (q_1, q_2, q_3)}{\partial q_{4\sigma}} \hat{\Pi}_i (q_1, q_2, q_3). \quad (7)$$

However, in this limit only 19 terms survive [11, 13] and using the symmetry $(q_1, \mu) \leftrightarrow (q_2, \nu)$ one obtains

$$a_{\mu \nu \lambda \sigma}^{HLbL} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_0^1 d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i (Q_1, Q_2, \tau) \hat{\Pi}_i (Q_1, Q_2, \tau). \quad (8)$$

The integration variable $\tau$ is defined via $Q_3^2 = Q_1^2 + Q_2^2 + 2\tau Q_1 Q_2$. Expressions for the $T_i$ can be found in [11], and the $\hat{\Pi}_i$ are related to the $\hat{\Pi}_i$ according to

$$\hat{\Pi}_1 = \hat{\Pi}_1, \quad \hat{\Pi}_2 = C_{23} \hat{\Pi}_1, \quad \hat{\Pi}_3 = \hat{\Pi}_4, \quad \hat{\Pi}_4 = C_{23} \hat{\Pi}_4,$$
$$\hat{\Pi}_5 = \hat{\Pi}_7, \quad \hat{\Pi}_6 = C_{12} \hat{\Pi}_7, \quad \hat{\Pi}_7 = C_{23} \hat{\Pi}_7,$$
$$\hat{\Pi}_8 = C_{13} \hat{\Pi}_7, \quad \hat{\Pi}_9 = \hat{\Pi}_7, \quad \hat{\Pi}_{10} = \hat{\Pi}_{39}$$
$$\hat{\Pi}_{11} = - C_{23} \hat{\Pi}_{54}, \quad \hat{\Pi}_{12} = \hat{\Pi}_{54}. \quad (9)$$

---

1 We realized during the course of this work that a similar method has been used for another contribution to $a_{\mu \nu \lambda \sigma}^{HLbL}$ in [10].
where $C_{ij}$ permutes the momenta according to $q_i \leftrightarrow q_j$ for $i, j \in \{1, 2, 3\}$. As can be seen, only the six functions $\hat{\Pi}_i$ for $i \in \{1, 4, 7, 17, 39, 54\}$ are needed.

3. The HLBL tensor in an external field

The HLBL tensor in (3) can be obtained from

$$\int d^4x \; d^4y \; e^{-i(q_1x + q_2y)} \langle 0| T \{ J^\mu(x) J^\nu(y) J^\lambda(0) \} | q(-q_3) \rangle \equiv -\Pi^{\mu\nu\lambda}(q_1, q_2, q_3) \equiv i\epsilon_{\alpha}(-q_4) \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3), \quad (10)$$

where we have captured the fourth photon vertex via the matrix element with a possibly off-shell photon and defined $q_3 = q_1 - q_2$.

In the static limit, $q_4 \to 0$, one can factor out the soft photon part according to

$$\Pi^{\mu\nu\lambda}(q_1, q_2, q_3) \equiv \Pi_F^{\mu\nu\lambda\rho\sigma}(q_1, q_2) \langle 0| F_{\rho\sigma} \gamma(-q_4) \rangle = i q_4 \epsilon_{\alpha}(-q_4) \Pi_F^{\mu\nu\lambda\rho\sigma}(q_1, q_2), \quad (11)$$

where $[\alpha \sigma]$ indicates antisymmetrization. Combining (11) with (7) and (10) one obtains

$$\lim_{q_4 \to 0} \frac{\partial \Pi^{\mu\nu\lambda\rho\sigma}}{\partial q_4}(q_1, q_2, q_3) = \Pi_F^{[\mu\nu\lambda]\rho\sigma}(q_1, q_2). \quad (12)$$

The momentum conservation in the static limit reads $q_1 + q_2 + q_3 = 0$. From the above equivalence, (12), together with (7), it is possible to obtain the required $\hat{\Pi}_i$ to calculate $\rho^{HLBL}$.

The short-distance quantity, $\Pi_F^{\mu\nu\lambda\rho\sigma}(q_1, q_2)$ does not depend on the soft-photon momentum and can be calculated directly using the methods of OPE in an external electromagnetic field of (15). By construction this procedure is free from infrared divergent propagators. The coupling to an external field can arise in two different ways, either via a soft insertion on a hard quark line or from the vacuum expectation values induced by the external electromagnetic field.

In order to simplify calculations we work in the radial gauge for the external electromagnetic field. This implies to first order, i.e. in the static limit,

$$A_\sigma(z) = \frac{1}{2} z^{\sigma} F_{\rho\sigma}(0) + \ldots, \quad (13)$$

allowing to calculate immediately in the $q_4 = 0$ limit. This gauge is particularly convenient for the soft QCD parts as well, since it allows to easily expand non-local terms such as $\langle \bar{q}(x) q(0) \rangle$ into gauge invariant local ones. This stems from the equivalence between partial derivatives and covariant derivatives in expansions of fields such as for instance $q(x) = q(0) + x^\mu D_\mu q(0) + \ldots$. A pedagogical introduction is in (18).

We first look at the contributions with a soft insertion on a hard line. The lowest order is illustrated in Fig. 3a. It is a quark loop with three hard insertions and one soft.

The calculation leads to the same result as the usual quark loop obtained from the calculation with Fig. 2a, including the dependence on the quark mass. We have calculated using both methods as well as compared with quark loop expressions from (19). The agreement is exact, both numerical and analytical. In future work we intend to calculate the gluonic corrections to this. This part shows that the quark loop at short distances is indeed the first term in a systematic expansion. We do not quote the analytical expressions since they are rather lengthy.

We now turn to the power corrections. The lowest dimensional contribution comes from

$$\langle q_{\sigma \alpha \beta} q \rangle \equiv e_q F_{\alpha \beta} X_q, \quad (14)$$

where $e_q$ is the one of the light quark charges in the matrix $Q_q$, and the $X_q$ are so-called tensor coefficients related to the magnetic susceptibility that are known from lattice QCD (28). Regarding the suppression of this condensate as compared to the leading term, the only scale to compensate dimensions is $\Lambda_{QCD}$. From naive dimensional analysis, this contribution is thus suppressed by at least a factor of $\frac{\Lambda_{QCD}^4}{\Lambda_{QCD}^2}$.

The contribution is schematically drawn in Fig. 3a. From chirality it follows that an extra insertion of a quark mass is needed, so we get a suppression compared to the quark loop of two powers of the hard scale. The analytical result for the leading power suppressed contributions are

$$\hat{\Pi}_1 = m_q X_q e_q^4 \frac{-4}{Q_1^2 Q_2^2 Q_3^2}, \quad \hat{\Pi}_7 = 0,$$

$$\hat{\Pi}_4 = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^2}, \quad \hat{\Pi}_{39} = 0,$$

$$\hat{\Pi}_17 = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^2}, \quad \hat{\Pi}_{54} = m_q X_q e_q^4 \frac{-4}{Q_1^2 Q_2^2 Q_3^2}. \quad (15)$$

Work is in progress to calculate the power corrections that are not suppressed by quark masses but these will be suppressed by more powers of the hard scale. Preliminary results indicate that the contributions not suppressed by quark masses occur first suppressed by four powers of the hard scale.

Figure 3: The two leading terms in the external field OPE: (a) The quark loop with loop momentum $p$, and (b) the condensate $\langle q_{\sigma \alpha \beta} q \rangle$. The presence of the external field is here represented by a crossed vertex. Note that there is no divergent propagator here as the momentum $q_4$ never enters the diagram explicitly.
Table 1: The total contributions to $a_{\mu}^{HLbL}$ from both the quark loop and the next term in the OPE. The condensate contributions have been divided into two parts, one for the up and down quarks and the other for the strange quark.

<table>
<thead>
<tr>
<th>$Q_{\text{min}}$</th>
<th>Quark Loop</th>
<th>$m_u X_u + m_d X_d$</th>
<th>$m_s X_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 GeV</td>
<td>$17.3 \times 10^{-11}$</td>
<td>$5.40 \times 10^{-13}$</td>
<td>$8.29 \times 10^{-13}$</td>
</tr>
<tr>
<td>2 GeV</td>
<td>$4.35 \times 10^{-11}$</td>
<td>$3.40 \times 10^{-14}$</td>
<td>$5.22 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

4. Numerical results

In this section we present numerical results obtained from the external field OPE. For the numerical integration of $a_{\mu}^{HLbL}$ in \cite{5}, we use the CUBA library \cite{21}, both employing a Monte Carlo algorithm (\texttt{Vegas}) as well as a deterministic algorithm (\texttt{Cuhre}) for a cross-check.

First of all we consider the quark loop. In order to compare with \cite{7}, we use constituent quark masses of $m_{u,d,s} = 240$ MeV and $N_c = 3$. This yields $a_{\mu}^{HLbL} = 80.30 \times 10^{-11}$, which is in excellent agreement with the result quoted in \cite{7}. This was of course expected given that our leading result analytically agrees with the quark loop.

We also numerically evaluate the contribution to $a_{\mu}^{HLbL}$ from the regime where our OPE is valid. In order to allow for future cross-checks, we first calculate it for two lower cut-offs $Q_{\text{min}} = 1, 2$ GeV such that $Q_{\text{min}1,2,3} \geq Q_{\text{min}}$. The condensates $X_q$ have been estimated in \cite{20} on the lattice, and the values are\footnote{The sign differs from \cite{20} due to differences in conventions.} \footnote{Given the numerical smallness of the result more precise values are not needed.}

$X_u = 40.7 \pm 1.3$ MeV, \quad $X_d = 39.4 \pm 1.4$ MeV, \quad $X_s = 53.0 \pm 7.2$ MeV. \quad (16)

The quark masses we use are $m_u = m_d = 5$ MeV and $m_s = 100$ MeV. The results are presented in Table 1. For an order of magnitude comparison also the quark loop with zero quark masses is included there with the same region of integration. As can be seen, the contributions from the condensates are strongly suppressed as compared to the quark loop. This is expected given the smallness of $m_q X_q$.

Finally, in addition to the above comparison we also look at $a_{\mu}^{HLbL}$ for a range of $Q_{\text{min}}$ in Figure 4. The running of the MS quark masses is implemented using the package \texttt{CRunDec} \cite{22}. In addition to the condensate contribution and massless quark loop, also the mass correction to the massless quark loop is plotted. As can be seen, both the condensate contribution and the mass correction scale the same way in $Q_{\text{min}}$. This $Q_{\text{min}}$ dependence goes as $1/Q_{\text{min}}^3$, while the massless quark loop scales perfectly as $1/Q_{\text{min}}^2$.

Note that higher-dimensional contributions contain condensates not suppressed by the small quark mass values. They are expected to dominate the power corrections when the cut-off $Q_{\text{min}}$ is small enough.

Figure 4: The $Q_{\text{min}}$ dependence of $a_{\mu}^{HLbL}$.  

5. Conclusions and outlook

Due to the long-standing deviation between the experimental value and the Standard Model prediction of the muon magnetic moment, there is at present much work going into reducing the errors on both quantities. One of the two main uncertainties in the Standard Model value comes from the HLbL contribution, $a_{\mu}^{HLbL}$. The loop integral in $a_{\mu}^{HLbL}$ is particularly complicated due to the various regions of virtual or internal photon momenta. In this letter we have focused on the region where the three (Euclidean) virtual photon momenta are large. We have shown how the standard OPE in the vacuum of the associated four-point correlation function breaks down beyond the leading order in the static limit in which $g - 2$ is defined. Instead, an OPE in the presence of an electromagnetic background field has been used. The photon associated to the soft momentum $q_4 \to 0$ can be emitted from both high-energy degrees of freedom, i.e. quarks, or from long distance ones parametrized by induced vacuum expectation values of QCD operators.

The leading order contribution arises from the radiation of a hard line and is analytically identical to the purely perturbative quark loop. This proves the expectation that the perturbative quark loop is the first term in a systematic expansion in this region. The first power correction in our OPE contains a condensate related to the magnetic susceptibility of the QCD vacuum. Our numerical study has shown that its contribution to $a_{\mu}^{HLbL}$ is suppressed, as compared to the quark loop, by roughly three orders of magnitude, as a consequence of the small values of the quark masses and the condensate itself. The leading contribution scales as suppressed by two powers of the hard scale while the first power correction is suppressed by four powers of the hard scale.

The higher order power corrections are not suppressed by the small quark masses. Together with the purely perturbative $\alpha_s$ correction, they should be enough to give a first reliable estimate of the onset of the asymptotic do-
main. Both calculations are underway and are expected to be presented in a forthcoming publication.

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