LU TP 17-37
November 2017

Scale hierarchies, symmetry breaking and SM-like fermions in SU(3)-family extended SUSY trinification

José E. Camargo-Molina[b] **António P. Morais**[a,b] **Astrid Ordell**[b] **Roman Pasechnik**[b] **Jonas Wessén**[b]

[a] *Departamento de Física, Universidade de Aveiro and CIDMA, Campus de Santiago, 3810-183 Aveiro, Portugal*

[b] *Department of Astronomy and Theoretical Physics, Lund University, 221 00 Lund, Sweden*

E-mail: eliel@thep.lu.se, aapmorais@ua.pt, astrid.ordell@thep.lu.se, roman.pasechnik@thep.lu.se, jonas.wessen@thep.lu.se

Abstract: A unification of left-right $\mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$, color $\mathrm{SU}(3)_\mathrm{C}$ and family $\mathrm{SU}(3)_\mathrm{F}$ symmetries in a maximal rank-8 subgroup of E_8 is proposed as a landmark for future explorations beyond the Standard Model (SM). While the product of the first three SU(3)'s is known as the gauge trinification originating from E_6, an addition of the $\mathrm{SU}(3)_\mathrm{F}$ family symmetry represents yet another possible but unexplored layer of high-scale unification due to an $\mathrm{E}_6 \times \mathrm{SU}(3)_\mathrm{F} \subset \mathrm{E}_8$ embedding. We discuss the implications of such an embedding in a supersymmetric (SUSY) model based on the trinification gauge $[\mathrm{SU}(3)]^3$ and global $\mathrm{SU}(3)_\mathrm{F}$ family symmetries with a constrained set of SUSY conserving and soft SUSY breaking operators. Among the key properties of this model are the unification of SM Higgs and lepton sectors predicting a common Yukawa coupling for chiral fermions, the absence of the μ-problem, gauge couplings unification and baryon number conservation at the Grand Unification scale, as well as light SM fermions. The minimal field content that may lead to a consistent SM-like effective theory at low energies is composed of one E_6 **27**-plet per generation as well as three gauge and one family SU(3) octets belonging to the fundamental sector of E_8. The details of the corresponding (SUSY and gauge) symmetry breaking scheme and the resulting effective low-energy scenarios are discussed.

Contents

1 Introduction

The search for a complete Grand-Unified Theory (GUT) containing the Standard Model (SM) as its low-energy effective field theory (EFT) limit has been ongoing for a few decades by now. While several candidate GUTs unifying the strong and electroweak interactions have been proposed and explored in some of the basic aspects so far, a detailed understanding of the SM origin, with all its parameters, hierarchies, symmetries and particle content known from phenomenology, remains a big challenge for the theoretical physics community.

Some of the most popular SM extensions are based on supersymmetric (SUSY) GUTs where the SM gauge interactions are unified under simple symmetry groups such as SU(5) and SO(10) [1–7] as well as E_6[1] and E_7 [11]. A particularly appealing scenario proposed by Glashow in 1984 [12] is based upon the rank-6 trinification symmetry $[\mathrm{SU}(3)]^3 \equiv \mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R} \times \mathrm{SU}(3)_\mathrm{C} \ltimes \mathbb{Z}_3 \subset \mathrm{E}_6$ (T-GUT, in what follows) where all matter fields are embedded in bi-triplet representations and due to the cyclic permutation symmetry $\mathbb{Z}_3$, the corresponding gauge couplings unify at the T-GUT Spontaneous Symmetry Breaking (SSB) scale, or GUT scale in what follows.

Many phenomenological and theoretical studies of T-GUTs in both SUSY and non-SUSY formulations have been inspired by their unique features (see e.g. Refs. [13–36]). For example, due to the fact that quarks and leptons belong to different gauge representations in T-GUT scenarios, the baryon number is naturally conserved by gauge interactions [15], only allowing for proton decay via Yukawa interactions. As was shown for a particular T-GUT realisation in Ref. [26], the proton decay rates were consistent with experimental limits in the case of low-scale SUSY, or completely unobservable in the case of split SUSY. Many T-GUTs can also accommodate any quark and lepton masses and mixing angles [15, 30] whereas neutrino masses are generated by a see-saw mechanism [23] of radiative [26] or inverse [28] type.

Despite a notable progress in exploring the gauge couplings unification, neutrino masses, Dark Matter candidates, TeV-scale Higgs partners, collider and other phenomenological implications of GUTs, there are several yet unresolved problems. A general challenge in GUT model building (and particularly so in T-GUT), has to do with the existence of an appropriate stable vacuum with spontaneously broken GUT symmetry down to the SM gauge group. Namely, the lower the number of free parameters at the GUT scale, the more

[1] The E_6-based models are typically motivated by heterotic string theories where massless sectors consistent with the chiral structure of the SM are naturally described by an $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge theory. For more details we refer the reader to Refs. [8–10]

difficult it is to find a realizable GUT scenario with a SM-like EFT limit at low energies. In turn, less unification at the GUT scale typically leads to a more technically cumbersome search for a SM-like EFT (with less predictive power). In typical T-GUT realizations, a large amount of fields and free parameters makes it challenging to derive even basic features of their low-energy EFT limit using the renormalization group (RG) analysis.

Another problem in the case of SUSY T-GUT model building is the longstanding issue of avoiding GUT scale masses for the would-be SM leptons. For this purpose, one usually introduces either several unmotivated E_6 **27** Higgs representations [15, 18, 20, 21, 25, 26, 28–31, 33, 37] acquiring vacuum expectation values (VEVs), which spontaneously break gauge trinification to the SM gauge symmetry, or higher-dimensional operators [20, 21, 25, 28, 38]. These approaches typically require a significant fine-tuning in high-scale parameter space (especially, in the Yukawa sector) to reach a SM-like limit [26]. Otherwise, they exhibit phenomenological issues with proton stability [15, 21, 26] and with a large amount of unobserved light states [12, 20, 30, 31, 34, 38]. Despite a continuous progress, the SM-like EFTs originating from T-GUTs still remain underdeveloped in comparison to other GUT models such as SU(5), SO(10) or even E_6 (see e.g. Ref. [32] and references therein).

In this paper, we explore in detail a recently proposed SUSY T-GUT model [39] with a global $SU(3)_F$ family symmetry inspired by the embedding of $E_6 \times SU(3)$ into E_8 and with the trinification subgroup fully contained in E_6. We will refer to this model as the SUSY Higgs-Unified Trinification (SHUT) model. As we will see, the SHUT model offers solutions to some of the problems faced by previous T-GUTs. As the light Higgs and lepton sectors are unified, the model can be embedded into a single E_8 representation. Furthermore, as the embedding suggests the introduction of adjoint scalars and a family $SU(3)_F$ which protects the SM fermions from acquiring masses before electroweak symmetry breaking (EWSB) due to an interplay between the gauge and family symmetries. The proton is stable at the GUT scale to all orders in perturbation theory while in the low-energy EFT limit, depending on the fermion mixing angles and on the effective Higgs sector, it can in principle be marginally destabilized via tree-level qubic scalar couplings and loop-induced Yukawa interactions (c.f. Ref. [15]). Besides, it provides a unification of the high-scale Yukawa sector into a single coupling, a unique feature of this model strongly constraining the particle spectra and interactions at low energies. This is in variance to well-known SO(10) and Pati-Salam models where the Yukawa unification is constrained to the third family only (see e.g. Refs. [40–52]).

The full Yukawa and gauge unification in the SHUT model largely reduces its parameter space, making a complete analysis of its low-energy EFT scenarios technically feasible. The model also has a particular feature of triggering the gauge symmetry breaking below the GUT scale through parameters of the soft SUSY breaking Lagrangian. As soft SUSY breaking parameters are protected from GUT-scale radiative corrections, it allows a strong hierarchy between the GUT scale and further gauge symmetry breaking scales that is preserved at quantum level.

In Sect. 2 we briefly discuss the key features of the SHUT model and its SSB scheme.

In Sect. 3 we introduce the high-scale SHUT model in its minimal setup in detail. In particular, we discuss its features and the details on how it solves the longstanding problems of previous T-GUT realizations and how the GUT-scale SSB in this model leads to a Left-Right (LR) symmetric SUSY theory. In Sect. 4 we discuss the inclusion of soft SUSY breaking interactions and how they lead to a breaking of the remaining gauge symmetries down to the SM gauge group. In Sect. 5 we provide a discussion of the possible low-energy EFTs, before concluding in Sect. 6.

A short note on notation

In this article we adopt the following notations:

- Supermultiplets are always written in bold (e.g. $\boldsymbol{\Delta}$). As usual, the scalar components of chiral supermultiplets and fermionic components of vector supermultiplets carry a tilde (e.g. $\widetilde{\Delta}$), except for the Higgs-Higgsino sector where the tilde serves to identify the fermion $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ bi-doublets (e.g. $\widetilde{H}$).
- Fundamental representations carry superscript indices while anti-fundamental representations carry subscript indices.
- $\mathrm{SU}(3)_{\mathrm{K}}$ and $\mathrm{SU}(2)_{\mathrm{K}}$ (anti-)fundamental indices are denoted by $k, k', k_1, k_2 \ldots$ for $K = L, R, C$, respectively.
- Indices belonging to (anti-)fundamental representations of $\mathrm{SU}(3)_{\mathrm{F}}$ are denoted by $i, j, k \ldots$.
- If a field transforms both under gauge and global symmetry groups, the index corresponding to the global one is placed within the parenthesis around the field, while the indices corresponding to the gauge symmetries are placed outside.
- Global symmetry groups will be indicated by $\{\ldots\}$.

2 Left-Right-Color-Family unification

In the Glashow's formulation of trinified $[\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{C}}] \ltimes \mathbb{Z}_3 \subset \mathrm{E}_6$ (LRC-symmetric) gauge theory [12], three families of the fermion fields from the SM are arranged over three **27**-plet copies of the E_6 group, namely,

$$\mathbf{27}^i \to \boldsymbol{L}^i \oplus \boldsymbol{Q}_{\mathrm{L}}^i \oplus \boldsymbol{Q}_{\mathrm{R}}^i \equiv (\mathbf{3}^l, \bar{\mathbf{3}}_r, \mathbf{1}^x)^i \oplus (\bar{\mathbf{3}}_l, \mathbf{1}^r, \mathbf{3}^x)^i \oplus (\mathbf{1}^l, \mathbf{3}^r, \bar{\mathbf{3}}_x)^i \,, \quad i = 1, 2, 3 \,,$$

while the Higgs fields responsible for a high-scale SSB are typically introduced via e.g. an additional **27**-plet. Here and below, the left, right, and color SU(3) indices are denoted by l, r, and x, respectively, while the fermion families are labeled by an index i.

The SHUT model first presented in Ref. [39], in variance to the Glashow's trinification, introduces the global family symmetry $\mathrm{SU}(3)_{\mathrm{F}}$ which acts in the space of fermion generations. In this case, the light Higgs and lepton superfields, as well as quarks and colored

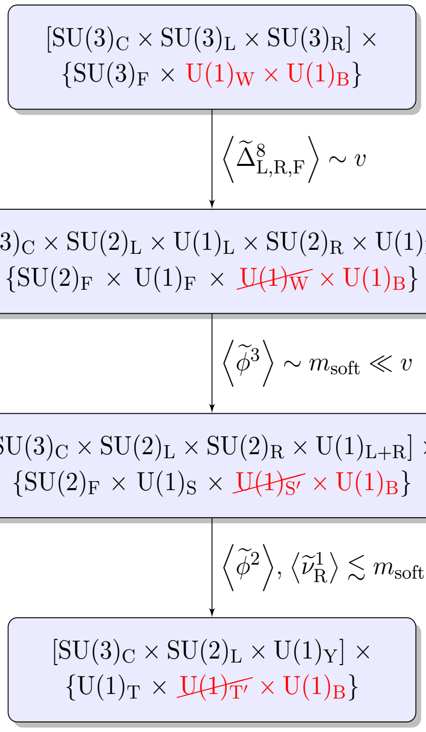

Figure 1. *The symmetry breaking scheme in the SHUT model studied in this work. The symmetry groups in red correspond to the accidental symmetries of the high-scale theory. The global accidental* $\mathrm{U}(1)_\mathrm{W}$ *and, consequently, its low-energy counterparts* $\mathrm{U}(1)_{\mathrm{S}',\mathrm{T}'}$ *discussed below are considered to be softly broken at low-energy scales and thus are shown as crossed-out symmetry groups.*

scalars, all are unified into a single $(\mathbf{27},\mathbf{3})$-plet under $\mathrm{E}_6 \times \mathrm{SU}(3)_\mathrm{F}$ symmetry, i.e.

$$(\mathbf{27},\mathbf{3}) \to \left(\boldsymbol{L}^i\right)^l{}_r \oplus \left(\boldsymbol{Q}^i_\mathrm{L}\right)^x{}_l \oplus \left(\boldsymbol{Q}^i_\mathrm{R}\right)^r{}_x \equiv (\mathbf{3}^l, \bar{\mathbf{3}}_r, \mathbf{1}^x, \mathbf{3}^i) \oplus (\bar{\mathbf{3}}_l, \mathbf{1}^r, \mathbf{3}^x, \mathbf{3}^i) \oplus (\mathbf{1}^l, \mathbf{3}^r, \bar{\mathbf{3}}_x, \mathbf{3}^i)\,.$$

The leptonic tri-triplet superfield $\left(\boldsymbol{L}^i\right)^l{}_r$ that unifies the SM left- and right-handed leptons and SM Higgs doublets can be conveniently represented as

$$\left(\boldsymbol{L}^i\right)^l{}_r = \begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} & \boldsymbol{e}_\mathrm{L} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} & \boldsymbol{\nu}_\mathrm{L} \\ \boldsymbol{e}^c_\mathrm{R} & \boldsymbol{\nu}^c_\mathrm{R} & \boldsymbol{\phi} \end{pmatrix}^i \,, \tag{2.1}$$

Besides, the left-quark $\left(\boldsymbol{Q}^i_\mathrm{L}\right)^x{}_l$ and right-quark $\left(\boldsymbol{Q}^i_\mathrm{R}\right)^r{}_x$ tri-triplets are

$$\begin{aligned} \left(\boldsymbol{Q}^i_\mathrm{L}\right)^x{}_l &= \begin{pmatrix} \boldsymbol{u}^x_\mathrm{L} & \boldsymbol{d}^x_\mathrm{L} & \boldsymbol{D}^x_\mathrm{L} \end{pmatrix}^i, \\ \left(\boldsymbol{Q}^i_\mathrm{R}\right)^r{}_x &= \begin{pmatrix} \boldsymbol{u}^c_{\mathrm{R}x} & \boldsymbol{d}^c_{\mathrm{R}x} & \boldsymbol{D}^c_{\mathrm{R}x} \end{pmatrix}^{\top\, i}. \end{aligned} \tag{2.2}$$

In addition, the SHUT model also incorporates the adjoint (namely, $\mathrm{SU}(3)_{\mathrm{L,R,C,F}}$ octet) superfields $\boldsymbol{\Delta}_{\mathrm{L,R,C,F}}$. The first SSB step in the SHUT model $\mathrm{SU}(3)_{\mathrm{L,R,F}} \to \mathrm{SU}(2)_{\mathrm{L,R,F}} \times \mathrm{U}(1)_{\mathrm{L,R,F}}$ is triggered at the GUT scale by the SUSY-preserving vacuum expectation values

(VEVs) in the scalar components of the corresponding octet superfields while all the subsequent low-scale SSB steps are triggered by VEVs in the leptonic tri-triplet $\left(\boldsymbol{L}^i\right)^l{}_r$ through parameters of the soft-SUSY breaking operators.

Along this work, we will be focused on the symmetry breaking scheme shown in Fig. 1. There it can be seen that an accidental global $\mathrm{U}(1)_\mathrm{B} \times \mathrm{U}(1)_\mathrm{W}$ symmetry (which is marked in red and will be discussed in detail in the next section) appears in the high-scale theory. As we will see, although alternative breaking schemes are possible, this is the one leading to the low energy SM-like scenarios we find most interesting. As we shall see in Sec. 5.1, dimension-3 operators that softy break $\mathrm{U}(1)_\mathrm{W}$, and consequently its low-energy decendants (that will be denoted below as $\mathrm{U}(1)_{\mathrm{S}',\mathrm{T}'}$), are needed for a phenomenologically viable low-scale fermion spectrum. Such interactions do not have a perturbative origin from the high-scale theory and are added to the effective theory that emerges once the heavy degrees of freedom of the SHUT model are integrated out. Provided that a small soft breaking of $\mathrm{U}(1)_\mathrm{B}$ is possible and can be made consistent with the proton decay constraints, without any loss of generality we leave $\mathrm{U}(1)_\mathrm{B}$ unbroken in this work, for simplicity.

3 Supersymmetric trinification with global $\mathrm{SU}(3)_\mathrm{F}$

We will start by reviewing the SHUT model, its symmetries, particle content and interactions in detail. Then, we discuss how the model addresses the shortcomings of previous T-GUTs and different possibilities for the minimal field content that could contain the would-be SM fields. Finally, we show what happens after the symmetries of the theory are broken spontaneously by adjoint field VEVs and discuss the features of the resulting LR-symmetric SUSY theory, including the remaining symmetries and how its parameters are related to those in the high-scale SHUT model.

3.1 Tri-triplet sector

In the following, we consider the SHUT model – a SUSY GUT theory based on the trinification gauge group with an accompanying global $\mathrm{SU}(3)_\mathrm{F}$ family symmetry, i.e.

$$G_{333\{3\}} \equiv [\mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R} \times \mathrm{SU}(3)_\mathrm{C}] \ltimes \mathbb{Z}_3^{(\mathrm{LRC})} \times \{\mathrm{SU}(3)_\mathrm{F}\}\,. \tag{3.1}$$

Here and below, curly brackets indicate global (non-gauge) symmetries. The minimal chiral superfield content (shown in Tab. 1) that can accommodate the SM (Higgs and fermion) fields, is comprised of three tri-triplet representations of $G_{333\{3\}}$ which we label as $\boldsymbol{L}$, $\boldsymbol{Q}_\mathrm{L}$ and $\boldsymbol{Q}_\mathrm{R}$ respectively (for their explicit relation to the SM field content, see Eqs. (2.1) and (2.2)). The $\mathbb{Z}_3^{(\mathrm{LRC})}$ in Eq. (3.1) is realized on the chiral and vector superfields as the simultaneous cyclic permutation within $\{\boldsymbol{L}, \boldsymbol{Q}_\mathrm{L}, \boldsymbol{Q}_\mathrm{R}\}$ and $\{\boldsymbol{V}_\mathrm{L}, \boldsymbol{V}_\mathrm{C}, \boldsymbol{V}_\mathrm{R}\}$ sets, respectively, where $\boldsymbol{V}_{\mathrm{L,R,C}}$ are the gauge (super)fields for the respective gauge $\mathrm{SU}(3)_{\mathrm{L,R,C}}$ groups. The $\mathbb{Z}_3^{(\mathrm{LRC})}$ symmetry enforces the gauge couplings of the $\mathrm{SU}(3)_{\mathrm{L,R,C}}$ groups to unify, i.e. $g_\mathrm{L} = g_\mathrm{R} = g_\mathrm{C} \equiv g_\mathrm{U}$.

Chiral supermultiplet fields					
Superfield		$\mathrm{SU}(3)_{\mathrm{C}}$	$\mathrm{SU}(3)_{\mathrm{L}}$	$\mathrm{SU}(3)_{\mathrm{R}}$	$\mathrm{SU}(3)_{\mathrm{F}}$
Lepton	$(\boldsymbol{L}^i)^l{}_r$	$\mathbf{1}$	$\mathbf{3}^l$	$\bar{\mathbf{3}}_r$	$\mathbf{3}^i$
Left-Quark	$(\boldsymbol{Q}_{\mathrm{L}}^i)^x{}_l$	$\mathbf{3}^x$	$\bar{\mathbf{3}}_l$	$\mathbf{1}$	$\mathbf{3}^i$
Right-Quark	$(\boldsymbol{Q}_{\mathrm{R}}^i)^r{}_x$	$\bar{\mathbf{3}}_x$	$\mathbf{1}$	$\mathbf{3}^r$	$\mathbf{3}^i$

Table 1. *Tri-triplet chiral superfields in the SHUT model and their quantum numbers.*

Considering the embedding of the trinification gauge group into E_6, the three bi-triplets in each generation can be thought of as originating from an E_6 **27**-plet that branches as

$$\mathbf{27} = (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \tag{3.2}$$

under $\mathrm{E}_6 \supset [\mathrm{SU}(3)]^3$. All fields in Tab. 1 can therefore be contained in a $(\mathbf{27}, \mathbf{3})$ representation of $\mathrm{E}_6 \times \mathrm{SU}(3)_{\mathrm{F}}$. In turn, the group $\mathrm{E}_6 \times \mathrm{SU}(3)_{\mathrm{F}}$ is a maximal subgroup of E_8,

$$\mathrm{E}_8 \supset \mathrm{E}_6 \times \mathrm{SU}(3)_{\mathrm{F}}\,, \tag{3.3}$$

where the $(\mathbf{27}, \mathbf{3})$ fits neatly into the **248** irrep of E_8 whose branching rule is given by

$$\mathbf{248} = (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \overline{\mathbf{3}})\ . \tag{3.4}$$

In this work, we treat $\mathrm{SU}(3)_{\mathrm{F}}$ as a global symmetry. While considerably simpler, the trinification model with global $\mathrm{SU}(3)_{\mathrm{F}}$ can be viewed as the principal part of the fully gauged version in the limit of a vanishingly small family-gauge coupling $g_{\mathrm{F}} \ll g_{\mathrm{U}}$. Such a restricted model can thus be a first step towards the fully gauged E_8-based version. As a consequence of this, in the current work we restrain ourselves from soft breaking of the family symmetry that is fully contained in E_8.

Considering only renormalizable interactions, the symmetry group $G_{333\{3\}}$ allows for just a single term in the superpotential with the tri-triplet superfields,

$$W = \lambda_{\mathbf{27}}\, \varepsilon_{ijk} (\boldsymbol{L}^i)^l{}_r (\boldsymbol{Q}_{\mathrm{L}}^j)^x{}_l (\boldsymbol{Q}_{\mathrm{R}}^k)^r{}_x\,. \tag{3.5}$$

where $\lambda_{\mathbf{27}}$ can be taken to be real without any loss of generality. As the light Higgs and lepton sectors are fully contained in the single tri-triplet $\boldsymbol{L}$, this construction provides an exact unification of Yukawa interactions of the fundamental superchiral sector and the corresponding scalar quartic couplings to a common origin, $\lambda_{\mathbf{27}}$.

The superpotential in Eq. (3.5) comes with an accidental continuous $\mathrm{U}(1)_{\mathrm{W}} \times \mathrm{U}(1)_{\mathrm{B}}$ symmetry as we can perform independent phase rotations on two of the tri-triplets as long as we do a compensating phase rotation on the third. We can arrange the charges of the tri-triplets under $\mathrm{U}(1)_{\mathrm{W}} \times \mathrm{U}(1)_{\mathrm{B}}$ as shown in Tab. 2, such that $\mathrm{U}(1)_{\mathrm{B}}$ is identified as the symmetry responsible for baryon number conservation.

	$\mathrm{U(1)_W}$	$\mathrm{U(1)_B}$
$\boldsymbol{L}$	$+1$	0
$\boldsymbol{Q}_\mathrm{L}$	$-1/2$	$+1/3$
$\boldsymbol{Q}_\mathrm{R}$	$-1/2$	$-1/3$

Table 2. Charge assignment of the tri-triplets under the accidental symmetries.

The model with the superpotential in Eq. (3.5) also exhibits an accidental symmetry under LR-parity. This is realized at the superspace level as

$$\left(\boldsymbol{L}^i\right)^s{}_t \leftrightarrow -\left(\boldsymbol{L}^*_i\right)_t{}^s \quad , \left(\boldsymbol{Q}^i_\mathrm{L}\right)^x{}_s \leftrightarrow \left(\boldsymbol{Q}^*_{\mathrm{R}i}\right)_s{}^x \quad , \boldsymbol{V}^a_\mathrm{L} \leftrightarrow -\boldsymbol{V}^a_\mathrm{R} \quad , \boldsymbol{V}^a_\mathrm{C} \leftrightarrow -\boldsymbol{V}^a_\mathrm{C} \tag{3.6}$$

accompanied by

$$x^\mu \leftrightarrow x_\mu \quad , \quad \theta^\alpha \to -\mathrm{i}\theta^\dagger_\alpha \,. \tag{3.7}$$

Here, α is the spinor index on the Grassman valued superspace coordinate θ. Note that s and t in Eq. (3.6) label both $\mathrm{SU(3)_{L,R}}$ indices as such representations are swapped under LR-parity. At the Lagrangian level, the LR-parity transformation rules become

$$\begin{aligned}
\left(\tilde{L}^i\right)^s{}_t \leftrightarrow -\left(\tilde{L}^*_i\right)_t{}^s, \quad \left[\left(L^i\right)^s{}_t\right]_\alpha \leftrightarrow -\mathrm{i}\left[\left(L^\dagger_i\right)_t{}^s\right]^\alpha, \\
\left(\tilde{Q}^i_\mathrm{L}\right)^x{}_s \leftrightarrow \left(\tilde{Q}^*_{\mathrm{R}i}\right)_s{}^x, \quad \left[\left(Q^i_\mathrm{L}\right)^x{}_s\right]_\alpha \leftrightarrow \mathrm{i}\left[\left(Q^\dagger_{\mathrm{R}i}\right)_s{}^x\right]^\alpha, \\
G^a_{\mathrm{L}\mu} \leftrightarrow G^{a\,\mu}_\mathrm{R}, \quad G^a_{\mathrm{C}\mu} \leftrightarrow G^{a\,\mu}_\mathrm{C}, \quad [\widetilde{\lambda}^a_\mathrm{L}]^\alpha \leftrightarrow \mathrm{i}[\widetilde{\lambda}^{a\,\dagger}_\mathrm{R}]_\alpha, \quad [\widetilde{\lambda}^a_\mathrm{C}]^\alpha \leftrightarrow \mathrm{i}[\widetilde{\lambda}^{a\,\dagger}_\mathrm{C}]_\alpha
\end{aligned} \tag{3.8}$$

which can be verified by expanding out the components of the superfields in Eq. (3.6). In this model, LR-parity exists already at the SU(3) level, unlike common $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ LR-symmetric realisations. Note also that there exist the corresponding accidental Right-Colour and Colour-Left parity symmetries due to the $\mathbb{Z}_3^{(\mathrm{LRC})}$ permutation symmetry imposed in the SHUT model.

As mentioned in the Introduction, one of the main drawbacks of a SUSY T-GUT (as well as any SUSY GUT with very few free parameters) is the difficulty for spontaneous breaking of large symmetries. For example, while the non-SUSY T-GUT in Ref. [36] has no problem with SSB down to a LR-symmetric theory, when including SUSY the additional relations between potential and gauge couplings make it so that there is no minimum of the potential allowing for that breaking. Moreover, even when relaxing the family symmetry, any VEV in e.g. $\widetilde{L}^i$ induces mass terms that mix the L^i fermions with the gauginos $\widetilde{\lambda}^a_\mathrm{L,R}$ through $\mathcal{D}$-term interactions of the type

$$\mathcal{L}_\mathcal{D} = -\sqrt{2}g_\mathrm{U}\left(\tilde{L}^*_i\right)_{l_1}{}^r \left(T_a\right)^{l_1}_{l_2} \left(L^i\right)^{l_2}{}_r \widetilde{\lambda}^a_\mathrm{L} \,. \tag{3.9}$$

This is a common problem in the previous T-GUT realizations as the number of light fields would not be enough to accommodate the particle content of the SM at low energies. While it is possible to get around this issue by adding extra Higgs multiplets to the theory and making them responsible for the SSB, this significantly increases the amount of light exotic fields that might be present at low energies but are unobserved. Such theories typically contain a very large number of free parameters and a fair amount of fine tuning which significantly reduces their predictive power.

The solution we propose in the SHUT model with the light Higgs-lepton unification is to add the adjoint $\mathrm{SU}(3)_{\mathrm{L,R,C,F}}$ chiral supermultiplets, $\boldsymbol{\Delta}_{\mathrm{L,R,C,F}}$, in addition to promoting the flavor index i of the tri-triplets to a fundamental index of the additional $\mathrm{SU}(3)_{\mathrm{F}}$ *family* symmetry such that

$$\mathbf{27}^i \to (\mathbf{27},\mathbf{3}) \quad \text{of} \quad \mathrm{E}_6\times\mathrm{SU}(3)_{\mathrm{F}}\subset\mathrm{E}_8\,. \tag{3.10}$$

By triggering the first SSB, while preserving SUSY, VEVs in scalar components of $\boldsymbol{\Delta}_{\mathrm{L,R,F}}$ do not lead to heavy "would-be" SM lepton fields. In addition, the scalar and fermion components of $\boldsymbol{\Delta}_{\mathrm{L,R,C}}$ all are automatically heavy after the breaking and thus do not remain in the low-energy theory whereas one $\mathrm{SU}(2)_{\mathrm{F}}$ doublet in $\boldsymbol{\Delta}_{\mathrm{F}}$ remains light. In the next section, we will also discuss how the adjoint supermultiplets can be motivated as well by the embedding of the T-GUT symmetry into $\mathrm{E}_6\times\mathrm{SU}(3)_{\mathrm{F}}\subset\mathrm{E}_8$ and the details of the SSB triggered by their VEVs.

3.2 SU(3) adjoint superfields

The addition of gauge adjoint superfields together with the family symmetry is the main feature preventing the SM leptons from getting a GUT-scale mass. As was briefly mentioned above, the gauge and family SU(3) adjoints are motivated by the $(\mathbf{78},\mathbf{1})$ and $(\mathbf{1},\mathbf{8})$ representations of $\mathrm{E}_6\times\mathrm{SU}(3)_{\mathrm{F}}$ (present in the branching rule of the **248**-rep in its embedding into E_8 as shown in Eq. (3.4)). Indeed, the **78**-rep, in turn, branches as

$$\mathbf{78}=(\mathbf{8},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{8})\oplus\left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right)\oplus\left(\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{3}\right)\,, \tag{3.11}$$

under $\mathrm{E}_6\supset[\mathrm{SU}(3)]^3$. We include three gauge-adjoint chiral superfields $\boldsymbol{\Delta}_{\mathrm{L,R,C}}$ corresponding to $(\mathbf{8},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{8},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{8})$ in Eq. (3.11), respectively, as well as the family $\mathrm{SU}(3)_{\mathrm{F}}$ adjoint, $\boldsymbol{\Delta}_{\mathrm{F}}$ (all listed in Table 3). The transformation rule for the $\mathbb{Z}_3^{(\mathrm{LRC})}$ symmetry in $G_{333\{3\}}$ is now accompanied by the cyclic permutation of $\{\boldsymbol{\Delta}_{\mathrm{L}},\boldsymbol{\Delta}_{\mathrm{C}},\boldsymbol{\Delta}_{\mathrm{R}}\}$ fields.

In order to keep the minimal setup, in this work we will not consider the fields that correspond to $\left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right)$ and $\left(\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{3}\right)$ from Eq. (3.11) for simplicity. Note, they can be very heavy and are not directly coupled to the fundamental reps containing the SM sector neither through superpotential terms nor through soft SUSY breaking interactions but only via gauge interactions.

By introducing the adjoint chiral superfields, we have to add the following terms

$$\begin{aligned} W\supset &\sum_{A=\mathrm{L,R,C}}\left[\frac{1}{2}\mu_{\mathbf{78}}\,\boldsymbol{\Delta}_A^a\boldsymbol{\Delta}_A^a+\frac{1}{3!}\lambda_{\mathbf{78}}\,d_{abc}\boldsymbol{\Delta}_A^a\boldsymbol{\Delta}_A^b\boldsymbol{\Delta}_A^c\right]+\frac{1}{2}\mu_{\mathbf{1}}\boldsymbol{\Delta}_{\mathrm{F}}^a\boldsymbol{\Delta}_{\mathrm{F}}^a \\ &+\frac{1}{3!}\lambda_{\mathbf{1}}d_{abc}\boldsymbol{\Delta}_{\mathrm{F}}^a\boldsymbol{\Delta}_{\mathrm{F}}^b\boldsymbol{\Delta}_{\mathrm{F}}^c\,, \end{aligned} \tag{3.12}$$

Chiral supermultiplet fields					
Superfield		$SU(3)_C$	$SU(3)_L$	$SU(3)_R$	$SU(3)_F$
Colour-adjoint	$\boldsymbol{\Delta}^a_C$	$\mathbf{8}^a$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
Left-adjoint	$\boldsymbol{\Delta}^a_L$	$\mathbf{1}$	$\mathbf{8}^a$	$\mathbf{1}$	$\mathbf{1}$
Right-adjoint	$\boldsymbol{\Delta}^a_R$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{8}^a$	$\mathbf{1}$
Family-adjoint	$\boldsymbol{\Delta}^a_F$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{8}^a$

Table 3. SU(3) *adjoint chiral superfields in the SHUT model and their representations.*

to the superpotential in Eq. (3.5). Here, $d_{abc} = 2\mathrm{Tr}[\{T_a, T_b\}T_c]$ are the totally symmetric SU(3) coefficients. We can pick the phase of $\boldsymbol{\Delta}_{\rm L,R,C,F}$ to make $\mu_{\mathbf{78}}$ and $\mu_{\mathbf{1}}$ real, which makes $\lambda_{\mathbf{78}}$ and $\lambda_{\mathbf{1}}$ complex, in general. Notice that the superpotential provides no renormalisable interaction terms between the adjoint superfields and the tri-triplets. The accidental $U(1)_W \times U(1)_B$ symmetry of the tri-triplet sector is not affected by $\boldsymbol{\Delta}_{\rm L,R,C,F}$ as we can take these fields simply to not transform under this symmetry. We can make the gauge interactions parity-invariant by defining the following transformation rule for the gauge adjoint fields,

$$\tilde{\Delta}^a_{\rm L} \leftrightarrow \tilde{\Delta}^{*a}_{\rm R}\,, \quad \tilde{\Delta}^a_{\rm C,F} \leftrightarrow \tilde{\Delta}^{*a}_{\rm C,F}\,, \quad [\Delta^a_{\rm L}]_\alpha \leftrightarrow \mathrm{i}[\Delta^{\dagger a}_{\rm R}]^\alpha\,, \quad [\Delta^a_{\rm C,F}]_\alpha \leftrightarrow \mathrm{i}[\Delta^{\dagger a}_{\rm C,F}]^\alpha \tag{3.13}$$

or, equivalently, $\boldsymbol{\Delta}^a_{\rm L,R,C,F} \leftrightarrow \boldsymbol{\Delta}^{*a}_{\rm R,L,C,F}$ at the superfield level. However, parity is not generally respected by the $\mathcal{F}$-term interactions unless $\lambda_{\mathbf{78}}$ is real. In what follows, we assume a real $\lambda_{\mathbf{78}}$, for simplicity, whereas the accidental LR-parity can be explicitly broken by the soft SUSY breaking sector of the theory, at or below the GUT scale.

Now, for illustration, let us discuss briefly the first symmetry breaking step which determines the GUT scale in the SHUT model (see Fig. 1). Eq. (3.12) leads to a scalar potential containing several SUSY-preserving minima with VEVs that are most conveniently put in the eighth component of $\tilde{\Delta}^8_{\rm L,R,F}$. In particular, there is a $SU(3)_C$ and LR-parity preserving minimum with

$$\langle \tilde{\Delta}^a_{\rm L,R} \rangle = \frac{v_{\rm L,R}}{\sqrt{2}}\delta^a_8 \quad \text{with} \quad v_{\rm L,R} = v \equiv 2\sqrt{6}\,\frac{\mu_{\mathbf{78}}}{\lambda_{\mathbf{78}}}, \quad v_{\rm C} = 0\,, \tag{3.14}$$

for the gauge-adjoints, and

$$\langle \tilde{\Delta}^a_{\rm F} \rangle = \frac{v_{\rm F}}{\sqrt{2}}\delta^a_8 \quad \text{with} \quad v_{\rm F} = 2\sqrt{6}\,\frac{\mu_{\mathbf{1}}}{\lambda_{\mathbf{1}}}\,, \tag{3.15}$$

for the family-adjoint, setting the GUT scale $v \sim v_{\rm F}$. The vacuum structure $\langle \tilde{\Delta}^8_{\rm L,R,F} \rangle \neq 0$ leads to the spontaneous breaking $SU(3)_{\rm L,R,F} \to SU(2)_{\rm L,R,F} \times U(1)_{\rm L,R,F}$ (see Appendix A for the corresponding generators and U(1) charges), resulting in the unbroken group

$$G_{32211\{21\}} \equiv SU(3)_{\rm C} \times [SU(2)_{\rm L} \times SU(2)_{\rm R} \times U(1)_{\rm L} \times U(1)_{\rm R}] \times \{SU(2)_{\rm F} \times U(1)_{\rm F}\}\,. \tag{3.16}$$

Indeed, LR-parity also remains unbroken since $v_{\rm L} = v^*_{\rm R}$, which is true as long as $\lambda_{\mathbf{78}}$ is taken to be real.

By making the shift

$$\boldsymbol{\Delta}^a_{\rm L,R} \to \boldsymbol{\Delta}^a_{\rm L,R} + \frac{v}{\sqrt{2}}\delta^a_8, \quad \boldsymbol{\Delta}^a_{\rm F} \to \boldsymbol{\Delta}^a_{\rm F} + \frac{v_{\rm F}}{\sqrt{2}}\delta^a_8 \tag{3.17}$$

and substituting $\mu_{\mathbf{78}} = \frac{\lambda_{\mathbf{78}}\, v}{2\sqrt{6}}$, $\mu_{\mathbf{1}} = \frac{\lambda_{\mathbf{1}}\, v_{\rm F}}{2\sqrt{6}}$ in the superpotential, we obtain

$$\begin{aligned} W \supset \sum_{B={\rm L,R}} &\left[\frac{\lambda_{\mathbf{78}}\, v}{2\sqrt{2}}\left(d_{aa8} + \frac{1}{2\sqrt{3}}\right)\boldsymbol{\Delta}^a_B\boldsymbol{\Delta}^a_B + \frac{1}{3!}\lambda_{\mathbf{78}}\, d_{abc}\boldsymbol{\Delta}^a_B\boldsymbol{\Delta}^b_B\boldsymbol{\Delta}^c_B\right] \\ &+\frac{\lambda_{\mathbf{1}}\, v_{\rm F}}{2\sqrt{2}}\left(d_{aa8} + \frac{1}{2\sqrt{3}}\right)\boldsymbol{\Delta}^a_{\rm F}\boldsymbol{\Delta}^a_{\rm F} + \frac{1}{3!}\lambda_{\mathbf{78}}\, d_{abc}\boldsymbol{\Delta}^a_{\rm F}\boldsymbol{\Delta}^b_{\rm F}\boldsymbol{\Delta}^c_{\rm F} \\ &+\frac{\lambda_{\mathbf{78}}\, v}{4\sqrt{6}}\boldsymbol{\Delta}^a_{\rm C}\boldsymbol{\Delta}^a_{\rm C} + \frac{1}{3!}\lambda_{\mathbf{78}}\, d_{abc}\boldsymbol{\Delta}^a_{\rm C}\boldsymbol{\Delta}^b_{\rm C}\boldsymbol{\Delta}^c_{\rm C} + {\rm const.} \end{aligned} \tag{3.18}$$

Note, due to $d_{aa8} = -1/(2\sqrt{3})$ for $a = 4, 5, 6, 7$ the quadratic terms in the superpotential vanish for $\boldsymbol{\Delta}^{4,5,6,7}_{\rm L,R,F}$, meaning that these fields receive no $\mathcal{F}$-term contribution to their masses (contrary to the other components of $\boldsymbol{\Delta}_{\rm L,R}$ and $\boldsymbol{\Delta}_{\rm F}$ which receive GUT-scale masses $m^2_\Delta \sim \lambda^2_{\mathbf{78}} v^2$ and $\lambda^2_{\mathbf{1}} v^2_{\rm F}$, respectively). While the global Goldstone bosons ${\rm Re}[\tilde{\Delta}^{4,5,6,7}_{\rm F}]$ are present in the physical spectrum, the gauge ones become the longitudinal polarisation states of the heavy gauge bosons related to the breaking $G_{333} \to G_{32211}$.

The presence of massless scalar degrees of freedom can only be avoided in the extended model with the gauged family symmetry. It is clear, however, that even in the case of an approximately global ${\rm SU(3)_F}$ with $g_{\rm F} \ll g_{\rm U}$ there are no massless Goldstones in the spectrum (provided that the accidental symmetries are softly broken at low energies) but a set of relatively light family gauge bosons very weakly interacting with the rest of the spectrum which is a viable solution.

By performing the shifts in Eq. (3.17) in the $\mathcal{D}$-terms, we obtain

$$\mathcal{D}^a_B \supset -{\rm i} f^{abc}\tilde{\Delta}^{b\,\dagger}_B\tilde{\Delta}^c_B \to -{\rm i}\frac{v}{\sqrt{2}} f^{a8b}\left(\tilde{\Delta}^b_B - \tilde{\Delta}^{b\,\dagger}_B\right) - {\rm i} f^{abc}\tilde{\Delta}^{b\,\dagger}_B\tilde{\Delta}^c_B, \tag{3.19}$$

for $B = {\rm L,R}$ leading to the universal GUT-scale mass term $m^2 = 3g^2_{\rm U}v^2/4$ for the gauge-adjoints ${\rm Im}[\tilde{\Delta}^{4,5,6,7}_{\rm L,R}]$, while $\tilde{\Delta}^{4,5,6,7}_{\rm F}$ have no $\mathcal{D}$-term contribution (or a small one in the case of approximately global ${\rm SU(3)_F}$ with $g_{\rm F} \ll g_{\rm U}$). Hence, all components of the gauge adjoints and $\tilde{\Delta}^{1,2,3,8}_{\rm F}$ receive masses of order GUT scale and are integrated out in the low-energy EFT. The remaining $\tilde{\Delta}^{4,5,6,7}_{\rm F}$, on the other hand, receive a much smaller mass from the soft SUSY breaking sector (and strongly suppressed $\mathcal{D}$-terms) and stay in the physical spectrum of the EFT. In what follows, we shall denote by $\boldsymbol{\mathcal{H}}^i_F$ the superfields containing ${\rm Im}[\tilde{\Delta}^{4,5,6,7}_{\rm L,R}]$, while ${\rm Re}[\tilde{\Delta}^{4,5,6,7}_{\rm L,R}]$ are embedded in the superfield $\boldsymbol{\mathcal{G}}^i_F$ alongside with its fermionic partners.

3.3 LR-symmetric SUSY theory

As shown in the previous section, all components of the gauge adjoint chiral superfields receive masses on the order of the GUT scale ($\mathcal{O}(v)$) in the vacuum given by Eq. (3.17).

This means that to study the low-energy predictions of the theory, we need to integrate out $\boldsymbol{\Delta}_{\rm L,R,C}$, as well as components 1, 2, 3 and 8 of $\boldsymbol{\Delta}_{\rm F}$.

For the gauge sector of the SHUT model, $\langle\tilde{\Delta}_{\rm L,R}\rangle$ naturally triggers a $\mathrm{SU}(3)_{\rm L,R}\to\mathrm{SU}(2)_{\rm L,R}\times\mathrm{U}(1)_{\rm L,R}$ breaking also for the tri-triplets (whose interactions with $\tilde{\Delta}_{\rm L,R}$ are mediated via $V^a_{\rm L,R}$ gauge bosons). For the global $\mathrm{SU}(3)_{\rm F}$ sector, there is no coupling of $\tilde{\Delta}_{\rm F}$ to the tri-triplets and, thus, the $\mathrm{SU}(3)_{\rm F}$ symmetry remains intact (or approximate in the case of $g_{\rm F}\ll g_{\rm U}$) in the tri-triplet sector, resulting in $G_{32211\{3\}}$ rather than $G_{32211\{21\}}$. Integrating out $\boldsymbol{\Delta}_{\rm L,R,C}$, and components 1, 2, 3 and 8 of $\boldsymbol{\Delta}_{\rm F}$, therefore leaves us with a supersymmetric theory based on the symmetry group $G_{32211\{3\}}$, with a chiral superfield content given by $\boldsymbol{\Delta}_{\rm F}^{4-7}$ and by the branching of $\boldsymbol{L}$, $\boldsymbol{Q}_{\rm L}$ and $\boldsymbol{Q}_{\rm R}$.

Writing the trinification tri-triplets in terms of $G_{32211\{3\}}$ representations one gets,

$$(\boldsymbol{L}^i)^l{}_r=\left(\begin{array}{cc|c}\boldsymbol{H}_{11} & \boldsymbol{H}_{12} & \boldsymbol{e}_{\rm L}\\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} & \boldsymbol{\nu}_{\rm L}\\ \hline \boldsymbol{e}^c_{\rm R} & \boldsymbol{\nu}^c_{\rm R} & \boldsymbol{\phi}\end{array}\right)^i,\quad \begin{array}{l}(\boldsymbol{Q}^i_{\rm L})^x{}_l=\left(\begin{array}{cc|c}\boldsymbol{u}^x_{\rm L} & \boldsymbol{d}^x_{\rm L} & \boldsymbol{D}^x_{\rm L}\end{array}\right)^i,\\ (\boldsymbol{Q}^i_{\rm R})^r{}_x=\left(\begin{array}{cc|c}\boldsymbol{u}^c_{{\rm R}x} & \boldsymbol{d}^c_{{\rm R}x} & \boldsymbol{D}^c_{{\rm R}x}\end{array}\right)^{\top\ i},\end{array} \tag{3.20}$$

where the vertical and horizontal lines denote the separation of the original tri-triplets into SU(2)-doublets and singlets after the first SSB step. We will refer to the lepton and quark $\mathrm{SU}(2)_{\rm L,R}$ doublets as $\boldsymbol{E}_{\rm L,R}$ and $\boldsymbol{q}_{\rm L,R}$. With this, we find that the most general superpotential consistent with $G_{32211\{3\}}$ is

$$\begin{aligned}W=\varepsilon_{ijk}\Big\{&y_1\boldsymbol{\phi}^i\boldsymbol{D}_{\rm L}{}^j\boldsymbol{D}_{\rm R}{}^k+y_2(\boldsymbol{H}^i)^L{}_R(\boldsymbol{q}_{\rm L}{}^j)_L(\boldsymbol{q}_{\rm R}{}^k)^R\\ &+y_3(\boldsymbol{E}_{\rm L}{}^i)^L(\boldsymbol{q}_{\rm L}{}^j)_L\boldsymbol{D}_{\rm R}{}^k+y_4(\boldsymbol{E}_{\rm R}{}^i)_R\boldsymbol{D}_{\rm L}{}^j(\boldsymbol{q}_{\rm R}{}^k)^R\Big\}\,.\end{aligned} \tag{3.21}$$

Note, in this effective SUSY LR theory one could naively add a mass term like $\varepsilon_{ij}\tilde{\mu}\boldsymbol{\mathcal{H}}^i_F\boldsymbol{\mathcal{G}}^i_F$ (that is symmetric under $\mathrm{SU}(2)_{\rm F}\times\mathrm{U}(1)_{\rm F}$ but not under full $\mathrm{SU}(3)_{\rm F}$) between the massless components of the family-adjoint superfield, $\boldsymbol{\mathcal{H}}^i_F$, and the massless superfield $\boldsymbol{\mathcal{G}}^i_F$ containing the Goldstone bosons. Such an effective μ-term is matched to zero at tree level at the GUT scale. Due to SUSY non-renormalisation theorems [53], in the exact SUSY limit this term cannot be regenerated radiatively at low energies so $\tilde{\mu}$ is identically zero and was not included in the superpotential given by Eq. (3.21). So, the resulting superpotential contains only fundamental superfields coming from $\boldsymbol{L}$, $\boldsymbol{Q}_{\rm L}$ and $\boldsymbol{Q}_{\rm R}$ and is indeed invariant under $\mathrm{SU}(3)_{\rm F}$.

In the high-scale theory, a complex $\lambda_{\mathbf{78}}$ would be the only source of LR-parity violation. In the low energy theory this would lead to $y_3\neq y_4^*$. Otherwise, $y_3=y_4^*$ and after matching is performed we can always make any $y_{1,2,3,4}$ real by field redefinitions. The same argument applies for the equality of the corresponding LR gauge couplings for $\mathrm{SU}(2)_{\rm L,R}\times\mathrm{U}(1)_{\rm L,R}$ symmetries.

Since we now have an effective LR-symmetric SUSY model containing gauge $\mathrm{U}(1)_{\rm L,R}$, there is a possibility of having a gauge kinetic mixing. The $\mathrm{U}(1)_{\rm L,R}$ D-term contribution to the

Lagrangian,

$$\mathcal{L} \supset \frac{1}{2}(\chi\,\mathcal{D}_{\rm L}\mathcal{D}_{\rm R} + \mathcal{D}_{\rm L}^2 + \mathcal{D}_{\rm R}^2) - \kappa(\mathcal{D}_{\rm L} - \mathcal{D}_{\rm R}) + X_{\rm L}\mathcal{D}_{\rm L} + X_{\rm R}\mathcal{D}_{\rm R}\,, \tag{3.22}$$

where the terms proportional to κ are the Fayet-Iliopoulos terms, while the D-terms and the expressions for $X_{\rm L,R}$ are shown in Appendix C.3.2.

The values of the parameters $\{y_{1,2,3,4}, g_{\rm C}, g_{\rm L,R}, g'_{\rm L,R}, \chi, \kappa\}$ in the LR-symmetric SUSY theory are determined by the values of the parameters $\{\lambda_{\mathbf{27}}, \lambda_{\mathbf{78}}, g_{\rm U}, v\}$ in the high-scale trinification theory at the GUT-scale boundary through a matching procedure[2]. The corresponding values at lower energies can then be obtained by an RG analysis. In particular, we note that the only dimensionful parameter in the effective theory is the Fayet-Iliopoulos parameter κ. This means that $\beta_\kappa \propto \kappa$ so that if $\kappa = 0$ at the matching scale (which is true, at least, at tree level), then κ will remain at a zero value throughout the RG flow yielding no spontaneous SUSY breaking. Thus, we stick to the concept of soft SUSY breaking in what follows.

4 Softly broken SUSY at the GUT scale

Assuming that SUSY is softly broken, an interesting point we would like to explore here is the fact that the soft SUSY breaking terms can lead to a VEV in $\big(\tilde{L}^3\big)^3{}_3 \equiv \widetilde{\phi}^3 \equiv \widetilde{\varphi}$ that is of the same order as the soft SUSY breaking scale, $m_{\rm soft}$. Such a VEV will, due to its smallness compared to the GUT scale v, barely contribute to the symmetry breaking caused by $\langle\tilde{\Delta}_{\rm L,R}\rangle$. However, it will break the gauge symmetry $\mathrm{U}(1)_{\rm L} \times \mathrm{U}(1)_{\rm R}$ to its center group $\mathrm{U}(1)_{\rm L+R}$ (a close analog of the $\mathrm{U}(1)_{\rm B-L}$ symmetry in the Pati-Salam models) and will also contribute to the breaking of the global $\mathrm{SU}(3)_{\rm F} \times \mathrm{U}(1)_{\rm W}$ symmetry.

4.1 The soft SUSY breaking Lagrangian

Supersymmetry is softly broken in the scalar sector via bilinear and trilinear interactions given by

$$\begin{aligned} V_{\rm soft}^{\rm G} = \Bigg\{ & m_{\mathbf{27}}^2 \big(\tilde{L}^i\big)^l{}_r \big(\tilde{L}^*_i\big)_l{}^r + m_{\mathbf{78}}^2 \tilde{\Delta}_{\rm L}^{*a}\tilde{\Delta}_{\rm L}^a + \left[\frac{1}{2}b_{\mathbf{78}}\tilde{\Delta}_{\rm L}^a\tilde{\Delta}_{\rm L}^a + \text{c.c}\right] \\ & + d_{abc}\left[\frac{1}{3!}A_{\mathbf{78}}\tilde{\Delta}_{\rm L}^a\tilde{\Delta}_{\rm L}^b\tilde{\Delta}_{\rm L}^c + \frac{1}{2}C_{\mathbf{78}}\tilde{\Delta}_{\rm L}^{*a}\tilde{\Delta}_{\rm L}^b\tilde{\Delta}_{\rm L}^c + \text{c.c.}\right] \\ & + \Big[A_{\rm G}\,\tilde{\Delta}_{\rm L}^a\,(T_a)^{l_1}{}_{l_2}\,\big(\tilde{L}^*_i\big)_{l_1}{}^r\big(\tilde{L}^i\big)^{l_2}{}_r + A_{\bar{\rm G}}\,\tilde{\Delta}_{\rm R}^a\,(T_a)^{r_2}{}_{r_1}\,\big(\tilde{L}^*_i\big)_l{}^{r_1}\big(\tilde{L}^i\big)^l{}_{r_2} + \text{c.c.}\Big] \\ & + (\mathbb{Z}_3^{(\rm LRC)}\ \text{permutations})\Bigg\} + \Big[A_{\mathbf{27}}\,\varepsilon_{ijk}\big(\tilde{Q}_{\rm L}^i\big)^x{}_l\big(\tilde{Q}_{\rm R}^j\big)^r{}_x\big(\tilde{L}^k\big)^l{}_r + \text{c.c.}\Big]\,, \end{aligned} \tag{4.1}$$

[2] Since $\lambda_{\mathbf{1}}$ and $v_{\rm F}$ parameters originate from interactions within the family-adjoint sector, they affect neither any couplings in the fundamental superfield sector nor the gauge couplings to any order in perturbation theory.

for the gauge-adjoints and pure tri-triplet terms, and

$$V_{\rm soft}^{\rm F} = m_{\bf 1}^2 \tilde\Delta_{\rm F}^{*a}\tilde\Delta_{\rm F}^{a} + \left[\frac{1}{2} b_{\bf 1}\tilde\Delta_{\rm F}^{a}\tilde\Delta_{\rm F}^{a} + {\rm c.c}\right] + d_{abc}\left[\frac{1}{3!}A_{\bf 1}\tilde\Delta_{\rm F}^{a}\tilde\Delta_{\rm F}^{b}\tilde\Delta_{\rm F}^{c} + \frac{1}{2}C_{\bf 1}\tilde\Delta_{\rm F}^{*a}\tilde\Delta_{\rm F}^{b}\tilde\Delta_{\rm F}^{c} + {\rm c.c.}\right]$$
$$+ \left[A_{\rm F}\tilde\Delta_{\rm F}^{a}\left(T_a\right)^i{}_j\left(\tilde L_i^*\right)_l{}^r\left(\tilde L^j\right)^l{}_r + {\rm c.c.} + (\mathbb{Z}_3^{\rm (LRC)}\text{ permutations})\right] . \tag{4.2}$$

for the family adjoint. All parameters here are assumed to be real for simplicity. We note that although trilinear terms with the gauge singlets (such as $\tilde\Delta_{\rm F}^*\tilde\Delta_{\rm F}\tilde\Delta_{\rm F}$ above) are not in general soft, due to the family symmetry and the fact that $\sum_a d_{aab} = 0$, the dangerous tadpole diagrams do indeed cancel and do not lead to quadratic divergences.

The terms in Eq. (4.1) and (4.2), which account for the most general soft SUSY breaking scalar potential consistent with $G_{333\{3\}}$, also respect the accidental $\mathrm{U(1)_W}\times\mathrm{U(1)_B}$ symmetry of the original SUSY theory. However, accidental LR-parity is, in general, softly-broken as long as $A_{\rm G}\neq A_{\bar{\rm G}}$, and this breaking can then be transmitted to the other sectors of the effective theory radiatively (e.g. via RG evolution and radiative corrections at the matching scale).

Note that the $A_{\rm F}$-term in the soft sector introduces small $\mathrm{SU(3)_F}$ violating (but $\mathrm{SU(2)_F}\times\mathrm{U(1)_F}$ preserving) effects on the interactions in the effective theory once $\langle\tilde\Delta_{\rm F}\rangle\neq 0$. Consider, for example, effective quartic interactions between components of $\tilde L$ that come from two $A_{\rm F}$ tri-linear vertices connected by an internal $\tilde\Delta_{\rm F}^{1,2,3}$ or $\tilde\Delta_{\rm F}^{8}$ propagator. The value of this diagram is $\sim {\rm i}A_{\rm F}^2/\lambda_{\bf 1}^2 v_{\rm F}^2$ neglecting the external momentum in the propagator. Then, since positivity of scalar masses squared in the high-scale theory requires

$$|A_{\rm F}| \lesssim \frac{m_{\bf 27}^2}{v_{\rm F}} \sim \frac{m_{\rm soft}^2}{v}\,, \tag{4.3}$$

this diagram behaves as $[m_{\rm soft}/v]^4$.

The scalar tree-level mass spectrum in the effective theory at the GUT scale (after integrating out the $\mathbf{\Delta}_{\rm L,R,C}$ and heavy components of $\mathbf{\Delta}_{\rm F}$) is fully controlled by soft parameters. Indeed, the scalar fields from the tri-triplets receive masses of the order of the soft SUSY breaking scale. The full expressions are given in Appendix B, from which we notice that local stability of the scalar potential requires

$$\left|A_{{\rm G},\bar{\rm G}}\right| v \sim |A_{\rm F}|\, v_{\rm F} \sim |A_{\bf 1}|\, v_{\rm F} \sim m_{\rm soft}^2 \Rightarrow \begin{cases} \left|A_{{\rm G},\bar{\rm G},{\rm F}}\right| & \lesssim \frac{m_{\bf 27}^2}{v}\,, \\ |A_{\bf 1}| & \lesssim \frac{m_{\bf 1}^2}{v_{\rm F}}\,. \end{cases} \tag{4.4}$$

For the full vacuum (meta)stability conditions, see Sect. B.1.1.

The possible fermion soft SUSY breaking terms are the Majorana mass terms for the gauginos and the Dirac mass terms between the gauginos and the fermion components of $\mathbf{\Delta}_{\rm L,R,C}$, namely,

$$\mathcal{L}_{\rm soft}^{\rm fermion} = \left[-\frac{1}{2}M_0\widetilde\lambda_{\rm L}^a\widetilde\lambda_{\rm L}^a - M_0'\widetilde\lambda_{\rm L}^a\Delta_{\rm L}^a + {\rm c.c.} + (\mathbb{Z}_3^{\rm (LRC)}\text{ permutations})\right], \tag{4.5}$$

From the transformation rules in Eqs. (3.6) and (3.13) it follows that LR-parity is not respected by $\mathcal{L}_{\rm soft}^{\rm fermion}$ unless $M_0'=0$.

4.2 Vacuum in the presence of soft SUSY breaking terms

As discussed above, the soft SUSY breaking terms could trigger a VEV in $\left(\tilde{L}^3\right)^3{}_3 \equiv \widetilde{\varphi}$ of the same order as the soft SUSY breaking scale. With $\langle\Delta^8_{\mathrm{L,R,F}}\rangle \equiv \frac{1}{\sqrt{2}} v_{\mathrm{L,R,F}}$ and $\langle\widetilde{\varphi}\rangle \equiv \frac{1}{\sqrt{2}} v_\varphi$ being the VEVs present, our potential evaluated in the vacuum is given by

$$
\begin{aligned}
V_{\mathrm{vac}} = & \left[\frac{1}{2} m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}\left(A_{\mathrm{G}} v_{\mathrm{L}} + A_{\bar{\mathrm{G}}} v_{\mathrm{R}} + A_{\mathrm{F}} v_{\mathrm{F}}\right)\right] v_\varphi^2 + \frac{1}{12} g_{\mathrm{U}}^2 v_\varphi^4 + \left\{\frac{1}{2}\left(m^2_{\mathbf{78}} + b_{\mathbf{78}}\right) v_{\mathrm{L}}^2\right. \\
& \left. - \frac{1}{\sqrt{6}}\left(\frac{1}{3!} A_{\mathbf{78}} + \frac{1}{2} C_{\mathbf{78}}\right) v_{\mathrm{L}}^3 + \frac{1}{2} v_{\mathrm{L}}^2 \left(\frac{1}{2\sqrt{6}} \lambda_{\mathbf{78}} v_{\mathrm{L}} - \mu_{\mathbf{78}}\right)^2 + (v_{\mathrm{L}} \to v_{\mathrm{R}})\right\} \\
& + \frac{1}{2}\left(m^2_{\mathbf{1}} + b_{\mathbf{1}}\right) v_{\mathrm{F}}^2 - \frac{1}{\sqrt{6}}\left(\frac{1}{3!} A_{\mathbf{1}} + \frac{1}{2} C_{\mathbf{1}}\right) v_{\mathrm{F}}^3 .
\end{aligned} \tag{4.6}
$$

As all other fields (that do not acquire VEVs) only enter in bi-linear combinations, it suffices to consider the above terms to solve the conditions for vanishing first derivatives of the scalar potential. We retain the notation $v \equiv v_{\mathrm{L}} = v_{\mathrm{R}} = 2\sqrt{6}\mu_{\mathbf{78}}/\lambda_{\mathbf{78}}$ for the VEVs of $\tilde{\Delta}^8_{\mathrm{L,R}}$ in the absence of soft terms. Adopting that the soft terms are much smaller than the GUT scale, i.e. $m_{\mathrm{soft}} \ll v$, we can resolve the extremum conditions for $v_{\mathrm{L,R},\varphi}$ by Taylor expanding them to the leading order in soft terms. Doing so we find

$$
\begin{aligned}
v_\varphi^2 & \approx \frac{3}{g_{\mathrm{U}}^2}\left[-m^2_{\mathbf{27}} + \sqrt{\frac{2}{3}}\left(A_{\mathrm{G}} + A_{\bar{\mathrm{G}}}\right) v + \sqrt{\frac{2}{3}} A_{\mathrm{F}} v_{\mathrm{F}}\right] , \\
v_{\mathrm{L,R}} & \approx v + \frac{24}{\lambda_{\mathbf{78}}}\left[-\frac{m^2_{\mathbf{78}} + b_{\mathbf{78}}}{v} + \sqrt{\frac{3}{2}}\left(\frac{1}{3!} A_{\mathbf{78}} + \frac{1}{2} C_{\mathbf{78}}\right) + \frac{1}{\sqrt{6}} A_{\mathrm{G},\bar{\mathrm{G}}}\left(\frac{v_\varphi}{v}\right)^2\right] .
\end{aligned} \tag{4.7}
$$

As is described above, the soft tri-linear couplings $A_{\mathrm{G},\bar{\mathrm{G}}}$, $A_{\mathbf{78}}$ and $C_{\mathbf{78}}$ need to be $\lesssim m^2_{\mathbf{27}}/v$ for the vacuum stability to hold. Adding the soft terms shifts the values of the VEVs $v_{\mathrm{L,R}}$ by a relative amount behaving as

$$
\sim \left[\frac{m_{\mathrm{soft}}}{v}\right]^2 . \tag{4.8}
$$

Furthermore, we note that the presence of v_φ slightly affects the equality of $v_{\mathrm{L,R}}$,

$$
v_{\mathrm{L}} - v_{\mathrm{R}} \approx \frac{4\sqrt{6}}{\lambda_{\mathbf{78}}}\left(\frac{v_\varphi}{v}\right)^2 \left(A_{\mathrm{G}} - A_{\bar{\mathrm{G}}}\right) . \tag{4.9}
$$

The relative difference between $v_{\mathrm{L,R}}$, therefore, behaves as

$$
\sim \left[\frac{m_{\mathrm{soft}}}{v}\right]^4 . \tag{4.10}
$$

That is, although the VEVs of $\tilde{\Delta}_{\mathrm{L,R}}$ are shifted by the soft terms, the effect is negligible and the presence of the $\mathbb{Z}_3^{(\mathrm{LRC})}$ symmetry in the soft SUSY breaking sector preserves the equality of $\langle\tilde{\Delta}_{\mathrm{L,R}}\rangle$ at tree-level (as neither A_{G} nor $A_{\bar{\mathrm{G}}}$ enter in the tree-level extremal conditions for these VEVs in the absence of v_φ).

With a non-zero $v_\varphi \sim m_{\rm soft} \ll v$, the symmetry is further broken as

$$\mathrm{U}(1)_{\rm L} \times \mathrm{U}(1)_{\rm R} \times \{\mathrm{U}(1)_{\rm F} \times \mathrm{U}(1)_{\rm W}\} \overset{\langle\varphi\rangle}{\to} \mathrm{U}(1)_{\rm L+R} \times \{\mathrm{U}(1)_{\rm S} \times \mathrm{U}(1)_{\rm S'}\} \tag{4.11}$$

where $\mathrm{U}(1)_{\rm L+R}$ consists of simultaneous $\mathrm{U}(1)_{\rm L,R}$ phase rotations by the same phase. $\mathrm{U}(1)_{\rm S}$ and $\mathrm{U}(1)_{\rm S'}$ are also simultaneous $\mathrm{U}(1)_{\rm L,R}$ phase rotations, but with opposite phase, which is compensated by an appropriate $\mathrm{U}(1)_{\rm F}$ and $\mathrm{U}(1)_{\rm W}$ transformation, respectively. This is further explained in Appendix A.

In the limit of vanishingly small $A_{\rm F} \to 0$ in Eq. (4.2), the model exhibits an exact global $\mathrm{SU}(3)_{\rm F'} \times \mathrm{SU}(3)_{\rm F''}$ symmetry as we could then perform independent SU(3) family rotations on $(\boldsymbol{L}, \boldsymbol{Q}_{\rm L,R})$ and $\boldsymbol{\Delta}_{\rm F}$. With non-zero v_φ and $v_{\rm F}$, we would in this case end up with Goldstone fields built up out of $\widetilde{\phi}^{1,2}$ and $\mathrm{Re}[\tilde{\Delta}_{\rm F}^{4,5,6,7}]$ from the spontaneous breaking of $\mathrm{SU}(3)_{\rm F'}$ and $\mathrm{SU}(3)_{\rm F''}$, respectively. With $A_{\rm F} \neq 0$ the $\mathrm{SU}(3)_{\rm F'} \times \mathrm{SU}(3)_{\rm F''}$ symmetry becomes softly broken to the familiar $\mathrm{SU}(3)_{\rm F}$. This causes $\widetilde{\phi}^{1,2}$ and $\mathrm{Re}[\tilde{\Delta}_{\rm F}^{4,5,6,7}]$ to arrange themselves into one pure Goldstone and one pseudo-Goldstone $\mathrm{SU}(2)_{\rm F}$ doublet (the mass of the latter is proportional to $A_{\rm F}$). Since $v_\varphi \ll v_{\rm F}$, the pure Goldstone is mostly $\mathrm{Re}[\tilde{\Delta}_{\rm F}^{4,5,6,7}]$ (it has a small $\mathcal{O}(v_\varphi/v)$ admixture of $\widetilde{\phi}^{1,2}$, while the pseudo-Goldstone mode is mostly $\widetilde{\phi}^{1,2}$ containing an $\mathcal{O}(v_\varphi/v)$ amount of $\mathrm{Re}[\tilde{\Delta}_{\rm F}^{4,5,6,7}]$).

4.3 Masses in presence of soft SUSY breaking terms

The inclusion of soft-SUSY breaking interactions results in the emergence of non-zero masses for the fundamental scalars contained in the $\boldsymbol{L}$, $\boldsymbol{Q}_{\rm L}$ and $\boldsymbol{Q}_{\rm R}$ superfields as well as for the gauginos. By construction, the soft SUSY breaking parameters are small in comparison to the GUT scale, i.e. $m_{\rm soft} \ll v$, which means that the heavy states in the SUSY theory discussed in Sect. 3 will remain heavy and only those that were massless will receive contributions whose size is relevant for the low-energy EFT.

The masses of the fundamental scalars are purely generated in the soft SUSY breaking sector. Furthermore, for a vacuum where only adjoint scalars acquire VEVs as in Eq. (3.17), there is no mixing among the components of the fundamental scalars corresponding to the physical eigenstates at the first breaking stage shown in Fig. 1.

The Higgs-slepton mass terms (no summation over the indices is implied) read

$$m^2_{\left(\tilde{L}^i\right)^l{}_r} = m^2_{\mathbf{27}} + 2\left[A_{\rm G} v \left(T^8\right)^l_l + A_{\bar{\rm G}} \left(T^8\right)^r_r + A_{\rm F} v_{\rm F} \left(T^8\right)^i_i\right] , \tag{4.12}$$

while the corresponding squark mass terms are given by

$$\begin{aligned} m^2_{\left(\tilde{Q}^i_{\rm L}\right)_l} &= m^2_{\mathbf{27}} + 2\left[A_{\rm G} v \left(T^8\right)^l_l + A_{\rm F} v_{\rm F} \left(T^8\right)^i_i\right] , \\ m^2_{\left(\tilde{Q}^i_{\rm R}\right)^r} &= m^2_{\mathbf{27}} + 2\left[A_{\bar{\rm G}} v \left(T^8\right)^r_r + A_{\rm F} v_{\rm F} \left(T^8\right)^i_i\right] . \end{aligned} \tag{4.13}$$

In Tab. 9 of appendix B we show the masses for each fundamental scalar component in the LR-parity symmetric limit corresponding to $A_{\rm G} = A_{\bar{\rm G}}$, for simplicity.

With the introduction of a soft SUSY breaking sector, $\widetilde{\mathcal{H}}_{\rm F}$ no longer stays massless receiving a contribution of the order $m_{\rm soft}$, namely,

$$m^2_{\widetilde{\mathcal{H}}_{\rm F}} \simeq 2m^2_{\mathbf{1}} + \mathcal{O}\left(m^4_{\rm soft}/v^2_{\rm F}\right) , \tag{4.14}$$

The exact expressions for scalar fields' masses squared can be found in Tab. 10 of appendix B.

The massless superpartners of the gauge bosons associated with the unbroken symmetries also acquire soft-scale masses. In particular, they mix with the chiral adjoint fermions via Dirac-terms whose strength, M_0' in Eq. (4.5), is also of the order $m_{\rm soft}$. Typically, for minimal Dirac-gaugino models, the ad-hoc introduction of adjoint chiral superfields has the undesirable side effect of spoiling the gauge couplings' unification. However, in the model studied in Refs. [54, 55], this problem is resolved by evoking trinification as the natural embedding for the required adjoint chiral scalars needed to form Dirac mass terms with gauginos. With this point in mind, we want to note that the SHUT model, with softly broken SUSY at the GUT scale, is on its own a Dirac-gaugino model and a possible high-scale framework for such a class of models.

The mass matrices for the adjoint fermions in the basis $\{\widetilde{\lambda}^{1,2,3}_{\rm L,R}, \widetilde{\Delta}^{1,2,3}_{\rm L,R}, \widetilde{\lambda}^{8}_{\rm L,R}, \widetilde{\Delta}^{8}_{\rm L,R}\}$ are then

$$\mathcal{M}_{\tilde{\lambda},\tilde{\Delta}} = \begin{pmatrix} M_0 & M_0' & 0 & 0 \\ M_0' & \frac{v\lambda_{\mathbf{78}}}{\sqrt{6}} + \mu_{\mathbf{78}} & 0 & 0 \\ 0 & 0 & M_0 & M_0' \\ 0 & 0 & M_0' & \frac{v\lambda_{\mathbf{78}}}{\sqrt{6}} - \mu_{\mathbf{78}} \end{pmatrix} , \tag{4.15}$$

which corresponds to $\{\mathcal{T}_{\rm L,R}, \mathcal{T}^{\perp}_{\rm L,R}, \mathcal{S}_{\rm L,R}, \mathcal{S}^{\perp}_{\rm L,R}\}$ in the diagonal basis. Here, $\mathcal{S}_{\rm L,R}$ and $\mathcal{T}_{\rm L,R}$ are the light (soft-scale) gauginos while $\mathcal{S}^{\perp}_{\rm L,R}$ and $\mathcal{T}^{\perp}_{\rm L,R}$ denote the heavy (GUT-scale) gauginos. Note, due to a small mixing, both the low- and high-scale gauginos are essentially Majorana-like. Indeed, as is seen from the exact expressions for the fermion masses in Tab. 11, the mass of the former ones are approximately given by M_0, while the high-scale gauginos $\mathcal{T}^{\perp}_{\rm L,R}$ and $\mathcal{S}^{\perp}_{\rm L,R}$ get their masses from $\mathcal{F}$-terms being approximately equal to $\left(\mathcal{M}_{\tilde{\lambda},\tilde{\Delta}}\right)^2_2$ and $\left(\mathcal{M}_{\tilde{\lambda},\tilde{\Delta}}\right)^4_4$, respectively.

The same effect is observed for the gluinos $\tilde{g}^a$ whose masses are M_0, for the light states, and $\mu_{\mathbf{78}}$, for the heavy states (denoted by $\perp$-label in Tab. 11). There is also an $\mathrm{SU}(2)_{\rm F}$-doublet fermion $\mathcal{H}_{\rm F}$ that acquires a mass of the order of soft SUSY breaking scale $m_{\rm soft}$. Note that $\mathcal{H}_{\rm F}$ as well as its superpartner $\widetilde{\mathcal{H}}_{\rm F}$ receive $\mathcal{D}$-term contributions if $\mathrm{SU}(3)_{\rm F}$ is gauged. Finally, the chiral fundamental fermions are massless at this stage.

5 Low-energy scenarios and possible paths towards the SM

After VEVs in $\tilde{\Delta}_{\rm L,R}$ break the trinification symmetry (including $\mathbb{Z}_3$), subsequent gauge symmetry breaking needs to occur in such way that a subset of the corresponding light fields can be identified with their SM counterparts, i.e. the light fields need to share quantum numbers with the SM particle content. In the following we will discuss how such a consistent SSB can happen without massless fermions below the EW SSB scale. Finally, one would like to find a particular EFT limit of the SHUT theory that is capable to explain patterns in the observed SM fermion mass spectrum in a way that does not contradict the existing constraints. Here we make the first step and discuss the basic features of the EFT scenarios as good candidates for further explorations.

Common to any possible scenario is the need for a VEV in $\left(\tilde{L}^i\right)^3{}_3 \equiv \tilde{\phi}^i$ and in the superpartner to a right-handed neutrino $\left(\tilde{L}^i\right)^3{}_2 \equiv \tilde{\nu}^i_{\mathrm{R}}$. While $\langle\phi^i\rangle$ initiates the breaking $\mathrm{U(1)_L} \times \mathrm{U(1)_R} \to \mathrm{U(1)_{L+R}}$, $\langle\tilde{\nu}^i_{\mathrm{R}}\rangle$ breaks the remaining gauge symmetry to that of the SM as $\mathrm{SU(2)_R} \times \mathrm{U(1)_{L+R}} \to \mathrm{U(1)_Y}$.

5.1 A consistent low-energy spectrum

As a first step, we offer in this section a short overview of the low-energy limits of the SHUT model. In particular, we investigate whether the SM-extended symmetry, $G_{\mathrm{SM}} \times \mathrm{U(1)_T} \times \mathrm{U(1)_{T'}}$ as represented at the bottom of Fig. 1, leaves enough freedom to realize the well-known SM particle spectrum and if it does not immediately contradict current observations.

5.1.1 Colour-neutral charged fermions

Once the $\mathrm{SU(2)_R} \times \mathrm{SU(2)_F}$ symmetries are broken, the tri-doublets $\widetilde{H}^{f\,l}_r$ and the bi-doublets $\widetilde{h}^l_r$ are split into three distinct generations of $\mathrm{SU(2)_L}$ doublets. We will then rename them as $\widetilde{H}^{f\,l}_{r=1} \equiv \widetilde{H}^{f\,l}_{\mathrm{u}}$, $\widetilde{h}^l_{r=1} \equiv \widetilde{h}^l_{\mathrm{u}}$, $\widetilde{H}^{f\,l}_{r=2} \equiv \widetilde{H}^{f\,l}_{\mathrm{d}}$ and $\widetilde{h}^l_{r=2} \equiv \widetilde{h}^l_{\mathrm{d}}$, while their scalar counterparts follow the same notation but without tildes.

Defining the components of the leptonic $\mathrm{SU(2)_L}$ doublets according to their electric charge (see Eq. (5.6)) as

$$\begin{aligned}
&\widetilde{H}^i_{\mathrm{u}} = \begin{pmatrix} \widetilde{H}^{i\,0}_{\mathrm{u}} \\ \widetilde{H}^{i\,+}_{\mathrm{u}} \end{pmatrix} \qquad \widetilde{H}^i_{\mathrm{d}} = \begin{pmatrix} \widetilde{H}^{i\,-}_{\mathrm{d}} \\ \widetilde{H}^{i\,0}_{\mathrm{d}} \end{pmatrix} \qquad E^i_{\mathrm{L}} = \begin{pmatrix} e^i_{\mathrm{L}} \\ \nu^i_{\mathrm{L}} \end{pmatrix} \\
&\widetilde{h}_{\mathrm{u}} = \begin{pmatrix} \widetilde{H}^{3\,0}_{\mathrm{u}} \\ \widetilde{H}^{3\,+}_{\mathrm{u}} \end{pmatrix} \qquad \widetilde{h}_{\mathrm{d}} = \begin{pmatrix} \widetilde{H}^{3\,-}_{\mathrm{d}} \\ \widetilde{H}^{3\,0}_{\mathrm{d}} \end{pmatrix} \qquad \mathcal{E}_{\mathrm{L}} = \begin{pmatrix} e^3_{\mathrm{L}} \\ \nu^3_{\mathrm{L}} \end{pmatrix}
\end{aligned} \tag{5.1}$$

where $i = 1, 2$, we can build mass terms for the charged lepton and charged Higgsinos as

$$\mathcal{L}_C = \left(e^1_{\mathrm{L}}\ e^2_{\mathrm{L}}\ e^3_{\mathrm{L}}\ \widetilde{H}^{1\,-}_{\mathrm{d}}\ \ \widetilde{H}^{2\,-}_{\mathrm{d}}\ \ \widetilde{H}^{3\,-}_{\mathrm{d}}\right) \mathcal{M}^C \left(e^1_{\mathrm{R}}\ e^2_{\mathrm{R}}\ e^3_{\mathrm{R}}\ \widetilde{H}^{1\,+}_{\mathrm{u}}\ \widetilde{H}^{2\,+}_{\mathrm{u}}\ \widetilde{H}^{3\,+}_{\mathrm{u}}\right)^{\top} + \mathrm{c.c.}\,. \tag{5.2}$$

Let us start by classifying all possible EW Higgs-doublet and complex-singlet bosons, whose VEVs may have a role in the SM-like fermion mass spectra. There are three types of Higgs-doublets distinguished in terms of their $\mathrm{U(1)_Y} \times \mathrm{U(1)_T}$ charges and one possibility for complex singlets (and their complex conjugates). In particular, we can have

1. (1, 1): $H_{\rm u}^{2,3}$, $H_{\rm d}^{*1}$, $\widetilde{E}_{\rm L}^{*2,3}$, with VEVs denoted as $\bullet$-type.
2. (1, 5): $H_{\rm u}^{1}$, $H_{\rm d}^{*2,3}$, with VEVs denoted as $\star$-type.
3. (1, −3): $\widetilde{E}_{\rm L}^{*1}$, with VEVs denoted as $*$-type.
4. (0, 4): $\widetilde{S}_{1,2}$, with VEVs denoted as $\diamond$-type.

Note that the two complex singlets emerge from the mixing $\left(\widetilde{\phi}^{*1},\, \widetilde{\nu}_{\rm R}^{2},\, \widetilde{\nu}_{\rm R}^{3}\right) \mapsto \left(\widetilde{S}_1,\, \widetilde{S}_2,\, \mathcal{G}_s\right)$ induced by the third breaking step in Fig. 1, with $\mathcal{G}_s$ being a complex Goldstone boson[3].

According to the quantum numbers shown in Tab. 8 of Appendix A, the matrix $\mathcal{M}^C$ has the structure

$$\mathcal{M}^C \sim \left(\begin{array}{ccc|ccc} 0 & \star & \star & 0 & 0 & 0 \\ \star & \bullet & \bullet & 0 & 0 & 0 \\ \star & \bullet & \bullet & 0 & 0 & 0 \\ \hline \star & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 & 0 \end{array}\right)\,, \tag{5.3}$$

In this case, the rank of the matrix $\mathcal{M}^C$ is three, which means that, while we may be able to identify the correct patterns for the masses of the charged leptons in the SM, the remaining charged massless Higgsinos are unacceptable at low energy. This effect is a direct consequence of the unbroken accidental $\mathrm{U}(1)_{\mathrm{T}'}$-symmetry that results from the accidental $\mathrm{U}(1)_{\mathrm{W}}$ present in the high-scale SHUT theory. In order to get a particle content consistent with the SM at lower energies one needs to break $\mathrm{U}(1)_{\mathrm{W}}$ at low energies.

As $\mathrm{U}(1)_{\mathrm{W}}$ is an accidental symmetry that results from the interplay between the original imposed symmetries and the choice of the (minimal) field content, there is no strict reason for keeping it intact below the GUT scale. Note, it is possible to incorporate the soft $\mathrm{U}(1)_{\mathrm{W}}$-violating interactions in the soft SUST breaking sector. Such effects can be parametrised by introducing the allowed soft $\mathrm{U}(1)_{\mathrm{W}}$-breaking terms

$$\begin{aligned} V_{\rm soft}^{\not{\psi}} = \Big(& A_{Hh\phi} H_r^{l\,f} h_{r'}^{l'} \widetilde{\phi}^{f'} + A_{hEE} h_r^{l} \widetilde{E}_{\rm L}^{f\,l'} \widetilde{E}_{{\rm R}\,r'}^{f'} \\ & + A_{hE\mathcal{E}} H_r^{f\,l} \widetilde{E}_{\rm L}^{f'l'} \widetilde{\mathcal{E}}_{{\rm R}\,r'} + \bar{A}_{hE\mathcal{E}} H_r^{f\,l} \widetilde{E}_{{\rm R}\,r'}^{f'} \widetilde{\mathcal{E}}_{\rm L}^{l'} \Big)\, \varepsilon_{ff'} \varepsilon_{ll'} \varepsilon^{rr'} + \text{c.c.}\,. \end{aligned} \tag{5.4}$$

Note that the trilinear couplings above are such that $A_{ijk} \ll v$. This can be justified as the case of softly-broken SUSY after the T-GUT symmetry breaking, $V_{\rm soft}^{\not{\psi}}$ would naturally appear in the soft-SUSY breaking Lagrangian of the effective LR-symmetric SUSY theory and thus should be present in the low-energy EFT[4]. This means that the SM symmetry is now solely extended by $\mathrm{U}(1)_{\mathrm{T}} \times \mathrm{U}(1)_{\mathrm{B}}$. As we shall see below, this largely opens up

[3] Note that the breaking $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{F}} \times \mathrm{U}(1)_{\mathrm{L+R}} \times \mathrm{U}(1)_{\mathrm{S}} \to \mathrm{U}(1)_{\mathrm{Y}} \times \mathrm{U}(1)_{\mathrm{T}}$ gives rise to six Goldstone bosons, three gauge and three global ones, where the former are $\mathrm{Im}\left[\widetilde{\nu}_{\rm R}^{1}\right]$, $\mathrm{Re}\left[\widetilde{e}_{\rm R}^{1}\right]$ and $\mathrm{Im}\left[\widetilde{e}_{\rm R}^{1}\right]$ while the latter ones are $\mathrm{Im}\left[\widetilde{\phi}^{2}\right]$ and $\mathcal{G}_s$.

[4] The accidental $\mathrm{U}(1)_{\mathrm{B}}$ symmetry can also be broken below the GUT scale. Indeed, in the same way that $\mathrm{U}(1)_{\mathrm{W}}$ is no longer a symmetry of the low-energy EFT if the soft-SUSY breaking interactions are to be

the possibilities for consistent low-energy spectra, whose details are driven by the specifics of the light Higgs sector with $U(1)_T$ being the family symmetry in the Higgs and fermion sectors.

The most general form of the new mass matrix $\mathcal{M}^C$ reads

$$\mathcal{M}^C \sim \left(\begin{array}{ccc|ccc} 0 & \star & \star & 0 & \diamond & \diamond \\ \star & \bullet & \bullet & \diamond & \blacklozenge & \blacklozenge \\ \star & \bullet & \bullet & \diamond & \blacklozenge & \blacklozenge \\ \hline \star & \bullet & \bullet & \diamond & \blacklozenge & \blacklozenge \\ \bullet & * & * & \blacklozenge & \diamond & \diamond \\ \bullet & * & * & \blacklozenge & \diamond & \diamond \end{array}\right), \tag{5.5}$$

where $\blacklozenge$-symbol labels allowed (different) Dirac mass terms generated before the EWSB such that their values are not related to the Higgs VEVs and can be well above the EW scale. We also see that the complex singlet VEVs can only have a sizable impact if their size is of the same order of as the $\blacklozenge$-type entries. We have now a mass matrix of rank-6 which means that no charged leptons and Higgsinos are left massless after EWSB. Note that before the EW symmetry is broken there are three massless lepton doublets. Furthermore, due to large $\blacklozenge$-type entries the structure of $\mathcal{M}^C$ allows for three exotic lepton eigenstates heavier than the EW scale. The charged leptons should then be identified as the light states resulting from the mixing between the charged components of the original $\left(E_L^f, \widetilde{H}_d^f\right)$ and $\left(e_R^f, \widetilde{H}_u^f\right)$ eigenstates. In particular, when any of the Higgs-doublets (one or more) develop VEVs one obtains the electromagnetic charge generator $T_Q = \frac{1}{2}T_Y - T_L^3$ and the electric charges

$$\hat{Q}_{em}\left[\begin{pmatrix} \boldsymbol{H}_u^0 & \boldsymbol{H}_d^- & \boldsymbol{e}_L \\ \boldsymbol{H}_u^+ & \boldsymbol{H}_d^0 & \boldsymbol{\nu}_L \\ \boldsymbol{e}_R^c & \boldsymbol{\nu}_R^c & \boldsymbol{\phi} \end{pmatrix}^i\right] = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^i, \tag{5.6}$$

in agreement with the SM.

We see from the structure of Eq. (5.5) that, while the maximal amount of $SU(2)_L$ Higgs doublets is nine, the minimal low-scale model needs at least two Higgs-doublets, where all entries but the $\star$, $\bullet$ and $\blacklozenge$-type ones in Eq. (5.5) are set to zero. It is also possible to extend each of the (2-9)HDMs by one or two complex singlets. Note also that the low-scale remnant of the family symmetry $U(1)_T$ is non-universal in the space of fermion generations. This means that distinct generations of Higgs bosons couple differently to different families of SM fermions and, since not all those interactions are present at tree-level, the size of the effective Yukawa couplings is generated at different orders in perturbation theory. As a consequence, depending on the choice of light Higgs-doublets in the SM-like EFT, even if sticking to the minimal 2HDM versions, the possibilities for a light Higgs sector yield different scenarios for the mass matrix (5.5).

generated after the T-GUT SSB, $U(1)_B$ will also be softly broken in that particular case by operators like $A_B\left[\widetilde{Q}_{L\,l}^{c\,f}\widetilde{q}_{L\,l'}^{c'}\widetilde{D}_L^{c''\,f'}\right]\varepsilon_{ff'}\varepsilon^{ll'}\varepsilon_{cc'c''}$ with $A_B \ll v$. As such the requirement of exact $U(1)_B$ adopted in this paper simplifies our consideration and does not affect our conclusions on overall consistency of the SHUT model.

Similarly, in the neutral sector, no massless states remain after EWSB. The main difference has to do with the generation of neutrino masses, for which several possibilities arise and will be studied in future work.

5.1.2 Quark sector

In the absence of the accidental $\mathrm{U}(1)_{\mathrm{T}'}$ symmetry, the low-energy limit of the SHUT model also offers good candidates for SM quarks without an emergence of massless states after EWSB. To see this we first note that once ϕ^3 develops a VEV at the second SSB stage shown in Fig. 1, two generations of D-quarks mix and acquire mass terms of the form $m_D D_{\mathrm{L}}^f D_{\mathrm{R}}^{f'} \varepsilon_{ff'}$, with $m_D = \mathcal{O}\big(m_{\mathrm{soft}}\big) \gg M_{\mathrm{EW}}$. Then, at the third breaking stage, the $\widetilde{\nu}_{\mathrm{R}}^1$ and $\widetilde{\phi}^2$ VEVs trigger a mixing between the R-type quarks D_{R}^i and d_{R}^i, which makes it convenient to move to the basis

$$
\begin{aligned}
d_{\mathrm{R}}^1 &\mapsto \mathpzc{d}_{\mathrm{R}}^1 && \big(\tfrac{2}{3}, 6\big) \,, \\
\big(D_{\mathrm{R}}^2,\, D_{\mathrm{R}}^3\big) &\mapsto \big(\mathpzc{d}_{\mathrm{R}}^2,\, \mathcal{D}_{\mathrm{R}}^1\big) && \big(\tfrac{2}{3}, -2\big) \,, \\
\big(D_{\mathrm{R}}^1,\, d_{\mathrm{R}}^2,\, d_{\mathrm{R}}^3\big) &\mapsto \big(\mathpzc{d}_{\mathrm{R}}^3,\, \mathcal{D}_{\mathrm{R}}^2,\, \mathcal{D}_{\mathrm{R}}^3\big) && \big(\tfrac{2}{3}, 2\big) \,,
\end{aligned}
\tag{5.7}
$$

where the fields on the l.h.s. represent the gauge eigenstates while the fields on the r.h.s. correspond to the mass/charge eigenstates whose $\mathrm{U}(1)_{\mathrm{Y}} \times \mathrm{U}(1)_{\mathrm{T}}$ charges are explicitly indicated in the end of each line. The massless states are denoted as $\mathpzc{d}_{\mathrm{R}}^i$.

Defining the components of the $\mathrm{SU}(2)_{\mathrm{L}}$ quark doublets as $Q_{\mathrm{L}}^{1,2} \equiv \big(u_{\mathrm{L}}^{1,2}, d_{\mathrm{L}}^{1,2}\big)^\top$ and $q_{\mathrm{L}} \equiv \big(u_{\mathrm{L}}^3, d_{\mathrm{L}}^3\big)^\top$, we can construct the Lagrangian for the SM quarks as

$$
\mathcal{L}_{\mathrm{quarks}} = \begin{pmatrix} u_{\mathrm{L}}^1 & u_{\mathrm{L}}^2 & u_{\mathrm{L}}^3 \end{pmatrix} \mathcal{M}^{\mathrm{u}} \begin{pmatrix} u_{\mathrm{R}}^1 \\ u_{\mathrm{R}}^2 \\ u_{\mathrm{R}}^3 \end{pmatrix} + \begin{pmatrix} d_{\mathrm{L}}^1 & d_{\mathrm{L}}^2 & d_{\mathrm{L}}^3 \end{pmatrix} \mathcal{M}^{\mathrm{d}} \begin{pmatrix} \mathpzc{d}_{\mathrm{R}}^1 \\ \mathpzc{d}_{\mathrm{R}}^2 \\ \mathpzc{d}_{\mathrm{R}}^3 \end{pmatrix} + \mathrm{h.c.} \tag{5.8}
$$

With the different possibilities found for the Higgs sector, the most generic structure for $\mathcal{M}^{\mathrm{u}}$ and $\mathcal{M}^{\mathrm{d}}$ matrices obey the following patterns:

$$
\mathcal{M}^{\mathrm{u}} \sim \begin{pmatrix} * & \bullet & \bullet \\ \bullet & \star & \star \\ \bullet & \star & \star \end{pmatrix} \tag{5.9}
$$

$$
\mathcal{M}^{\mathrm{d}} \sim \begin{pmatrix} 0 & \bullet & \star \\ \star & * & \bullet \\ \star & * & \bullet \end{pmatrix} \,. \tag{5.10}
$$

As it was observed earlier for the lepton sector, the low-scale limit of the SHUT model requires, at least, two Higgs-doublets, where both $\bullet$-type and $\star$-type ones are present. The electric charges of the Q_L and Q_R components read

$$\hat{Q}_\mathrm{em}\left[\begin{pmatrix} u_\mathrm{L}^x & d_\mathrm{L}^x & D_\mathrm{L}^x\end{pmatrix}^i\right] = \begin{pmatrix}\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3}\end{pmatrix}^i$$
$$\hat{Q}_\mathrm{em}\left[\begin{pmatrix} u_{\mathrm{R}x}^c & d_{\mathrm{R}x}^c & \mathcal{D}_{\mathrm{R}x}^c\end{pmatrix}^{\top\ i}\right] = \begin{pmatrix}-\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}^{\top\ i}$$

in agreement with what is needed to identify the light states with those of the SM.

We note that the Cabbibo mixing in the quark sector emerges at tree level with minimum two light Higgs doublets of up-type in the SM-like EFT. Indeed, consider the particular case of 2HDM with H_u^1 and H_u^2 such that at tree level none of the down-type quarks acquire mass (i.e. $\mathcal{M}^\mathrm{d}=0$) while one of the up-type quarks u remains massless due to

$$\mathcal{M}^\mathrm{u} = \begin{pmatrix} 0 & 0 & v_2 y_\mathrm{u}^{(2)} \\ 0 & 0 & v_1 y_\mathrm{u}^{(1)} \\ v_2 {y'}_\mathrm{u}^{(2)} & v_1 {y'}_\mathrm{u}^{(1)} & 0 \end{pmatrix}\,, \tag{5.11}$$

where $v_{1,2}$ and $y_\mathrm{u}^{(1,2)}$, ${y'}_\mathrm{u}^{(1,2)}$ are the corresponding Higgs VEVs and the Yukawa couplings matched to the high-scale y_4 coupling in Eq. (C.20), respectively. Then, by performing the singular value decomposition it is straightforward to show that the Cabibbo angle θ_C satisfies

$$\sin\theta_\mathrm{C} = \frac{y_\mathrm{u}^{(2)}}{\sqrt{\left(y_\mathrm{u}^{(2)}\right)^2 + \tan^2\beta\left(y_\mathrm{u}^{(1)}\right)^2}}\,, \qquad \tan\beta = \frac{v_1}{v_2}\,. \tag{5.12}$$

Adding an additional d-type Higgs doublet H_d^2 enables us to provide different tree-level masses to two out of three down-type quarks leaving d-quark massless while preserving the Cabibbo mixing as was found earlier in the non-SUSY formulation [36]. Interestingly enough, in the limit of unbroken LR-parity at low energies we recover $y_\mathrm{u}^{(1,2)} = {y'}_\mathrm{u}^{(1,2)}$ implying the degeneracy of charm and top quark masses, as well as strange and bottom quark masses. These degeneracies, however, are stongly violated in the SM indicating a notable breaking of LR-parity in the Yukawa sector. Note, an addition of extra Higgs doublets (and VEVs in them) to such 3HDM EFT does not distort this picture significantly any longer such that the first generation of quarks (u, d) can only acquire a small mass via radiative corrections. The latter are also responsible for an appearance of small mixing parameters in the Cabibbo–Kobayashi–Maskawa matrix.

The only tree-level Yukawa interactions between the would-be SM quarks and Higgs doublets responsible for the EWSB comes from the superpotential term $\boldsymbol{L}\boldsymbol{Q}_\mathrm{L}\boldsymbol{Q}_\mathrm{R}$ in Eq. (3.5). This is a consequence of the gauge trinification and family symmetries of the GUT-scale SHUT theory, and also means that the lepton and light quark Yukawa interactions in the low-energy EFT need to be radiatively generated at lower scales through a matching procedure. Indeed, the T-GUT tree-level Yukawa interactions are responsible for masses of the heavier quarks in the SM, with the details depending on which $\mathrm{SU}(2)_\mathrm{L}$ doublets in $\tilde{L}$

plays the role of Higgs doublets at the EW scale. This goes hand-in-hand with the mass hierarchies observed in the SM, as e.g. the top quark can receive mass through the T-GUT Yukawa interactions while the lighter quarks can receive masses through the effective Yukawa interactions generated at lower energies.

5.2 The role of the family adjoint in the SHUT model

With physics of $\boldsymbol{L}$, $\boldsymbol{Q}_{\mathrm{L,R}}$ and $\boldsymbol{\Delta}_{\mathrm{L,R,C}}$ largely unaffected by $\boldsymbol{\Delta}_{\mathrm{F}}$ in the SHUT model at the GUT scale, one might in principle consider a model without $\boldsymbol{\Delta}_{\mathrm{F}}$. Such a reduced model retains many positive features of the complete SHUT model presented in this paper (stable minima, SSB down to the SM gauge group, massless SM-like fermions until the EWSB, avoiding the hierarchy problem etc.), but differs in one critically important aspect.

Namely, since the family $\mathrm{SU}(3)_{\mathrm{F}}$ symmetry has to be broken by $\langle\tilde{\varphi}\rangle \neq 0$ (rather than via $\langle\tilde{\Delta}^8_{\mathrm{F}}\rangle \neq 0$), such a model would not allow for VEVs in $\phi^{1,2}$ as they become the Goldstone bosons. With one less VEV in the symmetry breaking chain, the model ends up with one extra U(1) at the EW scale also inhibiting some of the Higgsino masses below the EWSB. However, at variance with the complete case studied in the previous section, it would not be sufficient to break only the accidental symmetries in this reduced scenario. In fact, we would have to also explicitly break that additional family U(1) group in order to prevent the emergence of massless charged fermions at low energies. In this context, if we aim at making a solid bridge between this model and its direct completion with the exact E_8 embedding (where $\mathrm{SU}(3)_{\mathrm{F}}$ is a gauge symmetry) we should not explicitly break such would-be-local symmetries. As was already mentioned above, the considered SHUT model closely conforms the limit of nearly global $\mathrm{SU}(3)_{\mathrm{F}}$ with $g_{\mathrm{F}} \ll g_{\mathrm{U}}$ where neither the global Goldstone d.o.f.'s nor the issue of explict breaking of the global (non-accidental) family U(1)'s emerge in the low-energy EFT.

It may happen that in the SSB chain of Fig. 1 the $\widetilde{\nu}^1_{\mathrm{R}}$ and $\widetilde{\phi}^2$ VEVs are generated at well-separated scales, e.g. when $\langle\widetilde{\nu}^1_{\mathrm{R}}\rangle \gg \langle\widetilde{\phi}^2\rangle$. In this case one would need to consider an extra SSB step. The corresponding intermediate EFT symmetry would then have rank $6+2$. In Sect. A.2.1 of Appendix we present the quantum numbers for such a model, where the breaking chain and the corresponding generators are shown in Sect. A.1.1. This is one of the viable possibilities for further considerations.

5.3 Origin and stability of the electroweak scale

In Ref. [56] it was discussed that in a softly broken SUSY theory the soft parameters can be interpreted as a modification of the couplings of the rigid SUSY theory preserving its structure and renormalization properties, i.e. a softly broken theory does not have independent renormalizations. Therefore, if a given vacuum does not break SUSY spontaneously, which is the case of any of the VEVs $\langle\tilde{L}^j\rangle^l{}_r$, large (GUT-scale) radiative corrections to the parameters of the softly broken SUSY theory are absent (as they are renormalised at most logarithmically at the soft scale). Therefore, the soft SUSY breaking parameters and hence the corresponding SSB scales (such as EW and LR symmetry breaking scales) are protected by SUSY and do not acquire GUT-scale radiative corrections.

Note also that the SHUT model does not exhibit the so-called μ-problem. This is due to the fact that the T-GUT symmetry forbids bilinear terms of fundamental superfields in the superpotential. Such bilinear terms are present only for the adjoint superfields whose VEVs set the first scale where the T-GUT symmetry is spontaneously broken. All subsequent breaking steps occur at scales given by the soft parameters and so these scales are much smaller than the GUT scale. As such, we do not need to tune any dimensionfull superpotential parameter that naturally sits at (and defines!) the GUT scale to the EW SSB scale or to any other soft scale in the SHUT theory.

6 Summary

Here, we would like to summarise the basic features of the LRCF-symmetric SHUT theory considered in this paper:

- In variance to the previous GUT-scale formulations based upon gauge trinification, in the considered model all three fermion generations are unified into a single $(\mathbf{27},\mathbf{3})$-plet of $\mathrm{SU}(3)_{\mathrm{F}} \times \mathrm{E}_6$, and no copies of any fundamental E_6 reps are required for its consistent breaking down to the gauge symmetry of the SM. The considered $\mathrm{SU}(3)_{\mathrm{F}} \times \mathrm{E}_6$ symmetry can be naturally embedded into E_8 motivating the addition of $(\mathbf{1},\mathbf{8})$ and $(\mathbf{78},\mathbf{1})$ multiplets corresponding to four SU(3)-octet reps. The gauge couplings are enforced to unify by means of a cyclic permutation symmetry $\mathbb{Z}_3$ acting on the trinification subgroup of the LRCF-symmetry in the same way as in the Glashow's formulation.

- The chiral-adjoint sector $\boldsymbol{\Delta}^a_{\mathrm{F}} = (\mathbf{1},\mathbf{8})$ and $\boldsymbol{\Delta}^a_{\mathrm{L,R,C}} \subset (\mathbf{78},\mathbf{1})$ is necessary for a consistent breaking of the LRCF-symmetry down to the SM gauge symmetry in the softly-broken SUSY formulation of the theory while none of the adjoint fields remain at the EW scale. In our model, the fields developing VEVs at lower energies (the tri-triplets) happens to have the mass terms solely in the soft sector, while the fields whose VEVs spontaneously break the high-scale SHUT LRCF-symmetry (the adjoints) have their GUT-scale mass term in the superpotential. Hence, our model offers a natural solution to the μ-problem.

- The family symmetry, in combination with the first symmetry breaking being triggered at the GUT scale by VEVs in the adjoint (octet) scalars, forbids mass terms in the fundamentail ($\boldsymbol{L}$, $\boldsymbol{Q}_{\mathrm{L}}$, $\boldsymbol{Q}_{\mathrm{R}}$ tri-triplet) sector implying that the SM-like quark and leptons remain massless until EWSB. This enables the SM Higgs sector to consistently unify with the SM lepton sector within a single $\boldsymbol{L}$-supermultiplet in the SHUT theory.

- The Higgs-lepton unification implies a universal Yukawa and quartic coupling in the chiral fundamental sector of the SHUT theory since the only term in the superpotential $\boldsymbol{L}^i \boldsymbol{Q}_{\mathrm{L}}^j \boldsymbol{Q}_{\mathrm{R}}^k \epsilon_{ijk}$ is allowed giving rise to the SM quark Yukawa couplings and scalar quartic interactions at tree-level at low energies. The presence of the family $\mathrm{SU}(3)_{\mathrm{F}}$ has potential of giving rise to the Cabbibo quark mixing at tree-level while radiatively-generated Yukawa interactions in the SM-like EFT have good prospects for explaining the observed mass and mixing hierarchies in the SM fermion sector as was already noticed in the non-SUSY formulation [36].

- The symmetry breaking scales below the GUT scale (including the EW scale) are fully determined by the dynamics of the soft SUSY breaking interactions and thus are naturally protected from the GUT-scale radiative corrections. A particularly relevant multi-stage symmetry breaking scheme in the SHUT theory down to the SM-like gauge effective theory has been shown in Fig. 1. A strong hierarchy between $\langle \tilde{\phi}^2 \rangle$ and $\langle \tilde{\nu}_{\mathrm{R}}^1 \rangle$, as well as possible (e.g. $\mathrm{U}(1)_{\mathrm{T}}$-breaking) VEVs in the SM-singlet sector, could induce additional breaking steps relevant for further considerations of low-energy EFT limits of the SHUT theory.

- The LRCF-symmetric theory contains an accidential $\mathrm{U}(1)_{\mathrm{B}}$ baryon symmetry which forbids the baryon-number violating processes at the T-GUT scale (at least, at the pertubative level). In the current work, for technical simplicity we keep $\mathrm{U}(1)_{\mathrm{B}}$ unbroken leaving potential implications of its soft-breaking below the GUT scale for a future study. Other accidental $\mathrm{U}(1)_{\mathrm{W}}$ and LR-parity symmetries can be (softly) broken in the low-energy EFT ensuring the generation of the Higgsino masses above the EW scale and the breaking of $\mathrm{SU}(2)_{\mathrm{R}}$ and $\mathrm{SU}(2)_{\mathrm{L}}$ symmetries at different energy scales. A possible way to interpret these soft-breaking interactions is by invoking the soft-SUSY breaking sector below the GUT scale (in the sponteneously broken SHUT) leaving no GUT-scale accidental symmetries unbroken in the EFT limit.

Given the above properties, the SHUT model offers interesting new possibilities for deriving the structure and parameters of the SM from the GUT-scale dynamics. This is a very good motivation for investigations of this model, its multi-scale symmetry breaking patterns, loop-level matching and RG flow evolution. Among the first natural steps would be to uncover some of the features of the simplest SM-like low-energy EFTs in a symmetry-based study without invoking the full-fledged radiative analysis of the SHUT theory. The EFT scenarios studied in this work pave the ground for further phenomenological studies of trinification based GUTs and move beyond the most common issues of such theories in the past.

Acknowledgments

The authors would like to thank N.-E. Bomark, A. E. Cárcamo Hernández, C. Herdeiro, S. Kovalenko, W. Porod, J. Rathsman, H. Serodio and F. Staub for insightful discussions in the various stages of this work. J. E. C.-M., J.W. and R.P. thank Prof. C. Herdeiro for support of the project and hospitality during their visits at Aveiro univeristy. J. E. C.-M. was partially supported by the Crafoord Foundation and Lund University. A.M. is funded by FCT grant SFRH/BPD/97126/2013. R.P., A.O. and J.W. were partially supported by the Swedish Research Council, contract numbers 621-2013-428 and 2016-05996. R.P. was also partially supported by CONICYT grant PIA ACT1406. This work is also supported by the CIDMA project UID/MAT/04106/2013.

A Symmetry breaking schemes and charges

In this appendix we provide a comprehensive summary of two possible SSB schemes from the high-scale T-GUT symmetry down to that of the SM.

A.1 The common path

The breaking path from the T-GUT symmetry down to a LR-symmetric effective theory reads

$$\begin{aligned}
&[\mathrm{SU(3)_C \times SU(3)_L \times SU(3)_R}] \ltimes \mathbb{Z}_3^{(\mathrm{LRC})} \times \{\mathrm{SU(3)_F \times U(1)_W \times U(1)_B}\} \\
&\overset{v,v_{\mathrm{F}}}{\rightarrow} \quad \mathrm{SU(3)_C \times [SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R]} \\
&\times \{\mathrm{SU(2)_F \times U(1)_F \times U(1)_W \times U(1)_B}\} \\
&\overset{\langle\widetilde{\phi}^3\rangle}{\rightarrow} \quad \mathrm{SU(3)_C \times [SU(2)_L \times SU(2)_R] \times U(1)_{L+R} \times \{SU(2)_F \times U(1)_S} \\
&\times\ \mathrm{U(1)_{S'} \times U(1)_B}\} \equiv G_{3221\{21\}}\,,
\end{aligned} \tag{A.1}$$

where global symmetries (including the accidental ones) are indicated by $\{\cdots\}$. The generators of the U(1) groups after the T-GUT SSB are

$$\tfrac{2}{\sqrt{3}}T_{\mathrm{L}}^{8}\,, \qquad \tfrac{2}{\sqrt{3}}T_{\mathrm{R}}^{8}\,, \qquad \tfrac{2}{\sqrt{3}}T_{\mathrm{F}}^{8}\,, \qquad T_{\mathrm{W}}\,, \qquad T_{\mathrm{B}}\,, \tag{A.2}$$

whereas after the $\langle\widetilde{\phi}^3\rangle$ VEV we have

$$T_{\mathrm{L+R}} = \tfrac{2}{\sqrt{3}}\left(T_{\mathrm{L}}^{8} + T_{\mathrm{R}}^{8}\right)\,, \quad T_{\mathrm{S}} = \tfrac{2}{\sqrt{3}}\left(T_{\mathrm{L}}^{8} - T_{\mathrm{R}}^{8} - 2T_{\mathrm{F}}^{8}\right)\,, \quad T_{\mathrm{S'}} = \tfrac{2}{\sqrt{3}}\left(T_{\mathrm{L}}^{8} - T_{\mathrm{R}}^{8} + \tfrac{2}{\sqrt{3}}T_{\mathrm{W}}\right)\,. \tag{A.3}$$

with normalization factors conveniently chosen to provide integer charges for the leptons and scalar bosons. Note that, according to the discussion in Sect. 4.1 the LR-parity can be explicitly broken in the soft sector and is, therefore, absent in the effective theory.

A.1.1 Breaking one U(1) generator

It is possible to further break the symmetry and to obtain the gauge group of the SM if a sneutrino scalar, e.g. $\widetilde{\nu}_{\mathrm{R}}^{1}$, develops a VEV. For such a scenario the breaking scheme reads

$$G_{3221\{21\}} \overset{\langle\widetilde{\nu}^1_{\rm R}\rangle}{\to} \quad {\rm SU(3)_C \times SU(2)_L \times U(1)_Y} \times \{{\rm U(1)_V \times U(1)_{V''}} \times {\rm U(1)_{V'} \times U(1)_B}\}\,, \tag{A.4}$$

where the relevant unbroken U(1) generators are given by

$$T_{\rm Y} = -2\left(T^3_{\rm R} + \tfrac{1}{2}T_{\rm L+R}\right)\,, \quad T_{\rm V} = T_{\rm L+R} + 2T^3_{\rm F}\,, \quad T_{\rm V''} = T_{\rm S} + 2T^3_{\rm F}\,, \quad T_{\rm V'} = T_{\rm S'} - 2T^3_{\rm F}\,. \tag{A.5}$$

The low-energy theory is then described by a rank $6+2$ group that contains the SM group and where $+2$ denotes the contribution to the rank from the accidental $\rm U(1)_{T'}$ and $\rm U(1)_B$ symmetries.

A.1.2 Breaking two U(1) generators

It is possible to further reduce the rank of the global symmetry if, in addition to $\widetilde{\nu}^1_{\rm R}$, we also place a VEV in, e.g. $\widetilde{\phi}^2$. In such a case the breaking scheme takes the form

$$G_{3221\{21\}} \overset{\langle\widetilde{\nu}^1_{\rm R}\rangle,\langle\widetilde{\phi}^2\rangle}{\longrightarrow} \quad {\rm SU(3)_C \times SU(2)_L \times U(1)_Y} \times \{{\rm U(1)_T} \times {\rm U(1)_{T'} \times U(1)_B}\}\,, \tag{A.6}$$

where the hypercharge generator is identical to that in Eq. (A.5) while the $\rm U(1)_T$ and $\rm U(1)_{T'}$ ones read

$$T_{\rm T} = 4\left(T^3_{\rm F} - \tfrac{3}{2}T^3_{\rm R} - \tfrac{1}{4}T_{\rm S}\right)\,, \quad T_{\rm T'} = 2\left(T^3_{\rm F} - \tfrac{3}{4}T_{\rm S'} - \tfrac{1}{4}T_{\rm S}\right)\,. \tag{A.7}$$

In this case, the low-energy theory is described by a group of rank 5+2.

A.2 Quantum numbers

In this section we show tables for the representations and charges of the light states after each breaking step. We separate the rank 6+2 and 5+2 EFTs in order to account for a scenario with a sizable hierarchy between the $\langle\widetilde{\phi}^2\rangle$ and $\langle\widetilde{\nu}^1_{\rm R}\rangle$ VEVs. We consider as light states all fields that are decoupled from the T-GUT scale after the first SSB step. Subsequent breaking scales and mass hierarchies are not studied here. We will also consider the fields as defined in the trinification basis throughout.

In what follows, the Higgs bi-doublets are referred to as $H^{1,2,3}$, the singlet Higgs-lepton fields denoted as $\phi^{1,2,3}$, the lepton doublets are cast as $E^{1,2,3}_{\rm L,R}$, while the quark multiplets split up into $\mathcal{Q}^{1,2,3}_{\rm L,R}$ and $D^{1,2,3}_{\rm L,R}$, where $\mathcal{Q}$ are the 3×2 blocks and D the 3×1 blocks. The superscript $1,2,3$ is the generation number. Whenever convenient we will adopt a simplifying notation according to

$$\begin{aligned} H^3 &\to h\,, & \phi^3 &\to \varphi\,, \\ E^3_{\rm L,R} &\to \mathcal{E}_{\rm L,R}\,, & D^3_{\rm L,R} &\to \mathcal{B}_{\rm L,R}\,, \\ \mathcal{Q}^3_{\rm L,R} &\to q_{\rm L,R}\,, & X^{1,2} &\to X^f\,, \end{aligned} \tag{A.8}$$

where f is a family index running over the first two generations with X representing any of such $\rm SU(2)_F$ doublets.

Fermion	Boson	$\mathrm{SU(3)_C}$	$\mathrm{SU(2)_L}$	$\mathrm{SU(2)_R}$	$\{\mathrm{SU(2)_F}\}$	$\mathrm{U(1)_L}$	$\mathrm{U(1)_R}$	$\{\mathrm{U(1)_F}\}$	$\{\mathrm{U(1)_B}\}^{\mathrm{acc}}$	$\{\mathrm{U(1)_W}\}^{\mathrm{acc}}$
φ	$\widetilde{\varphi}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	-2	2	-2	0	1
ϕ^f	$\widetilde{\phi}^f$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	-2	2	1	0	1
$\mathcal{E}_{\mathrm{L}}^{l}$	$\widetilde{\mathcal{E}}_{\mathrm{L}}^{l}$	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{1}$	$\mathbf{1}$	1	2	-2	0	1
$E_{\mathrm{L}}^{f\,l}$	$\widetilde{E}_{\mathrm{L}}^{f\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{1}$	$\mathbf{2}^f$	1	2	1	0	1
$\mathcal{E}_{\mathrm{R}\,r}$	$\widetilde{\mathcal{E}}_{\mathrm{R}\,r}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_r$	$\mathbf{1}$	-2	-1	-2	0	1
$E_{\mathrm{R}\,r}^{f}$	$\widetilde{E}_{\mathrm{R}\,r}^{f}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_r$	$\mathbf{2}^f$	-2	-1	1	0	1
$\widetilde{h}_r^l$	h_r^l	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{2}_r$	$\mathbf{1}$	1	-1	-2	0	1
$\widetilde{H}_r^{f\,l}$	$H_r^{f\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{2}_r$	$\mathbf{2}^f$	1	-1	1	0	1
$q_{\mathrm{L}\,l}^{x}$	$\widetilde{q}_{\mathrm{L}\,l}^{x}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$\mathbf{1}$	$\mathbf{1}$	-1	0	-2	$1/3$	$-1/2$
$\mathcal{Q}_{\mathrm{L}\,l}^{x\,f}$	$\widetilde{\mathcal{Q}}_{\mathrm{L}\,l}^{x\,f}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$\mathbf{1}$	$\mathbf{2}^f$	-1	0	1	$1/3$	$-1/2$
$q_{\mathrm{R}\,x}^{r}$	$\widetilde{q}_{\mathrm{R}\,x}^{r}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{2}^r$	$\mathbf{1}$	0	1	-2	$-1/3$	$-1/2$
$\mathcal{Q}_{\mathrm{R}\,x}^{f\,r}$	$\widetilde{\mathcal{Q}}_{\mathrm{R}\,x}^{f\,r}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{2}^r$	$\mathbf{2}^f$	0	1	1	$-1/3$	$-1/2$
$\mathcal{B}_{\mathrm{L}}^{x}$	$\widetilde{\mathcal{B}}_{\mathrm{L}}^{x}$	$\mathbf{3}^x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	2	0	-2	$1/3$	$-1/2$
$\mathcal{B}_{\mathrm{R}\,x}$	$\widetilde{\mathcal{B}}_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-2	-2	$-1/3$	$-1/2$
$D_{\mathrm{L}}^{x\,f}$	$\widetilde{D}_{\mathrm{L}}^{x\,f}$	$\mathbf{3}^x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	2	0	1	$1/3$	$-1/2$
$D_{\mathrm{R}\,x}^{f}$	$\widetilde{D}_{\mathrm{R}\,x}^{f}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	0	-2	1	$-1/3$	$-1/2$
$\widetilde{g}^a$	$G_{\mathrm{C}}^{\mu\,a}$	$\mathbf{8}^a$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{T}_{\mathrm{L}}^{i}$	$G_{\mathrm{L}}^{\mu\,i}$	$\mathbf{1}$	$\mathbf{3}^i$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{T}_{\mathrm{R}}^{i}$	$G_{\mathrm{R}}^{\mu\,i}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}^i$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{S}_{\mathrm{L,R}}$	$G_{\mathrm{L,R}}^{\mu\,8}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{H}_{\mathrm{F}}^{f}$	$\widetilde{\mathcal{H}}_{\mathrm{F}}^{f}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	0	0	-1	0	0

Table 4. *Field content and quantum numbers of the LR-symmetric EFT after* $\widetilde{\Delta}_{\mathrm{L,R,F}}$ *VEVs in Eq.* (A.1). *Here and below,* $\{\dots\}^{\mathrm{acc}}$ *denote the accidental symmetries.*

The quantum numbers of the light eigenstates after the v and v_{F} VEVs are given in Tab. 4 while those of the model after $\widetilde{\phi}^3$ VEV are shown in Tab. 5.

Fermion	Boson	$\mathrm{SU(3)_C}$	$\mathrm{SU(2)_L}$	$\mathrm{SU(2)_R}$	$\{\mathrm{SU(2)_F}\}$	$\mathrm{U(1)_{L+R}}$	$\{\mathrm{U(1)_S}\}$	$\{\mathrm{U(1)_{S'}}\}^{\mathrm{acc}}$	$\{\mathrm{U(1)_B}\}^{\mathrm{acc}}$
φ	$\widetilde{\varphi}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
ϕ^f	$\widetilde{\phi}^f$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	0	-2	0	0
$\mathcal{E}^l_{\mathrm{L}}$	$\widetilde{\mathcal{E}}^l_{\mathrm{L}}$	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{1}$	$\mathbf{1}$	1	1	1	0
$E^{f\,l}_{\mathrm{L}}$	$\widetilde{E}^{f\,l}_{\mathrm{L}}$	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{1}$	$\mathbf{2}^f$	1	-1	1	0
$\mathcal{E}_{\mathrm{R}\,r}$	$\widetilde{\mathcal{E}}_{\mathrm{R}\,r}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_r$	$\mathbf{1}$	-1	1	1	0
$E^f_{\mathrm{R}\,r}$	$\widetilde{E}^f_{\mathrm{R}\,r}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_r$	$\mathbf{2}^f$	-1	-1	1	0
$\widetilde{h}^l_r$	h^l_r	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{2}_r$	$\mathbf{1}$	0	2	2	0
$\widetilde{H}^{f\,l}_r$	$H^{f\,l}_r$	$\mathbf{1}$	$\mathbf{2}^l$	$\mathbf{2}_r$	$\mathbf{2}^f$	0	0	2	0
$q^x_{\mathrm{L}\,l}$	$\widetilde{q}^x_{\mathrm{L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$\mathbf{1}$	$\mathbf{1}$	$-1/3$	1	-1	$1/3$
$\mathcal{Q}^{x\,f}_{\mathrm{L}\,l}$	$\widetilde{\mathcal{Q}}^{x\,f}_{\mathrm{L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$\mathbf{1}$	$\mathbf{2}^f$	$-1/3$	-1	-1	$1/3$
$q^r_{\mathrm{R}\,x}$	$\widetilde{q}^r_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{2}^r$	$\mathbf{1}$	$1/3$	1	-1	$-1/3$
$\mathcal{Q}^{f\,r}_{\mathrm{R}\,x}$	$\widetilde{\mathcal{Q}}^{f\,r}_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{2}^r$	$\mathbf{2}^f$	$1/3$	-1	-1	$-1/3$
$\mathcal{B}^x_{\mathrm{L}}$	$\widetilde{\mathcal{B}}^x_{\mathrm{L}}$	$\mathbf{3}^x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$2/3$	2	0	$1/3$
$\mathcal{B}_{\mathrm{R}\,x}$	$\widetilde{\mathcal{B}}_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2/3$	2	0	$-1/3$
$D^{x\,f}_{\mathrm{L}}$	$\widetilde{D}^{x\,f}_{\mathrm{L}}$	$\mathbf{3}^x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	$2/3$	0	0	$1/3$
$D^f_{\mathrm{R}\,x}$	$\widetilde{D}^f_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_f$	$-2/3$	0	0	$-1/3$
$\widetilde{g}^a$	$G^{\mu\,a}_{\mathrm{C}}$	$\mathbf{8}^a$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$\mathcal{T}^i_{\mathrm{L}}$	$G^{\mu\,i}_{\mathrm{L}}$	$\mathbf{1}$	$\mathbf{3}^i$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$\mathcal{T}^i_{\mathrm{R}}$	$G^{\mu\,i}_{\mathrm{R}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}^i$	$\mathbf{1}$	0	0	0	0
$\mathcal{S}_{\mathrm{L,R}}$	$G^{\mu\,8}_{\mathrm{L,R}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$\mathcal{H}^f_{\mathrm{F}}$	$\widetilde{\mathcal{H}}^f_{\mathrm{F}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}^f$	0	-2	0	0

Table 5. *Field content and quantum numbers of the LR-symmetric EFT after $\widetilde{\varphi}$ VEV as in the breaking path Eq. (A.1).*

Note that the $\langle\varphi\rangle$ VEV enables the mixing between the first and second generations of singlet (s)quarks. For example, it allows fermion mass terms of the form $m_D D^f_{\mathrm{L}} D^{f'}_{\mathrm{R}} \varepsilon_{ff'}$.

A.2.1 The rank 6+2 low scale symmetry

In Tabs. 6 and 7 we show the charges of the SM-like EFT after the $\widetilde{\nu}^1_{\mathrm{R}}$ VEV.

Fermion	Boson	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$\{U(1)_V\}$	$\{U(1)_{V''}\}$	$\{U(1)_{V'}\}^{acc}$	$\{U(1)_B\}^{acc}$
φ	$\widetilde{\varphi}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
ϕ^1	$\widetilde{\phi}^1$	$\mathbf{1}$	$\mathbf{1}$	0	1	-1	-1	0
ϕ^2	$\widetilde{\phi}^2$	$\mathbf{1}$	$\mathbf{1}$	0	-1	-3	1	0
$\mathcal{E}_L^l$	$\widetilde{\mathcal{E}}_L^l$	$\mathbf{1}$	$\mathbf{2}^l$	-1	1	1	1	0
$E_L^{1\,l}$	$\widetilde{E}_L^{1\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	2	0	0	0
$E_L^{2\,l}$	$\widetilde{E}_L^{2\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	0	-2	2	0
e_R^3	$\widetilde{e}_R^3$	$\mathbf{1}$	$\mathbf{1}$	2	-1	1	1	0
ν_R^3	$\widetilde{\nu}_R^3$	$\mathbf{1}$	$\mathbf{1}$	0	-1	1	1	0
e_R^1	$\widetilde{e}_R^1$	$\mathbf{1}$	$\mathbf{1}$	2	0	0	0	0
ν_R^1	$\widetilde{\nu}_R^1$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
e_R^2	$\widetilde{e}_R^2$	$\mathbf{1}$	$\mathbf{1}$	2	-2	-2	2	0
ν_R^2	$\widetilde{\nu}_R^2$	$\mathbf{1}$	$\mathbf{1}$	0	-2	-2	2	0
$\widetilde{h}_u^l$	h_u^l	$\mathbf{1}$	$\mathbf{2}^l$	1	0	2	2	0
$\widetilde{h}_d^l$	h_d^l	$\mathbf{1}$	$\mathbf{2}^l$	-1	0	2	2	0
$\widetilde{H}_u^{1\,l}$	$H_u^{1\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	1	1	1	1	0
$\widetilde{H}_d^{1\,l}$	$H_d^{1\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	1	1	1	0
$\widetilde{H}_u^{2\,l}$	$H_u^{2\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	1	-1	-1	3	0
$\widetilde{H}_d^{2\,l}$	$H_d^{2\,l}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	-1	-1	3	0

Table 6. *Field content and quantum numbers for the Higgs scalars and leptons in a SM-like EFT after $\widetilde{\nu}_R^1$ VEV as in the breaking path Eq.* (A.4).

Fermion	Boson	$\mathrm{SU}(3)_\mathrm{C}$	$\mathrm{SU}(2)_\mathrm{L}$	$\mathrm{U}(1)_\mathrm{Y}$	$\{\mathrm{U}(1)_\mathrm{V}\}$	$\{\mathrm{U}(1)_{\mathrm{V}''}\}$	$\{\mathrm{U}(1)_{\mathrm{V}'}\}^\mathrm{acc}$	$\{\mathrm{U}(1)_\mathrm{B}\}^\mathrm{acc}$
$q^x_{\mathrm{L}\,l}$	$\widetilde{q}^x_{\mathrm{L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$1/3$	$-1/3$	1	-1	$1/3$
$\mathcal{Q}^{x\,1}_{\mathrm{L}\,l}$	$\widetilde{\mathcal{Q}}^{x\,1}_{\mathrm{L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$1/3$	$2/3$	0	-2	$1/3$
$\mathcal{Q}^{x\,2}_{\mathrm{L}\,l}$	$\widetilde{\mathcal{Q}}^{x\,2}_{\mathrm{L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	$1/3$	$-4/3$	-2	0	$1/3$
$u^3_{\mathrm{R}\,x}$	$\widetilde{u}^3_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$-4/3$	$1/3$	1	-1	$-1/3$
$d^3_{\mathrm{R}\,x}$	$\widetilde{d}^3_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$2/3$	$1/3$	1	-1	$-1/3$
$u^1_{\mathrm{R}\,x}$	$\widetilde{u}^1_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$-4/3$	$4/3$	0	-2	$-1/3$
$d^1_{\mathrm{R}\,x}$	$\widetilde{d}^1_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$2/3$	$4/3$	0	-2	$-1/3$
$u^2_{\mathrm{R}\,x}$	$\widetilde{u}^2_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$-4/3$	$-2/3$	-2	0	$-1/3$
$d^2_{\mathrm{R}\,x}$	$\widetilde{d}^2_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$2/3$	$-2/3$	-2	0	$-1/3$
$\mathcal{B}^x_\mathrm{L}$	$\widetilde{\mathcal{B}}^x_\mathrm{L}$	$\mathbf{3}^x$	$\mathbf{1}$	$-2/3$	$2/3$	2	0	$1/3$
$\mathcal{B}_{\mathrm{R}\,x}$	$\widetilde{\mathcal{B}}_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$2/3$	$-2/3$	2	0	$-1/3$
$D^{x\,1}_\mathrm{L}$	$\widetilde{D}^{x\,1}_\mathrm{L}$	$\mathbf{3}^x$	$\mathbf{1}$	$-2/3$	$5/3$	1	-1	$1/3$
$D^{x\,2}_\mathrm{L}$	$\widetilde{D}^{x\,2}_\mathrm{L}$	$\mathbf{3}^x$	$\mathbf{1}$	$-2/3$	$-1/3$	-1	1	$1/3$
$D^1_{\mathrm{R}\,x}$	$\widetilde{D}^1_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$2/3$	$1/3$	1	-1	$-1/3$
$D^2_{\mathrm{R}\,x}$	$\widetilde{D}^2_{\mathrm{R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$2/3$	$-5/3$	-1	1	$-1/3$
$\widetilde{g}^a$	$G^{\mu\,a}_\mathrm{C}$	$\mathbf{8}^a$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{T}^i_\mathrm{L}$	$G^{\mu\,i}_\mathrm{L}$	$\mathbf{1}$	$\mathbf{3}^i$	0	0	0	0	0
$\mathcal{T}^\pm_\mathrm{R}$	$G^{\mu\,\pm}_\mathrm{R}$	$\mathbf{1}$	$\mathbf{1}$	± 2	0	0	0	0
$\mathcal{T}^0_\mathrm{R}$	$G^{\mu\,0}_\mathrm{R}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{S}_{\mathrm{L,R}}$	$G^{\mu\,8}_{\mathrm{L,R}}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0
$\mathcal{H}^1_\mathrm{F}$	$\widetilde{\mathcal{H}}^1_\mathrm{F}$	$\mathbf{1}$	$\mathbf{1}$	0	1	-1	-1	0
$\mathcal{H}^2_\mathrm{F}$	$\widetilde{\mathcal{H}}^2_\mathrm{F}$	$\mathbf{1}$	$\mathbf{1}$	0	-1	-3	1	0

Table 7. *Field content and quantum numbers of the quarks, squarks and adjoint fields of a SM-like EFT after $\widetilde{\nu}^1_\mathrm{R}$ VEV as in the breaking path Eq. (A.4).*

A.2.2 The rank 5+2 low scale symmetry

In Tabs. 8 we show the charges of the SM-like EFT after the $\widetilde{\nu}^1_{\rm R}$ and $\widetilde{\phi}^2$ VEVs which may either occur simultaneously or at separate scales.

Fermion	Boson	$\mathrm{SU(3)_C}$	$\mathrm{SU(2)_L}$	$\mathrm{U(1)_Y}$	$\{\mathrm{U(1)_T}\}$	$\{\mathrm{U(1)_{T'}}\}^{\rm acc}$	$\{\mathrm{U(1)_B}\}^{\rm acc}$
ϕ^1	$\widetilde{\phi}^1$	$\mathbf{1}$	$\mathbf{1}$	0	4	2	0
ϕ^2, φ	$\widetilde{\phi}^2$, $\widetilde{\varphi}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$E^{1\,l}_{\rm L}$	$\widetilde{E}^{1\,l}_{\rm L}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	3	0	0
$E^{2\,l}_{\rm L}$, $\mathcal{E}^l_{\rm L}$	$\widetilde{E}^{2\,l}_{\rm L}$, $\widetilde{\mathcal{E}}^l_{\rm L}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	-1	-2	0
$e^1_{\rm R}$	$\widetilde{e}^1_{\rm R}$	$\mathbf{1}$	$\mathbf{1}$	2	6	0	0
$\nu^1_{\rm R}$	$\widetilde{\nu}^1_{\rm R}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$e^{2,3}_{\rm R}$	$\widetilde{e}^{2,3}_{\rm R}$	$\mathbf{1}$	$\mathbf{1}$	2	2	-2	0
$\nu^{2,3}_{\rm R}$	$\widetilde{\nu}^{2,3}_{\rm R}$	$\mathbf{1}$	$\mathbf{1}$	0	-4	-2	0
$\widetilde{H}^{1\,l}_{\rm u}$	$H^{1\,l}_{\rm u}$	$\mathbf{1}$	$\mathbf{2}^l$	1	5	-2	0
$\widetilde{H}^{1\,l}_{\rm d}$	$H^{1\,l}_{\rm d}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	-1	-2	0
$\widetilde{H}^{2\,l}_{\rm u}$, $\widetilde{h}^l_{\rm u}$	$H^{2\,l}_{\rm u}$, $h^l_{\rm u}$	$\mathbf{1}$	$\mathbf{2}^l$	1	1	-4	0
$\widetilde{H}^{2\,l}_{\rm d}$, $\widetilde{h}^l_{\rm d}$	$H^{2\,l}_{\rm d}$, $h^l_{\rm d}$	$\mathbf{1}$	$\mathbf{2}^l$	-1	-5	-4	0
$\mathcal{Q}^{x\,1}_{{\rm L}\,l}$	$\widetilde{\mathcal{Q}}^{x\,1}_{{\rm L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	1/3	3	3	1/3
$\mathcal{Q}^{x\,2}_{{\rm L}\,l}$, $q^x_{{\rm L}\,l}$	$\widetilde{\mathcal{Q}}^{x\,2}_{{\rm L}\,l}$, $\widetilde{q}^x_{{\rm L}\,l}$	$\mathbf{3}^x$	$\mathbf{2}_l$	1/3	-1	1	1/3
$u^1_{{\rm R}\,x}$	$\widetilde{u}^1_{{\rm R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$-4/3$	0	3	$-1/3$
$d^1_{{\rm R}\,x}$	$\widetilde{d}^1_{{\rm R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	2/3	6	3	$-1/3$
$u^{2,3}_{{\rm R}\,x}$	$\widetilde{u}^{2,3}_{{\rm R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	$-4/3$	-4	1	$-1/3$
$d^{2,3}_{{\rm R}\,x}$	$\widetilde{d}^{2,3}_{{\rm R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	2/3	2	1	$-1/3$
$D^{x\,1}_{\rm L}$	$\widetilde{D}^{x\,1}_{\rm L}$	$\mathbf{3}^x$	$\mathbf{1}$	$-2/3$	2	1	1/3
$D^{x\,2}_{\rm L}$, $\mathcal{B}^x_{\rm L}$	$\widetilde{D}^{x\,2}_{\rm L}$, $\widetilde{\mathcal{B}}^x_{\rm L}$	$\mathbf{3}^x$	$\mathbf{1}$	$-2/3$	-2	-1	1/3
$D^1_{{\rm R}\,x}$	$\widetilde{D}^1_{{\rm R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	2/3	2	1	$-1/3$
$D^2_{{\rm R}\,x}$, $\mathcal{B}_{{\rm R}\,x}$	$\widetilde{D}^2_{{\rm R}\,x}$, $\widetilde{\mathcal{B}}_{{\rm R}\,x}$	$\mathbf{3}_x$	$\mathbf{1}$	2/3	-2	-1	$-1/3$
$\widetilde{g}^a$	$G^{\mu\,a}_{\rm C}$	$\mathbf{8}^a$	$\mathbf{1}$	0	0	0	0
$\mathcal{T}^i_{\rm L}$	$G^{\mu\,i}_{\rm L}$	$\mathbf{1}$	$\mathbf{3}^i$	0	0	0	0
$\mathcal{T}^\pm_{\rm R}$	$G^{\mu\,\pm}_{\rm R}$	$\mathbf{1}$	$\mathbf{1}$	±2	0	0	0
$\mathcal{T}^0_{\rm R}$	$G^{\mu\,0}_{\rm R}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$\mathcal{S}_{\rm L,R}$	$G^{\mu\,8}_{\rm L,R}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0
$\mathcal{H}^1_{\rm F}$	$\widetilde{\mathcal{H}}^1_{\rm F}$	$\mathbf{1}$	$\mathbf{1}$	0	4	2	0
$\mathcal{H}^2_{\rm F}$	$\widetilde{\mathcal{H}}^2_{\rm F}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0

Table 8. *Field content and quantum numbers for a SM-like rank 5+2 EFT as in the breaking path* (A.6).

B Particle masses in the high-scale theory

B.1 Scalar spectra and minimisation conditions

The extrema conditions obtained after taking the first derivatives of the scalar potential of the SHUT model can be solved, e.g. w.r.t. the soft parameters $m^2_{\mathbf{78}}$ and $m^2_{\mathbf{1}}$ from where we obtain

$$\begin{aligned} m^2_{\mathbf{78}} &= -\,b_{\mathbf{78}} + \tfrac{v}{12}\left(\sqrt{6}A_{\mathbf{78}} + 3\sqrt{6}C_{\mathbf{78}} - v\lambda^2_{\mathbf{78}}\right) + \tfrac{\sqrt{6}}{4}v\lambda_{\mathbf{78}}\mu_{\mathbf{78}} - \mu^2_{\mathbf{78}}\,, \\ m^2_{\mathbf{1}} &= -\,b_{\mathbf{1}} + \tfrac{v_{\mathrm{F}}}{12}\left(\sqrt{6}A_{\mathbf{1}} - v_{\mathrm{F}}\lambda^2_{\mathbf{1}}\right) + \tfrac{\sqrt{6}}{4}v_{\mathrm{F}}\lambda_{\mathbf{1}}\mu_{\mathbf{1}} - \mu^2_{\mathbf{1}}\,. \end{aligned} \tag{B.1}$$

The minimization equations are then plugged into the Hessian matrix whose eigenvalues corresponding to the fundamental and adjoint scalar sectors are shown in Tabs. 9 and 10, respectively. Note that, for simplicity, we use the LR-symmetric case with $A_{\bar{\mathrm{G}}} = A_{\mathrm{G}}$.

# of real d.o.f.'s	$(mass)^2$	Scalar components
8	$m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}(A_{\mathrm{G}}v + 2A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{\nu}^{(3)}_{\mathrm{R}}, \widetilde{e}^{(3)}_{\mathrm{R}}, \widetilde{\nu}^{(3)}_{\mathrm{L}}, \widetilde{e}^{(3)}_{\mathrm{L}}$
2	$m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}(4A_{\mathrm{G}}v + 2A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{\phi}^{(3)}$
8	$m^2_{\mathbf{27}} + \frac{1}{\sqrt{6}}(2A_{\mathrm{G}}v - 2A_{\mathrm{F}}v_{\mathrm{F}})$	$H^{(3)}_{11}, H^{(3)}_{21}, H^{(3)}_{12}, H^{(3)}_{22}$
4	$m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}(4A_{\mathrm{G}}v - A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{\phi}^{(1,2)}$
16	$m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}(A_{\mathrm{G}}v - A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{\nu}^{(1,2)}_{\mathrm{R}}, \widetilde{e}^{(1,2)}_{\mathrm{R}}, \widetilde{\nu}^{(1,2)}_{\mathrm{L}}, \widetilde{e}^{(1,2)}_{\mathrm{L}}$
16	$m^2_{\mathbf{27}} + \frac{1}{\sqrt{6}}(2A_{\mathrm{G}}v + A_{\mathrm{F}}v_{\mathrm{F}})$	$H^{(1,2)}_{11}, H^{(1,2)}_{21}, H^{(1,2)}_{12}, H^{(1,2)}_{22}$
24	$m^2_{\mathbf{27}} + \frac{1}{\sqrt{6}}(A_{\mathrm{G}}v - 2A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{u}^{(3)}_{\mathrm{L}}, \widetilde{d}^{(3)}_{\mathrm{L}}, \widetilde{u}^{(3)}_{\mathrm{R}}, \widetilde{d}^{(3)}_{\mathrm{R}}$
12	$m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}(2A_{\mathrm{G}}v + 2A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{D}^{(3)}_{\mathrm{L}}, \widetilde{D}^{(3)}_{\mathrm{R}}$
48	$m^2_{\mathbf{27}} + \frac{1}{\sqrt{6}}(A_{\mathrm{G}}v + A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{u}^{(1,2)}_{\mathrm{L}}, \widetilde{d}^{(1,2)}_{\mathrm{L}}, \widetilde{u}^{(1,2)}_{\mathrm{R}}, \widetilde{d}^{(1,2)}_{\mathrm{R}}$
24	$m^2_{\mathbf{27}} - \frac{1}{\sqrt{6}}(2A_{\mathrm{G}}v - A_{\mathrm{F}}v_{\mathrm{F}})$	$\widetilde{D}^{(1,2)}_{\mathrm{L}}, \widetilde{D}^{(1,2)}_{\mathrm{R}}$

Table 9. *Scalar masses squared in the SHUT model for fields in the fundamental (tri-triplet) representation of the* $[\mathrm{SU}(3)]^3 \times \mathrm{SU}(3)_{\mathrm{F}}$ *symmetry.*

The branching rule for a fundamental representation of $\mathrm{SU}(3)_{\mathrm{A}}$, $\mathrm{A} = \mathrm{L}, \mathrm{R}, \mathrm{F}$ when it is broken down to $\mathrm{SU}(2)_{\mathrm{A}} \times \mathrm{U}(1)_{\mathrm{A}}$ reads

$$\mathbf{3} \to \mathbf{2}_1 \oplus \mathbf{1}_{-2}\,, \tag{B.2}$$

where, up to an overall normalization factor, the subscripts represent the $\mathrm{U}(1)_{\mathrm{A}}$ charge. Therefore, after the SSB, the eigenstates shown in Tab. 9 form representations of $G_{32211\{21\}}$ symmetry given in Eq. (3.16) and transform as singlets, doublets, bi-doublets and tri-doublets under the $\mathrm{SU}(2)_{\mathrm{L,R,F}}$ symmetries, as schematically represented by the blocks in Eq. (3.20)[5]. The LR-parity discussed in Sect. 3.1 yields identical masses for the $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ eigenstates at the trinification SSB scale.

[5]The family $\mathrm{SU}(3)_{\mathrm{F}}$ triplets are also split up into $\mathrm{SU}(2)_{\mathrm{F}}$ doublets, containing the first and second generations, and singlets corresponding to the third generation.

# of real d.o.f.'s	$(mass)^2$	Label
12	0	$\widetilde{G}_{\rm L,R,F}$
3	$\sqrt{\frac{3}{2}}\frac{v_{\rm F}}{2}\left(3\lambda_{\mathbf{1}}\mu_{\mathbf{1}}+A_{\mathbf{1}}\right)$	$\widetilde{\mathcal{T}}_{\rm F}$
1	$\frac{v_{\rm F}}{12}\left(2v_{\rm F}\lambda_{\mathbf{1}}^2-3\sqrt{6}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}-\sqrt{6}A_{\mathbf{1}}\right)$	$\widetilde{\mathcal{S}}_{\rm F}$
1	$-2b_{\mathbf{1}}+\frac{v_{\rm F}}{12}\left(\sqrt{6}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}+3\sqrt{6}A_{\mathbf{1}}\right)$	$\widetilde{\mathcal{S}}'_F$
4	$-2b_{\mathbf{1}}+\frac{v_{\rm F}}{12}\left(2\sqrt{6}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}-v_{\rm F}\lambda_{\mathbf{1}}^2+2\sqrt{6}A_{\mathbf{1}}\right)$	$\mathcal{H}_{\rm F}$
3	$-2b_{\mathbf{1}}+\frac{v_{\rm F}}{12}\left(5\sqrt{6}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}+2v_{\rm F}\lambda_{\mathbf{1}}^2-\sqrt{6}A_{\mathbf{1}}\right)$	$\widetilde{\mathcal{T}}'_{\rm F}$
6	$\sqrt{\frac{3}{2}}\frac{v}{2}\left(3\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+A_{\mathbf{78}}+3C_{\mathbf{78}}\right)$	$\widetilde{\mathcal{T}}_{\rm L,R}$
8	$\frac{v}{12}\left(-v\lambda_{\mathbf{78}}^2+3\sqrt{6}\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+\sqrt{6}A_{\mathbf{78}}+3\sqrt{6}C_{\mathbf{78}}\right)$	${\rm Re}[\widetilde{\Delta}_{\rm C}^{1,\cdots,8}]$
2	$\frac{v}{12}\left(2v\lambda_{\mathbf{78}}^2-3\sqrt{6}\lambda_{\mathbf{78}}\mu_{\mathbf{78}}-\sqrt{6}A_{\mathbf{78}}-3\sqrt{6}C_{\mathbf{78}}\right)$	$\widetilde{\mathcal{S}}_{\rm L,R}$
2	$-2b_{\mathbf{78}}+\frac{\sqrt{6}}{12}v\left(\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+3A_{\mathbf{78}}+C_{\mathbf{78}}\right)$	$\widetilde{\mathcal{S}}'_{\rm L,R}$
8	$-2b_{\mathbf{78}}+\frac{3}{4}g_{\rm U}^2v^2+\frac{v^2}{12}\lambda_{\mathbf{78}}^2+\frac{\sqrt{6}}{6}v\left(\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+A_{\mathbf{78}}+C_{\mathbf{78}}\right)$	$\mathcal{H}_{\rm L,R}$
8	$-2b_{\mathbf{78}}-\frac{v^2}{12}\lambda_{\mathbf{78}}^2+\frac{\sqrt{6}}{12}v\left(3\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+A_{\mathbf{78}}+3C_{\mathbf{78}}\right)$	${\rm Im}[\widetilde{\Delta}_{\rm C}^{1,\cdots,8}]$
6	$-2b_{\mathbf{78}}+\frac{v^2}{6}\lambda_{\mathbf{78}}^2+\frac{\sqrt{6}}{12}v\left(5\lambda_{\mathbf{78}}\mu_{\mathbf{78}}-A_{\mathbf{78}}+5C_{\mathbf{78}}\right)$	$\widetilde{\mathcal{T}}'_{\rm L,R}$

Table 10. *Scalar masses squared in the SHUT model for fields in the adjoint representations of the* $\mathrm{SU}(3)_{\rm L,R,C,F}$ *symmetries.*

The adjoint scalars $\widetilde{\Delta}^a_{\rm L,R,F}$ are complex octets whose branching rule is given by

$$\mathbf{8}\to\mathbf{3}_0\oplus\mathbf{2}_1\oplus\mathbf{2}_{-1}\oplus\mathbf{1}_0\,. \tag{B.3}$$

After the SSB this provides two real triplets, two real singlets and two complex doublets. Each broken symmetry provides four Goldstone degrees of freedom out of which eight correspond to breaking of the local symmetries whereas four of them – to the global ones. While the triplet mass eigenstates, $\mathbf{3}_0$, can be written as

$$\widetilde{\mathcal{T}}_{\rm A}\equiv\frac{1}{\sqrt{2}}\begin{pmatrix}{\rm Re}[\widetilde{\Delta}^2_{\rm A}]+{\rm iRe}[\widetilde{\Delta}^1_{\rm A}]\\ \sqrt{2}{\rm Re}[\widetilde{\Delta}^3_{\rm A}]\\ {\rm Re}[\widetilde{\Delta}^2_{\rm A}]-{\rm iRe}[\widetilde{\Delta}^1_{\rm A}]\end{pmatrix}\,,\;\widetilde{\mathcal{T}}'_{\rm A}\equiv\frac{1}{\sqrt{2}}\begin{pmatrix}{\rm Im}[\widetilde{\Delta}^2_{\rm A}]+{\rm iIm}[\widetilde{\Delta}^1_{\rm A}]\\ \sqrt{2}{\rm Im}[\widetilde{\Delta}^3_{\rm A}]\\ {\rm Im}[\widetilde{\Delta}^2_{\rm A}]-{\rm iIm}[\widetilde{\Delta}^1_{\rm A}]\end{pmatrix}\,, \tag{B.4}$$

the two real singlets $\mathbf{1}_0$ read

$$\widetilde{\mathcal{S}}_{\rm A}\equiv{\rm Re}[\widetilde{\Delta}^8_{\rm A}]\,,\;\widetilde{\mathcal{S}}'_{\rm A}\equiv{\rm Im}[\widetilde{\Delta}^8_{\rm A}]\,. \tag{B.5}$$

Finally, there are two independent doublets transforming as $\mathbf{2}_1$ and $\mathbf{2}_{-1}$, which are given by

$$\widetilde{G}_{\rm A}\equiv\frac{1}{\sqrt{2}}\begin{pmatrix}{\rm Re}[\widetilde{\Delta}^5_{\rm A}]+{\rm iRe}[\widetilde{\Delta}^4_{\rm A}]\\ {\rm Re}[\widetilde{\Delta}^7_{\rm A}]+{\rm iRe}[\widetilde{\Delta}^6_{\rm A}]\end{pmatrix}\,,\;\mathcal{H}_{\rm A}\equiv\frac{1}{\sqrt{2}}\begin{pmatrix}{\rm Im}[\widetilde{\Delta}^6_{\rm A}]+{\rm iIm}[\widetilde{\Delta}^7_{\rm A}]\\ {\rm Im}[\widetilde{\Delta}^4_{\rm A}]+{\rm iIm}[\widetilde{\Delta}^5_{\rm A}]\end{pmatrix}\,, \tag{B.6}$$

respectively. Equivalently, we could have defined the Goldstones $\widetilde{G}_{\rm A}$ to transform as $\mathbf{2}_{-1}$ and the $\mathcal{H}_{\rm A}$ states as $\mathbf{2}_1$ but with $\mathrm{SU}(2)_{\rm A}$ isospin components flipped (up to a global phase).

B.1.1 Scalar mass spectrum

It is possible to derive the exact analytical minimisation conditions recasting the scalar masses in a convenient way. In particular, the fundamental scalar masses can be collectively written as

$$m^2_{\widetilde{\varphi}_i} = m^2_{\mathbf{27}} + c^i_1 A_{\rm G} v + c^i_2 A_{\rm F} v_{\rm F} \,, \tag{B.7}$$

where $c^i_{1,2}$ are constants with index i running over all fundamental scalar eigenstates. For simplicity, the soft SUSY breaking parameters and the family breaking VEV can be redefined in terms of v as follows

$$v_{\rm F} = \beta v \,, \;\; m^2_{\mathbf{27}} = \alpha_{\mathbf{27}} v^2 \,, \;\; A_{\rm G} = \sigma_{\rm G} v \,, \;\; A_{\rm F} = \sigma_{\rm F} v \,, \tag{B.8}$$

where, in the limit of low-scale SUSY breaking, $\alpha_{\mathbf{27}}$, $\sigma_{\rm G}$, $\sigma_{\rm F} \ll 1$ and $\beta \sim \mathcal{O}(1)$ such that both gauge and family SSBs occur simultaneously at the T-GUT scale. Eq. (B.8) allows one to rewrite the scalar masses as

$$m^2_{\widetilde{\varphi}_i} = v^2 \left(\alpha_{\mathbf{27}} + c^i_1 \sigma_{\rm G} + c^i_2 \beta \sigma_{\rm F} \right) \equiv v^2 \omega_{\widetilde{\varphi}_i} \,, \quad \omega_{\widetilde{\varphi}_i} \ll 1 \,, \tag{B.9}$$

such that the fundamental scalar spectrum can be characterized by three independent dimensionless parameters

$$\omega_{\widetilde{H}^{(3)}} \equiv \xi \,, \; \omega_{\widetilde{E}^{(1,2)}_{\rm L,R}} \equiv \delta \,, \; \omega_{\widetilde{H}^{(1,2)}} \equiv \kappa \,, \tag{B.10}$$

from where we can recast the scalar mass terms in the resulting EFT as

$$\begin{aligned}
m^2_{\widetilde{H}^{(3)}} &= v^2 \xi \,, & m^2_{\widetilde{H}^{(1,2)}} &= v^2 \kappa \,, \\
m^2_{\widetilde{E}^{(3)}_{\rm L,R}} &= v^2 \left(\delta + \xi - \kappa \right) \,, & m^2_{\widetilde{E}^{(1,2)}_{\rm L,R}} &= v^2 \delta \,, \\
m^2_{\widetilde{\phi}^{(3)}} &= v^2 \left(2\delta + \xi - 2\kappa \right) \,, & m^2_{\widetilde{\phi}^{(1,2)}} &= v^2 \left(2\delta - \kappa \right) \,, \\
m^2_{\widetilde{Q}^{(3)}_{\rm L,R}} &= \tfrac{1}{3} v^2 \left(\delta + 3\xi - \kappa \right) \,, & m^2_{\widetilde{Q}^{(1,2)}_{\rm L,R}} &= \tfrac{1}{3} v^2 \left(\delta + 2\kappa \right) \,, \\
m^2_{\widetilde{D}^{(3)}_{\rm L,R}} &= \tfrac{1}{3} v^2 \left(4\delta + 3\xi - 4\kappa \right) \,, & m^2_{\widetilde{D}^{(1,2)}_{\rm L,R}} &= \tfrac{1}{3} v^2 \left(4\delta - \kappa \right) \,.
\end{aligned} \tag{B.11}$$

Using Eq. (B.11) the general set of conditions necessary to set the positivity of the fundamental scalar mass spectrum reads

$$\kappa > 0 \wedge \left[\left(\frac{\kappa}{2} \leq \delta \leq \kappa \wedge \xi > -2\delta + 2\kappa \right) \vee \left(\delta > \kappa \wedge \xi > 0 \right) \right] . \tag{B.12}$$

Following the same procedure, we may redefine the parameters of the adjoint sector in terms of the T-GUT SSB scale v as follows

$$\begin{aligned}
b_{\mathbf{1}} &= \tau_{\mathbf{1}} v^2 \,, & & \\
b_{\mathbf{78}} &= \tau_{\mathbf{78}} v^2 \,, & A_{\mathbf{1}} &= \sigma_{\mathbf{1}} v \,, \\
\mu_{\mathbf{1}} &= \alpha_{\mathbf{1}} v \,, & A_{\mathbf{78}} &= \sigma_{\mathbf{78}} v \,, \\
\mu_{\mathbf{78}} &= \alpha_{\mathbf{78}} v \,, & C_{\mathbf{78}} &= \theta_{\mathbf{78}} v \,.
\end{aligned} \tag{B.13}$$

Substituting Eqs. (B.13) in Tab. 10 and, similarly to Eq. (B.9), choosing

$$\omega_{\tilde{\mathcal{T}}_{\mathrm{F}}} \equiv \eta_{\mathrm{F}}\,,\; \omega_{\mathcal{H}_{\mathrm{F}}} \equiv \rho_{\mathrm{F}}\,,\; \omega_{\tilde{\mathcal{T}}'_{\mathrm{F}}} \equiv \eta'_{\mathrm{F}}\,,\; \omega_{\tilde{\mathcal{T}}_{\mathrm{L,R}}} \equiv \eta\,,\; \omega_{\tilde{\mathcal{H}}_{\mathrm{L,R}}} \equiv \rho\,,\; \omega_{\tilde{\Delta}'_{\mathrm{C}}} \equiv \vartheta\,, \tag{B.14}$$

where now $\omega_{\tilde{\varphi}_i \neq \mathcal{H}_{\mathrm{F}}} \sim O(1)$ since only $\mathcal{H}_{\mathrm{F}}$ does not contain large $\mathcal{F}$- and $\mathcal{D}$-term contributions. Solving the system of equations w.r.t $\sigma_{\mathbf{1}}$, $\tau_{\mathbf{1}}$, $\alpha_{\mathbf{1}}$, $\sigma_{\mathbf{78}}$, $\tau_{\mathbf{78}}$, $\alpha_{\mathbf{78}}$ we obtain

$$\begin{aligned}
m^2_{\tilde{\mathcal{T}}_{\mathrm{F}}} &= \eta_{\mathrm{F}} v^2\,, & m^2_{\tilde{\mathcal{T}}_{\mathrm{L,R}}} &= \eta v^2\,,\\
m^2_{\tilde{\mathcal{T}}'_{\mathrm{F}}} &= \eta'_{\mathrm{F}} v^2\,, & m^2_{\tilde{\mathcal{T}}'_{\mathrm{L,R}}} &= \tfrac{1}{4} v^2 \left(\lambda^2_{\mathbf{78}} + 6 g^2_{\mathrm{U}} + 12\vartheta - 8\rho\right)\,,\\
m^2_{\tilde{\mathcal{S}}_{\mathrm{F}}} &= \tfrac{1}{6} v^2 \left(\beta^2 \lambda^2_{\mathbf{1}} - 2\eta_{\mathrm{F}}\right)\,, & m^2_{\tilde{\mathcal{S}}_{\mathrm{L,R}}} &= \tfrac{1}{6} v^2 \left(\lambda^2_{\mathbf{78}} - 2\eta\right)\,,\\
m^2_{\tilde{\mathcal{S}}'_{\mathrm{F}}} &= \tfrac{1}{6} v^2 \left(\beta^2 \lambda^2_{\mathbf{1}} - 2\eta'_{\mathrm{F}} + 8\rho_{\mathrm{F}}\right)\,, & m^2_{\tilde{\mathcal{S}}'_{\mathrm{L,R}}} &= \tfrac{1}{12} v^2 \left(\lambda^2_{\mathbf{78}} - 18 g^2_{\mathrm{U}} - 12\vartheta + 24\rho\right)\,,\\
m^2_{\mathcal{H}_{\mathrm{F}}} &= \rho_{\mathrm{F}} v^2\,, & m^2_{\mathcal{H}_{\mathrm{L,R}}} &= \rho v^2\,,\\
m^2_{\tilde{\Delta}_{\mathrm{C}}} &= \tfrac{1}{12} v^2 \left(4\eta - \lambda^2_{\mathbf{78}}\right)\,, & m^2_{\tilde{\Delta}'_{\mathrm{C}}} &= \vartheta v^2\,.
\end{aligned} \tag{B.15}$$

The scalar field components of the gauge and family adjoint sectors are teated separately. Noting that $\rho_{\mathrm{F}} \ll 1$, the general stability condition for the masses of the family sector read

$$\rho_{\mathrm{F}} \geq 0 \,\wedge\, \left(\eta'_{\mathrm{F}} > 4\rho_{\mathrm{F}} \,\wedge\, x > 2\eta'_{\mathrm{F}} - 8\rho_{\mathrm{F}} \,\wedge\, \eta_{\mathrm{F}} < \frac{x}{2}\right)\,, \tag{B.16}$$

where we have defined $\beta^2 \lambda^2_{\mathbf{1}} \equiv x > 0$. Finally, the positivity conditions for the gauge sector are

$$\eta > 0 \,\wedge\, 2\eta < y < 4\eta \,\wedge\, \vartheta > 0 \,\wedge\, \frac{1}{24}\left(z - y + 12\vartheta\right) < \rho < \frac{1}{8}\left(y + 6z + 12\vartheta\right)\,, \tag{B.17}$$

where we have defined $\lambda^2_{\mathbf{78}} \equiv y > 0$ and $g^2_{\mathrm{U}} \equiv z > 0$. When conditions (B.12), (B.16) and (B.17) are simultaneously satisfied, the tree-level vacuum of the SHUT model is stable.

B.2 Fermion Masses

The masses of the fermions that originate from the gauge-adjoint sector are somewhat more complicated. For the sake of transparency, we use a shortened notation and show the exact expressions for the fermion masses squared in Tab. 11. In particular, we parametrize the octet masses by $X^{\mathbf{8}}_{\mathrm{C}}$, $Y^{\mathbf{8}}_{\mathrm{C}}$ and $Z^{\mathbf{8}}_{\mathrm{C}}$, where the number in the superscript denotes the representation under the symmetry labeled in the subscript. The explicit form of such parameters reads

$$X^{\mathbf{8}}_{\mathrm{C}} = 4M_0^2 + 2M_0'^2 + \mu^2_{\mathbf{78}}\,, \tag{B.18}$$

$$Y^{\mathbf{8}}_{\mathrm{C}} = 4M_0'^2 \left(2M_0 + \mu_{\mathbf{78}}\right)^2\,, \tag{B.19}$$

$$Z^{\mathbf{8}}_{\mathrm{C}} = \left(\mu^2_{\mathbf{78}} - 4M_0^2\right)^2\,. \tag{B.20}$$

# of Weyl spinors	$(mass)^2$	Fermionic components
81	0	$\phi^{(1,2,3)}\,,\widetilde{H}^{(1,2,3)}\,,E^{(1,2,3)}_{\rm L,R}\,,\mathcal{Q}^{(1,2,3)}_{\rm L,R}\,,D^{(1,2,3)}_{\rm L,R}$
1	$\frac{1}{6}\left(v_{\rm F}^2\lambda_{\mathbf{1}}^2-2\sqrt{6}v_{\rm F}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}+6\mu_{\mathbf{1}}^2\right)$	$\Delta^8_{\rm F}\equiv\mathcal{S}_{\rm F}$
3	$\frac{1}{6}\left(v_{\rm F}^2\lambda_{\mathbf{1}}^2+2\sqrt{6}v_{\rm F}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}+6\mu_{\mathbf{1}}^2\right)$	$\Delta^{1,2,3}_{\rm F}\equiv\mathcal{T}_{\rm F}$
4	$\frac{1}{24}\left(v_{\rm F}^2\lambda_{\mathbf{1}}^2-4\sqrt{6}v_{\rm F}\lambda_{\mathbf{1}}\mu_{\mathbf{1}}+24\mu_{\mathbf{1}}^2\right)$	$\Delta^{4,5,6,7}_{\rm F}\equiv\widetilde{\mathcal{H}}_{\rm F}$
8	$\frac{1}{2}\left(X^{\mathbf{8}}_{\rm C}-\sqrt{Y^{\mathbf{8}}_{\rm C}+Z^{\mathbf{8}}_{\rm C}}\right)$	$c_{\theta_{\mathbf{8}}}\widetilde{\lambda}^a_{\rm C}-s_{\theta_{\mathbf{8}}}\Delta^a_{\rm C}\equiv\widetilde{g}^a$
8	$\frac{1}{2}\left(X^{\mathbf{8}}_{\rm C}+\sqrt{Y^{\mathbf{8}}_{\rm C}+Z^{\mathbf{8}}_{\rm C}}\right)$	$s_{\theta_{\mathbf{8}}}\widetilde{\lambda}^a_{\rm C}+c_{\theta_{\mathbf{8}}}\Delta^a_{\rm C}\equiv\widetilde{g}^a_\perp$
2	$\frac{1}{24}\left(X^{\mathbf{1}}_{\rm L,R}-\sqrt{Y^{\mathbf{1}}_{\rm L,R}+Z^{\mathbf{1}}_{\rm L,R}}\right)$	$c_{\theta_{\mathbf{1}}}\widetilde{\lambda}^8_{\rm L,R}-s_{\theta_{\mathbf{1}}}\Delta^8_{\rm L,R}\equiv\mathcal{S}_{\rm L,R}$
2	$\frac{1}{24}\left(X^{\mathbf{1}}_{\rm L,R}+\sqrt{Y^{\mathbf{1}}_{\rm L,R}+Z^{\mathbf{1}}_{\rm L,R}}\right)$	$s_{\theta_{\mathbf{1}}}\widetilde{\lambda}^8_{\rm L,R}+c_{\theta_{\mathbf{1}}}\Delta^8_{\rm L,R}\equiv\mathcal{S}^\perp_{\rm L,R}$
6	$\frac{1}{24}\left(X^{\mathbf{3}}_{\rm L,R}-\sqrt{Y^{\mathbf{3}}_{\rm L,R}+Z^{\mathbf{3}}_{\rm L,R}}\right)$	$c_{\theta_{\mathbf{3}}}\widetilde{\lambda}^{1,2,3}_{\rm L,R}-s_{\theta_{\mathbf{3}}}\Delta^{1,2,3}_{\rm L,R}\equiv\mathcal{T}_{\rm L,R}$
6	$\frac{1}{24}\left(X^{\mathbf{3}}_{\rm L,R}+\sqrt{Y^{\mathbf{3}}_{\rm L,R}+Z^{\mathbf{3}}_{\rm L,R}}\right)$	$s_{\theta_{\mathbf{3}}}\widetilde{\lambda}^{1,2,3}_{\rm L,R}+c_{\theta_{\mathbf{3}}}\Delta^{1,2,3}_{\rm L,R}\equiv\mathcal{T}^\perp_{\rm L,R}$
8	$\frac{1}{48}\left(X^{\mathbf{2}}_{\rm L,R}-\sqrt{Y^{\mathbf{2}}_{\rm L,R}+Z^{\mathbf{2}}_{\rm L,R}}\right)$	$\varrho_1\Delta^{4,6}_{\rm L,R}+\varrho_2\Delta^{5,7}_{\rm L,R}+\varrho_3\widetilde{\lambda}^{4,6}_{\rm L,R}+\varrho_4\widetilde{\lambda}^{5,7}_{\rm L,R}\equiv\widetilde{\mathcal{H}}^{1,2}_{\rm L,R}$
8	$\frac{1}{48}\left(X^{\mathbf{2}}_{\rm L,R}+\sqrt{Y^{\mathbf{2}}_{\rm L,R}+Z^{\mathbf{2}}_{\rm L,R}}\right)$	$\overline{\varrho}_1\Delta^{4,6}_{\rm L,R}+\overline{\varrho}_2\Delta^{5,7}_{\rm L,R}+\overline{\varrho}_3\widetilde{\lambda}^{4,6}_{\rm L,R}+\overline{\varrho}_4\widetilde{\lambda}^{5,7}_{\rm L,R}\equiv\widetilde{\mathcal{H}}^{1,2\perp}_{\rm L,R}$

Table 11. *Fermion masses squared and left singular eigenvectors in the SHUT model. The $c_{\theta_{\boldsymbol{R}}}$ and $s_{\theta_{\boldsymbol{R}}}$ coefficients denote the cosine and sine of the 2×2 mixing angles for the representation $\boldsymbol{R}$. Here, $\varrho_{1,2,3,4}$ and $\overline{\varrho}_{1,2,3,4}$ are coefficients that parametrize a unitary mixing. The fermion masses, for a given irrep $\boldsymbol{R}$ and gauge group* L, R, C, *are determined in terms of the $X^{\boldsymbol{R}}_{\rm A}$, $Y^{\boldsymbol{R}}_{\rm A}$ and $Z^{\boldsymbol{R}}_{\rm A}$ coefficients, with explicit expressions given in Eqs.* (B.18)-(B.26).

The singlet and triplet fermion masses depend on the $X^{\mathbf{1,3}}_{\rm L,R}$, $Y^{\mathbf{1,3}}_{\rm L,R}$ and $Z^{\mathbf{1,3}}_{\rm L,R}$ parameters which are given by

$$X^{\mathbf{1,3}}_{\rm L,R}=\left[2v^2\lambda_{\mathbf{78}}^2\mp 4\sqrt{6}v\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+12\left(4M_0^2+2M_0'^2+\mu_{\mathbf{78}}^2\right)\right]\,, \tag{B.21}$$

$$Y^{\mathbf{1,3}}_{\rm L,R}=\left[\pm 2\sqrt{6}v\lambda_{\mathbf{78}}\mu_{\mathbf{78}}-v^2\lambda_{\mathbf{78}}^2-6\left(4M_0^2+2M_0'^2+\mu_{\mathbf{78}}^2\right)\right]^2\,, \tag{B.22}$$

$$\begin{aligned}Z^{\mathbf{1,3}}_{\rm L,R}=192&\Big[3M_0'^4\pm 2M_0M_0'^2\left(\sqrt{6}v\lambda_{\mathbf{78}}\mp 6\mu_{\mathbf{78}}\right)\\&+2M_0^2\left(v^2\lambda_{\mathbf{78}}^2\mp 2\sqrt{6}v\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+6\mu_{\mathbf{78}}^2\right)\Big]\,.\end{aligned} \tag{B.23}$$

For the new doublet fermions, the mass eigenstates are written in terms of $X^{\mathbf{2}}_{\rm L,R}$, $Y^{\mathbf{2}}_{\rm L,R}$ and $Z^{\mathbf{2}}_{\rm L,R}$ that read

$$X^{\mathbf{2}}_{\rm L,R}=96M_0^2+48M_0'^2+36v^2g_{\rm U}^2+v^2\lambda_{\mathbf{78}}^2-4\sqrt{6}v\lambda_{\mathbf{78}}\mu_{\mathbf{78}}+24\mu_{\mathbf{78}}^2\,, \tag{B.24}$$

$$Y^{\mathbf{2}}_{\rm L,R}=v^4\lambda_{\mathbf{78}}^4-8\sqrt{6}v^3\lambda_{\mathbf{78}}^3\mu_{\mathbf{78}}+24v^2\lambda_{\mathbf{78}}^2\left(4M_0'^2-8M_0^2+3v^2g_{\rm U}^2+6\mu_{\mathbf{78}}^2\right)\,, \tag{B.25}$$

$$\begin{aligned}Z^{\mathbf{2}}_{\rm L,R}=96&\Big\{6\left[4M_0'^2+(\mu_{\mathbf{78}}-2M_0)^2\right]\left[3v^2g_{\rm U}^2+(\mu_{\mathbf{78}}+2M_0)^2\right]\\&+\sqrt{6}v\lambda_{\mathbf{78}}\left(6v^2g_{\rm U}^2M_0-8M_0M_0'^2+8M_0^2\mu_{\mathbf{78}}-4M_0'^2\mu_{\mathbf{78}}-3v^2g_{\rm U}^2\mu_{\mathbf{78}}-2\mu_{\mathbf{78}}^3\right)\Big\}\,.\end{aligned} \tag{B.26}$$

Note that the doublets $\widetilde{\mathcal{H}}_{\rm A}$, which are the left-handed Weyl fermions defined to transform as $\mathbf{2}_1$, form mass terms of the form $m\widetilde{\mathcal{H}}_{\rm A}\widetilde{\mathcal{H}}'_{\rm A}$ with $\widetilde{\mathcal{H}}'_{\rm A}$ being also the left-handed Weyl fermions transforming as $\mathbf{2}_{-1}$.

B.3 Gauge boson masses

The gauge bosons of the $\mathrm{SU}(3)_\mathrm{C}$ group remain massless and are identified with the SM gluons whereas the massive gauge bosons are generated upon the SSB of the $\mathrm{SU}(3)_\mathrm{L,R}$ symmetries. The covariant derivative of the T-GUT symmetry reads

$$\boldsymbol{D}^\mu = \partial^\mu \mathbb{1}_L \otimes \mathbb{1}_R \otimes \mathbb{1}_C - \mathrm{i} g_\mathrm{U} \sum_{a=1}^{8} \left[G_\mathrm{L}^{\mu a} \boldsymbol{T}_\mathrm{L}^a \otimes \mathbb{1}_R \otimes \mathbb{1}_C + G_\mathrm{R}^{\mu a} \boldsymbol{T}_\mathrm{R}^a \otimes \mathbb{1}_L \otimes \mathbb{1}_C + G_\mathrm{C}^{\mu a} \boldsymbol{T}_\mathrm{C}^a \otimes \mathbb{1}_L \otimes \mathbb{1}_R \right], \quad \text{(B.27)}$$

where $G_\mathrm{L}^{\mu a}$ are the gauge fields of the $\mathrm{SU}(3)_\mathrm{L}$ symmetry which cyclically transform into $G_\mathrm{R}^{\mu a}$ and $G_\mathrm{C}^{\mu a}$ by means of $\mathbb{Z}_3$-permutations. Considering the gauge-breaking VEVs $\left\langle \widetilde{\Delta}_\mathrm{L,R}^c \right\rangle = \delta_8^c\, v$, the relevant kinetic terms that couple the vector and scalar fields evaluated in the vacuum of the theory are given by

$$\left| D^\mu \left\langle \widetilde{\Delta}_\mathrm{L,R}^b \right\rangle \right|^2 = \frac{3}{4} g_\mathrm{U}^2 v^2 \sum_{a=4}^{7} \eta_{\mu\nu} G_\mathrm{L,R}^{\mu a} G_\mathrm{L,R}^{\nu a} \,. \quad \text{(B.28)}$$

Therefore, there are eight massive gauge bosons in the model which transform as complex $\mathbf{2}_1$ representations of $\mathrm{SU}(2)_\mathrm{L,R} \times \mathrm{U}(1)_\mathrm{L,R}$ whose charge eigenstates read

$$\mathcal{G}_\mathrm{L,R}^\mu \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} G_\mathrm{L,R}^{\mu 5} + \mathrm{i} G_\mathrm{L,R}^{\mu 4} \\ G_\mathrm{L,R}^{\mu 7} + \mathrm{i} G_\mathrm{L,R}^{\mu 6} \end{pmatrix}, \quad \text{(B.29)}$$

with mass $m_\mathcal{G}^2 = \frac{3}{4} g_\mathrm{U}^2 v^2$. In addition to the unbroken colour sector, the remaining gauge bosons are also massless at the SHUT SSB scale.

C Full effective Lagrangians

The field content of the EFT is derived from the mass spectrum after the T-GUT symmetry breaking. As a general rule, the light fields, i.e. those with a mass scale much smaller than the GUT scale v, are kept in the EFT spectrum whereas those with masses of the same order of magnitude as v are integrated out.

The light field components and their group transformations under the LR-symmetry obtained after v and v_F VEVs (see Eq. (A.1)) are shown in Tab. 4, where we use the notation given in Eq. (A.8).

C.1 The scalar potential of the LR-symmetric effective model

The scalar potential of the effective LR-symmetric theory generated after the T-GUT breaking can be summarized by

$$V_\mathrm{LR} = V_2 + V_3 + V_4 \,, \quad \text{(C.1)}$$

where V_2, V_3 and V_4 denote the quadratic, cubic and quartic scalar self-interactions, respectively. For simplicity, we will suppress colour indices in $V_{\rm LR}$ and, for all those terms that can be written from LR-parity transformations on the fields, we will show them within square brackets as $\widehat{\mathcal{P}}_{\rm LR}[\cdots]$. Note that here we use this notation for both the cases of invariance or not under LR-parity. For instance, while for the LR-parity symmetric case we should preserve the couplings, for the LR-parity broken case we should also read $m \to \bar{m}$, $A \to \bar{A}$, $\lambda \to \bar{\lambda}$ whenever LR-parity transformation is applied.

We start by writing the scalar mass terms,

$$\begin{aligned} V_2 =& m_H^2 H^{*\,r}_{f\,l} H^{f\,l}_r + m_h^2 h^{*\,r}_l h^l_r + m_\phi^2 \widetilde{\phi}^*_f \widetilde{\phi}^f + m_\varphi^2 \widetilde{\varphi}^* \widetilde{\varphi} + m_\Delta^2 \widetilde{\mathcal{H}}^*_{{\rm F}\,f} \widetilde{\mathcal{H}}^f_{\rm F} \\ &+ \widehat{\mathcal{P}}_{\rm LR} \Big[m_E^2 \widetilde{E}^*_{{\rm L}\,f\,l} \widetilde{E}^{f\,l}_{\rm L} + m_{\mathcal{E}}^2 \widetilde{\mathcal{E}}^*_{{\rm L}\,l} \widetilde{\mathcal{E}}^l_{\rm L} + m_{\mathcal{Q}}^2 \widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,f} \widetilde{\mathcal{Q}}^f_{{\rm L}\,l} \\ &+ m_q^2 \widetilde{q}^{*\,l}_{\rm L} \widetilde{q}_{{\rm L}\,l} + m_D^2 \widetilde{D}^*_{{\rm L}\,f} \widetilde{D}^f_{\rm L} + m_{\mathcal{B}}^2 \widetilde{\mathcal{B}}^*_{\rm L} \widetilde{\mathcal{B}}_{\rm L} \Big] \end{aligned} \tag{C.2}$$

whereas the trilinear interactions are expressed as

$$\begin{aligned} V_3 =& \varepsilon_{ff'} \Big\{ \widehat{\mathcal{P}}_{\rm LR} \Big[A_1 \widetilde{\mathcal{Q}}^{f\,r}_{\rm R} h^l_r \widetilde{\mathcal{Q}}^{f'}_{{\rm L}\,l} + A_2 \widetilde{D}^f_{\rm R} \widetilde{\varphi} \widetilde{D}^{f'}_{\rm L} \Big] + \widehat{\mathcal{P}}_{\rm LR} \Big[A_3 \widetilde{q}^r_{\rm R} H^{f\,l}_r \widetilde{\mathcal{Q}}^{f'}_{{\rm L}\,l} + A_4 \widetilde{\mathcal{B}}_{\rm R} \widetilde{\phi}^{f'} \widetilde{D}^f_{\rm L} \\ &+ A_5 \widetilde{\mathcal{B}}_{\rm R} \widetilde{\mathcal{Q}}^f_{{\rm L}\,l} \widetilde{E}^{f'\,l}_{\rm L} + A_6 \widetilde{D}^f_{\rm R} \widetilde{\mathcal{Q}}^{f'}_{{\rm L}\,l} \widetilde{\mathcal{E}}^l_{\rm L} + A_7 \widetilde{D}^f_{\rm R} \widetilde{q}_{{\rm L}\,l} \widetilde{E}^{f'\,l}_{\rm L} + {\rm c.c.} \Big] \Big\} \end{aligned} \tag{C.3}$$

Due to a large number of possible contractions of four scalar fields in the effective LR-symmetric model, we will employ a condensed notation to express the scalar quartic self-interactions. We describe below the five possible types of terms.

For the first type, which we denote "sc1", we consider terms with *one* reoccurring index, where we define the reoccurring index as an index possessed by all the four fields. For such a combination there are three possible contractions, out of which two of them are linearly independent. In particular, we have

$$\begin{aligned} V_{\rm sc1} \supset& \lambda_{k_1} \widetilde{D}^*_{{\rm L}\,x\,f'} \widetilde{D}^{x\,f'}_{\rm L} H^{*\,r}_{f\,l} H^{f\,l}_r + \lambda_{k_2} \widetilde{D}^*_{{\rm L}\,x\,f'} \widetilde{D}^{x\,f}_{\rm L} H^{*\,r}_{f\,l} H^{f'\,l}_r \\ \equiv& \lambda_{k_1-k_2} \widetilde{D}^*_{{\rm L}\,f'} \widetilde{D}^{f'}_{\rm L} H^{*\,r}_{f\,l} H^{f\,l}_r\,, \end{aligned} \tag{C.4}$$

where colour indices are suppressed in the condensed form.

For terms with *two* reoccurring indices, denoted as "sc2", no matter if they are SU(2) indices or SU(3) indices[6], there are four linearly independent contractions that read

$$\begin{aligned} V_{\rm sc2} \supset& \lambda_{n_1} \widetilde{E}^*_{{\rm L}\,l'\,f'} \widetilde{E}^{l'\,f'}_{\rm L} \widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,x\,f} \widetilde{\mathcal{Q}}^{x\,f}_{{\rm L}\,l} + \lambda_{n_2} \widetilde{E}^*_{{\rm L}\,l'\,f'} \widetilde{E}^{l'\,f}_{\rm L} \widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,x\,f} \widetilde{\mathcal{Q}}^{x\,f'}_{{\rm L}\,l} \\ &+ \lambda_{n_3} \widetilde{E}^*_{{\rm L}\,l'\,f'} \widetilde{E}^{l\,f'}_{\rm L} \widetilde{\mathcal{Q}}^{*\,l'}_{{\rm L}\,x\,f} \widetilde{\mathcal{Q}}^{x\,f}_{{\rm L}\,l} + \lambda_{n_4} \widetilde{E}^*_{{\rm L}\,l'\,f'} \widetilde{E}^{l\,f}_{\rm L} \widetilde{\mathcal{Q}}^{*\,l'}_{{\rm L}\,x\,f} \widetilde{\mathcal{Q}}^{x\,f'}_{{\rm L}\,l} \\ \equiv& \lambda_{n_1-n_4} \widetilde{E}^*_{{\rm L}\,l'\,f} \widetilde{E}^{l'\,f}_{\rm L} \widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,f'} \widetilde{\mathcal{Q}}^{f'}_{{\rm L}\,l}\,. \end{aligned} \tag{C.5}$$

[6] The two types coincide since for SU(2) the three combinations reduce down to two, using that $\varepsilon_{ij}\varepsilon^{kl} = \delta^k_i \delta^l_j - \delta^l_i \delta^k_j$, while for SU(3) there are only two possible contractions to begin with, and no Levi-Civita tensor to impose a reduction.

The third type involves terms with *two* reoccurring indices (either SU(2) or SU(3) indices) but *identical* fields. We denote this case as "sc3" and observe that there are only two linearly independent terms of the form

$$\begin{aligned} V_{\rm sc3} \supset & \lambda_{j_1} \widetilde{D}^*_{{\rm L}\, x'\, f'} \widetilde{D}_{\rm L}^{x'\, f'} \widetilde{D}^*_{{\rm L}\, x\, f} \widetilde{D}_{\rm L}^{x\, f} + \lambda_{j_2} \widetilde{D}^*_{{\rm L}\, x'\, f'} \widetilde{D}_{\rm L}^{x\, f'} \widetilde{D}^*_{{\rm L}\, x\, f} \widetilde{D}_{\rm L}^{x'\, f} \\ \equiv & \lambda_{j_1 - j_2} \widetilde{D}^*_{{\rm L}\, f'} \widetilde{D}_{\rm L}^{f'} \widetilde{D}^*_{{\rm L}\, f} \widetilde{D}_{\rm L}^{f} \,, \end{aligned} \tag{C.6}$$

where colour contractions are once again implicit.

For terms with *three* reoccurring indices and identical fields, labeled as "sc4", there are four linearly independent combinations that we write as

$$\begin{aligned} V_{\rm sc4} \supset & \lambda_{m_1} H^{*\,r'}_{f'\,l'} H^{f'\,l'}_{r'} H^{*\,r}_{f\,l} H^{f\,l}_{r} + \lambda_{m_2} H^{*\,r'}_{f'\,l'} H^{f'\,l'}_{r} H^{*\,r}_{f\,l} H^{f\,l}_{r'} \\ & + \lambda_{m_3} H^{*\,r'}_{f'\,l'} H^{f\,l'}_{r'} H^{*\,r}_{f\,l} H^{f'\,l}_{r} + \lambda_{m_4} H^{*\,r'}_{f'\,l'} H^{f'\,l}_{r'} H^{*\,r}_{f\,l} H^{f\,l'}_{r} \\ \equiv & \lambda_{m_1 - m_4} H^{*\,r'}_{f'\,l'} H^{f'\,l'}_{r'} H^{*\,r}_{f\,l} H^{f\,l}_{r} \end{aligned} \tag{C.7}$$

Note that the case with three reoccurring indices and different fields does not exist and the only case with one reoccurring index and identical fields is the one involving the gauge singlet ϕ^f.

Finally, the fifth type ("sc5") involves terms without reoccurring indices or terms with one reoccurring index but four identical fields such as

$$V_{\rm sc5} \supset \lambda_i h^{*\,r}_l h^l_r \widetilde{\phi}^*_f \widetilde{\phi}^f + \lambda_j \widetilde{\phi}^*_{f'} \widetilde{\phi}^{f'} \widetilde{\phi}^*_f \widetilde{\phi}^f \,. \tag{C.8}$$

Note that, for ease of notation, we assume that combinatorial factors were absorbed by various λ_i and λ_{i-j}.

We will then consider five different scenarios organized according to the type of index contractions as described in detail in Eqs. (C.4), (C.5), (C.6), (C.7) and (C.8):

$$V_4 = V_{\rm sc1} + V_{\rm sc2} + V_{\rm sc3} + V_{\rm sc4} + V_{\rm sc5} \,. \tag{C.9}$$

The first contribution reads

$$
\begin{aligned}
V_{\mathrm{sc1}} =& \lambda_{1-2}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{q}^{*}_{\mathrm{R}\,r}\widetilde{q}^{r}_{\mathrm{R}} + \lambda_{3-4}\widetilde{\mathcal{B}}^{*}_{\mathrm{L}}\widetilde{\mathcal{B}}_{\mathrm{L}}\widetilde{\mathcal{B}}^{*}_{\mathrm{R}}\widetilde{\mathcal{B}}_{\mathrm{R}} + \lambda_{5-6}H^{*\,r}_{f'\,l}H^{f'\,l}_{r}\widetilde{\phi}^{*}_{f}\widetilde{\phi}^{f} \\
&+ \lambda_{7-8}\widetilde{E}^{*}_{\mathrm{L}\,f'\,l}\widetilde{E}^{f'\,l}_{\mathrm{L}}\widetilde{E}^{*\,r}_{\mathrm{R}\,f}\widetilde{E}^{f}_{\mathrm{R}\,r} + \widehat{\mathcal{P}}_{\mathrm{LR}}\Big[\lambda_{9-10}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{\mathcal{Q}}^{*}_{\mathrm{R}\,f\,r}\widetilde{\mathcal{Q}}^{f\,r}_{\mathrm{R}} \\
&+\lambda_{11-12}\widetilde{\mathcal{B}}^{*}_{\mathrm{L}}\widetilde{\mathcal{B}}_{\mathrm{L}}\widetilde{D}^{*}_{\mathrm{R}\,f}\widetilde{D}^{f}_{\mathrm{R}} + \lambda_{13-14}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{D}^{*}_{\mathrm{L}\,f}\widetilde{D}^{f}_{\mathrm{L}} + \lambda_{15-16}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l}\widetilde{\mathcal{B}}^{*}_{\mathrm{L}}\widetilde{\mathcal{B}}_{\mathrm{L}} \\
&+\lambda_{17-18}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{\mathcal{B}}^{*}_{\mathrm{L}}\widetilde{\mathcal{B}}_{\mathrm{L}} + \lambda_{19-20}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{D}^{*}_{\mathrm{R}\,f}\widetilde{D}^{f}_{\mathrm{R}} + \lambda_{21-22}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l}\widetilde{\mathcal{B}}^{*}_{\mathrm{R}}\widetilde{\mathcal{B}}_{\mathrm{R}} \\
&+\lambda_{23-24}\widetilde{\mathcal{Q}}^{*\,l'}_{\mathrm{L}\,f}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l'}h^{*\,r}_{l}h^{l}_{r} + \lambda_{25-26}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{\mathcal{B}}^{*}_{\mathrm{R}}\widetilde{\mathcal{B}}_{\mathrm{R}} + \lambda_{27-28}\widetilde{q}^{*\,l'}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l'}H^{*\,r}_{f\,l}H^{f\,l}_{r} \\
&+\lambda_{29-30}\widetilde{q}^{*\,l'}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l'}h^{*\,r}_{l}h^{l}_{r} + \lambda_{31-32}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{D}^{f'}_{\mathrm{L}}H^{*\,r}_{f\,l}H^{f\,l}_{r} + \lambda_{33-34}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{E}^{*}_{\mathrm{L}\,f\,l'}\widetilde{E}^{f\,l'}_{\mathrm{L}} \\
&+\lambda_{35-36}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l}\widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l'}\widetilde{\mathcal{E}}^{l'}_{\mathrm{L}} + \lambda_{37-38}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l}\widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l'}\widetilde{\mathcal{E}}^{l'}_{\mathrm{L}} + \lambda_{39-40}\widetilde{\mathcal{Q}}^{*\,r}_{\mathrm{R}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{R}\ \,r}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}} \\
&+\lambda_{41-42}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{D}^{f'}_{\mathrm{L}}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{43-44}\widetilde{D}^{*}_{\mathrm{R}\,f'}\widetilde{D}^{f'}_{\mathrm{R}}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{45-46}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{\phi}^{*}_{f}\widetilde{\phi}^{f} \\
&+\lambda_{47-48}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{D}^{f'}_{\mathrm{L}}\widetilde{\phi}^{*}_{f}\widetilde{\phi}^{f} + \lambda_{49-50}h^{*\,r}_{l'}h^{l'}_{r}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{51-52}\widetilde{D}^{*}_{\mathrm{L}\,f}\widetilde{D}^{f}_{\mathrm{L}}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}} \\
&+\lambda_{53-54}H^{*\,r}_{f\,l'}H^{f\,l'}_{r}\widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l}\widetilde{\mathcal{E}}^{l}_{\mathrm{L}} + \lambda_{55-56}h^{*\,r}_{l'}h^{l'}_{r}\widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l}\widetilde{\mathcal{E}}^{l}_{\mathrm{L}} + \lambda_{57-58}\widetilde{\mathcal{B}}^{*}_{\mathrm{L}}\widetilde{\mathcal{B}}_{\mathrm{L}}\widetilde{D}^{*}_{\mathrm{L}\,f}\widetilde{D}^{f}_{\mathrm{L}} \\
&+\lambda_{59-60}\widetilde{E}^{*}_{\mathrm{L}\,f'\,l}\widetilde{E}^{f'\,l}_{\mathrm{L}}\widetilde{\phi}^{*}_{f}\widetilde{\phi}^{f} + \lambda_{61-62}\widetilde{\phi}^{f}H^{f'\,l}_{r}\widetilde{E}^{*}_{\mathrm{L}\,f'\,l}\widetilde{E}^{*\,r}_{\mathrm{R}\,f} + \Big(\lambda_{63-64}h^{*\,r}_{l'}H^{f\,l'}_{r}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \\
&+\lambda_{65-66}h^{*\,r}_{l'}H^{f\,l'}_{r}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f}\widetilde{q}_{\mathrm{L}\,l} + \lambda_{67-68}\widetilde{E}^{*}_{\mathrm{L}\,f\,l'}\widetilde{\mathcal{E}}^{l'}_{\mathrm{L}}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} + \lambda_{69-70}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{E}^{*\,r}_{\mathrm{R}\,f}H^{f\,l}_{r} \\
&+\lambda_{71-72}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{E}^{f\,l}_{\mathrm{L}}\widetilde{\phi}^{*}_{f} + \lambda_{73-74}\widetilde{\mathcal{B}}_{\mathrm{L}}\widetilde{\mathcal{Q}}^{f}_{\mathrm{R}\ \,r}\widetilde{D}^{*}_{\mathrm{L}\,f}\widetilde{q}^{*}_{\mathrm{R}\,r} + \text{c.c.}\Big)\Big] + V^{\mathrm{gen}}_{\mathrm{sc1}}\,,
\end{aligned}
\tag{C.10}
$$

with $V^{\mathrm{gen}}_{\mathrm{sc1}}$ corresponding to the interactions generated only after the matching procedure, i.e. not directly obtained by expansion of the Lagrangian of the original theory, and given by

$$
\begin{aligned}
V^{\mathrm{gen}}_{\mathrm{sc1}} =& \widehat{\mathcal{P}}_{\mathrm{LR}}\Big[\delta_{1-2}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f}\widetilde{\mathcal{H}}^{f}_{\mathrm{F}} + \delta_{3-4}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{D}^{f'}_{\mathrm{L}}\widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f}\widetilde{\mathcal{H}}^{f}_{\mathrm{F}} + \delta_{5-6}\widetilde{E}^{*}_{\mathrm{L}\,f'\,l}\widetilde{E}^{f'\,l}_{\mathrm{L}}\widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f}\widetilde{\mathcal{H}}^{f}_{\mathrm{F}}\Big] \\
&+ \delta_{7-8}H^{*\,r}_{f'\,l}H^{f'\,l}_{r}\widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f}\widetilde{\mathcal{H}}^{f}_{\mathrm{F}}\,.
\end{aligned}
\tag{C.11}
$$

The effective quartic interactions with two reoccurring indices are given by

$$
\begin{aligned}
V_{\mathrm{sc2}} =& \lambda_{75-78}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{\mathcal{Q}}^{*}_{\mathrm{R}\,f\,r}\widetilde{\mathcal{Q}}^{f\,r}_{\mathrm{R}} + \lambda_{79-82}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{D}^{f'}_{\mathrm{L}}\widetilde{D}^{*}_{\mathrm{R}\,f}\widetilde{D}^{f}_{\mathrm{R}} \\
&+ \lambda_{83-86}h^{*\,r'}_{l'}h^{l'}_{r'}H^{*\,r}_{f\,l}H^{f\,l}_{r} + \widehat{\mathcal{P}}_{\mathrm{LR}}\Big[\lambda_{75-78}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{D}^{*}_{\mathrm{L}\,f}\widetilde{D}^{f}_{\mathrm{L}} \\
&+\lambda_{87-90}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{D}^{*}_{\mathrm{R}\,f}\widetilde{D}^{f}_{\mathrm{R}} + \lambda_{91-94}\widetilde{q}^{*\,l'}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l'}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} \\
&+\lambda_{95-98}\widetilde{\mathcal{Q}}^{*\,l'}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l'}H^{*\,r}_{f\,l}H^{f\,l}_{r} + \lambda_{99-102}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l}\widetilde{E}^{*}_{\mathrm{L}\,f\,l'}\widetilde{E}^{f\,l'}_{\mathrm{L}} \\
&+\lambda_{103-106}H^{*\,r}_{f'\,l'}H^{f'\,l'}_{r}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}}\Big]\,.
\end{aligned}
\tag{C.12}
$$

The third contribution, which accounts for identical multiplets and two reoccurring indices, has the form

$$
\begin{aligned}
V_{\mathrm{sc3}} =& \lambda_{107-108}h^{*\,r'}_{l'}h^{l'}_{r'}h^{*\,r}_{l}h^{l}_{r} + \widehat{\mathcal{P}}_{\mathrm{LR}}\Big[\lambda_{101-102}\widetilde{q}^{*\,l'}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l'}\widetilde{q}^{*\,l}_{\mathrm{L}}\widetilde{q}_{\mathrm{L}\,l} \\
&+\lambda_{109-110}\widetilde{D}^{*}_{\mathrm{L}\,f'}\widetilde{D}^{f'}_{\mathrm{L}}\widetilde{D}^{*}_{\mathrm{L}\,f}\widetilde{D}^{f}_{\mathrm{L}} + \lambda_{111-112}\widetilde{E}^{*}_{\mathrm{L}\,f'\,l'}\widetilde{E}^{f'\,l'}_{\mathrm{L}}\widetilde{E}^{*}_{\mathrm{L}\,f\,l}\widetilde{E}^{f\,l}_{\mathrm{L}}\Big]\,,
\end{aligned}
\tag{C.13}
$$

while the forth scenario, where identical fields with three reoccurring indices are considered, reads

$$
V_{\mathrm{sc4}} = \lambda_{113-116}H^{*\,r'}_{f'\,l'}H^{f'\,l'}_{r'}H^{*\,r}_{f\,l}H^{f\,l}_{r} + \widehat{\mathcal{P}}_{\mathrm{LR}}\Big[\lambda_{117-120}\widetilde{\mathcal{Q}}^{*\,l'}_{\mathrm{L}\,f'}\widetilde{\mathcal{Q}}^{f'}_{\mathrm{L}\,l'}\widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f}\widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l}\Big]\,. \tag{C.14}
$$

Finally, for those terms that contain only one independent type of contraction we have

$$
\begin{aligned}
V_{\mathrm{sc5}} =& \lambda_{121} h^{*\,r}_{l} h^{l}_{r} \widetilde{\phi}^{*}_{f} \widetilde{\phi}^{f} + \lambda_{122} H^{*\,r}_{f\,l} H^{f\,l}_{r} \widetilde{\varphi}^{*} \widetilde{\varphi} + \lambda_{123} h^{*\,r}_{l} h^{l}_{r} \widetilde{\varphi}^{*} \widetilde{\varphi} + \lambda_{124} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} \widetilde{\mathcal{E}}_{\mathrm{R}\,r} \\
&+ \lambda_{125} \widetilde{\phi}^{*}_{f'} \widetilde{\phi}^{f'} \widetilde{\phi}^{*}_{f} \widetilde{\phi}^{f} + \lambda_{126} \widetilde{\phi}^{*}_{f} \widetilde{\phi}^{f} \widetilde{\varphi}^{*} \widetilde{\varphi} + \lambda_{127} \widetilde{\varphi}^{*} \widetilde{\varphi} \widetilde{\varphi}^{*} \widetilde{\varphi} \\
&+ \widehat{\mathcal{P}}_{\mathrm{LR}} \Big[\lambda_{128} \widetilde{\varphi} h^{l}_{r} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} + \lambda_{129} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{\mathcal{B}}^{*}_{\mathrm{R}} \widetilde{D}^{f}_{\mathrm{R}} + \lambda_{130} \widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f} \widetilde{q}_{\mathrm{L}\,l} \widetilde{q}^{*}_{\mathrm{R}\,r} \widetilde{\mathcal{Q}}^{f\,r}_{\mathrm{R}} \\
&+ \lambda_{131} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} + \lambda_{132} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} H^{*\,r}_{f\,l} H^{f\,l}_{r} + \lambda_{133} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{D}^{f}_{\mathrm{L}} h^{*\,r}_{l} h^{l}_{r} \\
&+ \lambda_{134} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} h^{*\,r}_{l} h^{l}_{r} + \lambda_{135} \widetilde{q}^{*}_{\mathrm{R}\,r} \widetilde{q}^{r}_{\mathrm{R}} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{136} \widetilde{\mathcal{Q}}^{*}_{\mathrm{R}\,f\,r} \widetilde{\mathcal{Q}}^{f\,r}_{\mathrm{R}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \\
&+ \lambda_{137} \widetilde{q}^{*}_{\mathrm{R}\,r} \widetilde{q}^{r}_{\mathrm{R}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} + \lambda_{138} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{D}^{f}_{\mathrm{L}} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{139} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{D}^{f}_{\mathrm{L}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \\
&+ \lambda_{140} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} + \lambda_{141} \widetilde{\mathcal{B}}^{*}_{\mathrm{R}} \widetilde{\mathcal{B}}_{\mathrm{R}} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{142} \widetilde{D}^{*}_{\mathrm{R}\,f} \widetilde{D}^{f}_{\mathrm{R}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \\
&+ \lambda_{143} \widetilde{\mathcal{B}}^{*}_{\mathrm{R}} \widetilde{\mathcal{B}}_{\mathrm{R}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} + \lambda_{144} \widetilde{q}^{*\,l}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{\phi}^{*}_{f} \widetilde{\phi}^{f} + \lambda_{145} \widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f} \widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} \widetilde{\varphi}^{*} \widetilde{\varphi} \\
&+ \lambda_{146} \widetilde{q}^{*\,l}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{\varphi}^{*} \widetilde{\varphi} + \lambda_{147} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{\phi}^{*}_{f} \widetilde{\phi}^{f} + \lambda_{148} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{D}^{f}_{\mathrm{L}} \widetilde{\varphi}^{*} \widetilde{\varphi} \\
&+ \lambda_{149} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{\varphi}^{*} \widetilde{\varphi} + \lambda_{150} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l'} \widetilde{\mathcal{E}}^{l'}_{\mathrm{L}} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} + \lambda_{151} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l'} \widetilde{\mathcal{E}}^{l'}_{\mathrm{L}} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \\
&+ \lambda_{152} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{E}^{*\,r}_{\mathrm{R}\,f} \widetilde{E}^{f}_{\mathrm{R}\,r} + \lambda_{153} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\phi}^{*}_{f} \widetilde{\phi}^{f} + \lambda_{154} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} \widetilde{\varphi}^{*} \widetilde{\varphi} \\
&+ \lambda_{155} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\varphi}^{*} \widetilde{\varphi} + \Big(\lambda_{156} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\varphi}^{*} \widetilde{\phi}^{f} + \lambda_{157} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{\phi}^{f} h^{l}_{r} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} \\
&+ \lambda_{158} H^{f\,l}_{r} \widetilde{E}^{*\,r}_{\mathrm{R}\,f} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\varphi} + \lambda_{159} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{D}^{f}_{\mathrm{L}} + \lambda_{160} \widetilde{\varphi}^{*} \widetilde{\phi}^{f} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{\mathcal{B}}_{\mathrm{L}} \\
&+ \lambda_{161} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{q}^{*\,l}_{\mathrm{L}} \widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} + \lambda_{162} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{E}^{*\,r}_{\mathrm{R}\,f} H^{f\,l}_{r} + \lambda_{163} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} \widetilde{E}^{*\,r}_{\mathrm{R}\,f} h^{l}_{r} \\
&+ \lambda_{164} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} h^{l}_{r} + \lambda_{165} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} \widetilde{\phi}^{*}_{f} + \lambda_{166} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\phi}^{*}_{f} \\
&+ \lambda_{167} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\varphi}^{*} + \lambda_{168} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{D}^{f}_{\mathrm{L}} \widetilde{E}^{*\,r}_{\mathrm{R}\,f} \widetilde{\mathcal{E}}_{\mathrm{R}\,r} + \lambda_{169} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{q}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} H^{f\,l}_{r} \\
&+ \lambda_{170} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} h^{l}_{r} + \lambda_{171} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{q}_{\mathrm{L}\,l} \widetilde{E}^{f\,l}_{\mathrm{L}} \widetilde{\varphi}^{*} + \lambda_{172} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{\mathcal{Q}}^{f}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\varphi}^{*} + \text{c.c.} \Big) \Big] \\
&+ V^{\mathrm{gen}}_{\mathrm{sc5}} \, .
\end{aligned}
\tag{C.15}
$$

Here, the terms generated after the breaking are

$$
\begin{aligned}
V^{\mathrm{gen}}_{\mathrm{sc5}} =& \lambda_{173} h^{*\,r}_{l} H^{f\,l}_{r} \widetilde{\phi}^{*}_{f} \widetilde{\varphi} + \lambda_{174} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\mathcal{E}}^{*\,r}_{\mathrm{R}} \widetilde{E}^{f}_{\mathrm{R}\,r} + \delta_{9} h^{*\,r}_{l} h^{l}_{r} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} \\
&+ \delta_{10} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f'} \widetilde{\mathcal{H}}^{f'}_{\mathrm{F}} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} + \delta_{11} \widetilde{\varphi}^{*} \widetilde{\varphi} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} + \delta_{12} \widetilde{\phi}^{*}_{f'} \widetilde{\phi}^{f'} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} \\
&+ \widehat{\mathcal{P}}_{\mathrm{LR}} \Big[\lambda_{175} h^{*\,r}_{l} H^{f\,l}_{r} \widetilde{D}^{*}_{\mathrm{L}\,f} \widetilde{\mathcal{B}}_{\mathrm{L}} + \lambda_{176} \widetilde{\varphi}^{*} \widetilde{\phi}^{f} \widetilde{\mathcal{Q}}^{*\,l}_{\mathrm{L}\,f} \widetilde{q}_{\mathrm{L}\,l} + \lambda_{177} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{D}^{f}_{\mathrm{L}} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \\
&+ \lambda_{178} \widetilde{E}^{*}_{\mathrm{L}\,f\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{q}^{*}_{\mathrm{R}\,r} \widetilde{\mathcal{Q}}^{f\,r}_{\mathrm{R}} + \delta_{13} \widetilde{q}^{*\,l}_{\mathrm{L}} \widetilde{q}_{\mathrm{L}\,l} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} + \delta_{14} \widetilde{\mathcal{B}}^{*}_{\mathrm{L}} \widetilde{\mathcal{B}}_{\mathrm{L}} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} \\
&+ \delta_{15} \widetilde{\mathcal{E}}^{*}_{\mathrm{L}\,l} \widetilde{\mathcal{E}}^{l}_{\mathrm{L}} \widetilde{\mathcal{H}}^{*}_{\mathrm{F}\,f} \widetilde{\mathcal{H}}^{f}_{\mathrm{F}} \Big] \, .
\end{aligned}
\tag{C.16}
$$

C.2 The fermion sector of the LR-symmetric EFT

The part of the Lagrangian of the effective LR-symmetric theory that involves purely quadratic fermion interactions as well as the Yukawa terms reads

$$
\mathcal{L}_{\mathrm{fermi}} = \mathcal{L}_{\mathrm{M}} + \mathcal{L}_{\mathrm{Yuk}} \, . \tag{C.17}
$$

For the mass terms we have

$$\begin{aligned}\mathcal{L}_{\rm M} =&\widehat{\mathcal{P}}_{\rm LR}\left[\tfrac{1}{2}m_{\mathcal{S}_{\rm L}}\mathcal{S}_{\rm L}\mathcal{S}_{\rm L}+\tfrac{1}{2}m_{\mathcal{T}_{\rm L}}\mathcal{T}^i_{\rm L}\mathcal{T}^i_{\rm L}+{\rm c.c.}\right]\\ &+\widehat{\mathcal{P}}_{\rm LR}\left[\tfrac{1}{2}m_{\widetilde{g}}\widetilde{g}^a\widetilde{g}^a+m_{\rm LR}\mathcal{S}_{\rm L}\mathcal{S}_{\rm R}+\tfrac{1}{2}m_{\mathcal{H}}\mathcal{H}^*_{{\rm F}f}\mathcal{H}^f_{\rm F}\right]\,,\end{aligned} \tag{C.18}$$

while for the Yukawa ones we write for convenience,

$$\mathcal{L}_{\rm Yuk}=\mathcal{L}_{\rm 3c}+\mathcal{L}_{\rm 2c}+\mathcal{L}_{\rm 1c}+\mathcal{L}_{\mathcal{S}}+\mathcal{L}_{\mathcal{T}}+\mathcal{L}_{\widetilde{\rm g}}\,, \tag{C.19}$$

where the first three terms, which involve only the fields from the fundamental representations of the trinification group, denote three, two and one SU(2) contractions, respectively, whereas the last ones describe the Yukawa interactions of the singlet $\mathcal{S}$, triplet $\mathcal{T}$ and octet $\widetilde{g}^a$ fermions.

The terms with three SU(2) contractions are given by

$$\begin{aligned}\mathcal{L}_{\rm 3c} =&\varepsilon_{ff'}\Big(\widehat{\mathcal{P}}_{\rm LR}\left[{\rm y}_1\mathcal{Q}^{f\,r}_{\rm R}h^l_r\mathcal{Q}^{f'}_{{\rm L}\,l}\right]+\widehat{\mathcal{P}}_{\rm LR}\Big[{\rm y}_2\widetilde{q}^r_{\rm R}\widetilde{H}^{f\,l}_r\mathcal{Q}^{f'}_{{\rm L}\,l}+{\rm y}_3\widetilde{\mathcal{Q}}^{f\,r}_{\rm R}\widetilde{h}^l_r\mathcal{Q}^{f'}_{{\rm L}\,l}\\ &+{\rm y}_4q^r_{\rm R}H^{f\,l}_r\mathcal{Q}^{f'}_{{\rm L}\,l}+{\rm c.c.}\Big]\Big)\,,\end{aligned} \tag{C.20}$$

those with two SU(2) contractions are written as

$$\begin{aligned}\mathcal{L}_{\rm 2c} =&\varepsilon_{ff'}\widehat{\mathcal{P}}_{\rm LR}\Big[{\rm y}_5\widetilde{\mathcal{B}}_{\rm R}\mathcal{Q}^f_{{\rm L}\,l}E^{f'\,l}_{\rm L}+{\rm y}_6\widetilde{D}^f_{\rm R}\mathcal{Q}^{f'}_{{\rm L}\,l}\mathcal{E}^l_{\rm L}+{\rm y}_7\widetilde{D}^f_{\rm R}q_{{\rm L}\,l}E^{f'\,l}_{\rm L}+{\rm y}_8\mathcal{B}_{\rm R}\widetilde{\mathcal{Q}}^f_{{\rm L}\,l}E^{f'\,l}_{\rm L}\\ &+{\rm y}_9D^f_{\rm R}\widetilde{\mathcal{Q}}^{f'}_{{\rm L}\,l}\mathcal{E}^l_{\rm L}+{\rm y}_{10}D^f_{\rm R}\widetilde{q}_{{\rm L}\,l}E^{f'\,l}_{\rm L}+{\rm y}_{11}\mathcal{B}_{\rm R}\mathcal{Q}^f_{{\rm L}\,l}\widetilde{E}^{f'\,l}_{\rm L}+{\rm y}_{12}D^f_{\rm R}\mathcal{Q}^{f'}_{{\rm L}\,l}(\widetilde{\mathcal{E}}_{\rm L})^l\\ &+{\rm y}_{13}D^f_{\rm R}q_{{\rm L}\,l}\widetilde{E}^{f'\,l}_{\rm L}+{\rm c.c.}\Big]\,,\end{aligned} \tag{C.21}$$

and for those with one SU(2) contraction we have

$$\begin{aligned}\mathcal{L}_{\rm 1c} =&\varepsilon_{ff'}\Big(\widehat{\mathcal{P}}_{\rm LR}\left[{\rm y}_{14}D^f_{\rm R}\widetilde{\varphi}D^{f'}_{\rm L}\right]+\widehat{\mathcal{P}}_{\rm LR}\Big[{\rm y}_{15}\widetilde{\mathcal{B}}_{\rm R}\phi^fD^{f'}_{\rm L}+{\rm y}_{16}\widetilde{D}^f_{\rm R}\phi^{f'}\mathcal{B}_{\rm L}\\ &+{\rm y}_{17}\widetilde{D}^f_{\rm R}\varphi D^{f'}_{\rm L}+{\rm y}_{18}\mathcal{B}_{\rm R}\widetilde{\phi}^fD^{f'}_{\rm L}+{\rm c.c.}\Big]\Big)\,.\end{aligned} \tag{C.22}$$

The part of the Lagrangian involving the singlets $\mathcal{S}_{\rm L,R}$ reads

$$\begin{aligned}\mathcal{L}_{\mathcal{S}} =&\widehat{\mathcal{P}}_{\rm LR}\Big[{\rm y}_{19}\widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,f}\mathcal{S}_{\rm L}\mathcal{Q}^f_{{\rm L}\,l}+{\rm y}_{20}\widetilde{q}^{*\,l}_{\rm L}\mathcal{S}_{\rm L}q_{{\rm L}\,l}+{\rm y}_{21}\widetilde{D}^*_{{\rm L}\,f}\mathcal{S}_{\rm L}D^f_{\rm L}+{\rm y}_{22}\widetilde{\mathcal{B}}^*_{\rm L}\mathcal{S}_{\rm L}\mathcal{B}_{\rm L}\\ &+{\rm y}_{23}H^{*\,r}_{f\,l}\mathcal{S}_{\rm L}\widetilde{H}^{f\,l}_r+{\rm y}_{24}h^{*\,r}_l\mathcal{S}_{\rm L}\widetilde{h}^l_r+{\rm y}_{25}\widetilde{E}^*_{{\rm L}\,f\,l}\mathcal{S}_{\rm L}E^{f\,l}_{\rm L}+{\rm y}_{26}\widetilde{\mathcal{E}}^*_{{\rm L}\,l}\mathcal{S}_{\rm L}\mathcal{E}^l_{\rm L}\\ &+{\rm y}_{27}\widetilde{\phi}^*_f\mathcal{S}_{\rm L}\phi^f+{\rm y}_{28}\widetilde{\varphi}^*\mathcal{S}_{\rm L}\varphi+y\mathcal{H}^*_{{\rm F}f}\mathcal{S}_{\rm L}\mathcal{H}^f_{\rm F}+{\rm y}_{29}\widetilde{E}^*_{{\rm L}\,f\,l}\mathcal{S}_{\rm R}E^{f\,l}_{\rm L}+{\rm y}_{30}\widetilde{\mathcal{B}}^*_{\rm L}\mathcal{S}_{\rm R}\mathcal{B}_{\rm L}\\ &+{\rm y}_{31}\widetilde{D}^*_{{\rm L}\,f}\mathcal{S}_{\rm R}D^f_{\rm L}+{\rm y}_{32}\widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,f}\mathcal{S}_{\rm R}\mathcal{Q}^f_{{\rm L}\,l}+{\rm y}_{33}\widetilde{q}^{*\,l}_{\rm L}\mathcal{S}_{\rm R}q_{{\rm L}\,l}+{\rm y}_{34}\widetilde{\mathcal{E}}^*_{{\rm L}\,l}\mathcal{S}_{\rm R}\mathcal{E}^l_{\rm L}+{\rm c.c.}\Big]\,,\end{aligned} \tag{C.23}$$

while those interactions that couple to $\mathcal{T}^i_{\rm L,R}$ read

$$\begin{aligned}\mathcal{L}_{\mathcal{T}} =&\widehat{\mathcal{P}}_{\rm LR}\Big[(\sigma_i)^l_{l'}\Big({\rm y}_{35}\widetilde{\mathcal{Q}}^{*\,l'}_{{\rm L}\,f}\mathcal{T}^i_L\mathcal{Q}^f_{{\rm L}\,l}+{\rm y}_{36}\widetilde{q}^{*\,l'}_{\rm L}\mathcal{T}^i_Lq_{{\rm L}\,l}+{\rm y}_{37}H^{*\,r}_{f\,l}\mathcal{T}^i_L\widetilde{H}^{f\,l'}_r+{\rm y}_{38}h^{*\,r}_l\mathcal{T}^i_L\widetilde{h}^{l'}_r\\ &+{\rm y}_{39}\widetilde{E}^*_{{\rm L}\,f\,l}\mathcal{T}^i_LE^{f\,l'}_{\rm L}+{\rm y}_{40}\widetilde{\mathcal{E}}^*_{{\rm L}\,l}\mathcal{T}^i_L\mathcal{E}^{l'}_{\rm L}+{\rm c.c.}\Big)\Big]\,.\end{aligned} \tag{C.24}$$

Finally, the Yukawa interactions involving gluinos are given by

$$\mathcal{L}_{\widetilde{g}} =\widehat{\mathcal{P}}_{\rm LR}\Big[{\rm y}_{41}\widetilde{\mathcal{Q}}^{*\,l}_{{\rm L}\,f}\boldsymbol{T}^a\widetilde{g}^a\mathcal{Q}^f_{{\rm L}\,l}+{\rm y}_{41}\widetilde{q}^{*\,l}_{\rm L}\boldsymbol{T}^a\widetilde{g}^aq_{{\rm L}\,l}+{\rm y}_{43}\widetilde{D}^*_{{\rm L}\,f}\boldsymbol{T}^a\widetilde{g}^aD^f_{\rm L}+{\rm y}_{44}\widetilde{\mathcal{B}}^*_{\rm L}\boldsymbol{T}^a\widetilde{g}^a\mathcal{B}_{\rm L}+{\rm c.c.}\Big]\,. \tag{C.25}$$

C.3 The gauge sector of the LR-symmetric EFT

In this section, we consider interactions involving the gauge bosons of the effective SHUT-LR model. For ease of reading, we separate those into the gauge-scalar (gs), gauge-fermion (gf) and pure-gauge (pg) interaction types,

$$\mathcal{L}_{\rm gauge} = \mathcal{L}_{\rm gs} + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm pg}\,, \tag{C.26}$$

where Eqs. (C.28) and (C.29) of appendix C.3.1 can be employed to write

$$\begin{aligned}
\mathcal{L}_{\rm gs} =& \left(\boldsymbol{D}_\mu\widetilde{\varphi}\right)^* \left(\boldsymbol{D}^\mu\widetilde{\varphi}\right) + \left(\boldsymbol{D}_\mu\widetilde{\phi}\right)^*_f \left(\boldsymbol{D}^\mu\widetilde{\phi}\right)^f + \left(\boldsymbol{D}_\mu h\right)^{\dagger\,r}_l \left(\boldsymbol{D}^\mu h\right)^l_r + \left(\boldsymbol{D}_\mu H\right)^{\dagger\,r}_{f\,l} \left(\boldsymbol{D}^\mu H\right)^{f\,l}_r \\
&+ \eta_{\mu\nu}\widehat{\mathcal{P}}_{\rm LR}\left[\left(\boldsymbol{D}^\nu\widetilde{\mathcal{E}}_{\rm L}\right)^\dagger_l \left(\boldsymbol{D}^\mu\widetilde{\mathcal{E}}_{\rm L}\right)^l + \left(\boldsymbol{D}^\nu\widetilde{E}_{\rm L}\right)^\dagger_{f\,l} \left(\boldsymbol{D}^\mu\widetilde{E}_{\rm L}\right)^{f\,l} + \left(\boldsymbol{D}^\nu\widetilde{q}_{\rm L}\right)^{\dagger\,l} \left(\boldsymbol{D}^\mu\widetilde{q}_{\rm L}\right)_l \right.\\
&\left.+ \left(\boldsymbol{D}^\nu\widetilde{\mathcal{Q}}_{\rm L}\right)^{\dagger\,l}_f \left(\boldsymbol{D}^\mu\widetilde{\mathcal{Q}}_{\rm L}\right)^f_l + \left(\boldsymbol{D}^\nu\widetilde{\mathcal{B}}_{\rm L}\right)^\dagger \left(\boldsymbol{D}^\mu\widetilde{\mathcal{B}}_{\rm L}\right) + \left(\boldsymbol{D}^\nu\widetilde{D}_{\rm L}\right)^\dagger_f \left(\boldsymbol{D}^\mu\widetilde{D}_{\rm L}\right)^f\right] \\
\mathcal{L}_{\rm gf} =& {\rm i}\varphi^\dagger\overline{\sigma}_\mu\boldsymbol{D}^\mu\varphi + {\rm i}\phi^\dagger_f\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\phi\right)^f + {\rm i}\widetilde{h}^{\dagger\,r}_l\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\widetilde{h}\right)^l_r + {\rm i}\widetilde{H}^{\dagger\,r}_{f\,l}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\widetilde{H}\right)^{f\,l}_r \\
&+ \widehat{\mathcal{P}}_{\rm LR}\left[{\rm i}\mathcal{E}^\dagger_{{\rm L}\,l}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\mathcal{E}_{\rm L}\right)^l + {\rm i}E^\dagger_{{\rm L}\,f\,l}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu E_{\rm L}\right)^{f\,l} + {\rm i}q^{\dagger\,l}_{\rm L}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu q_{\rm L}\right)_l \right.\\
&\left.+{\rm i}\mathcal{Q}^{\dagger\,l}_{{\rm L}\,f}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\mathcal{Q}_{\rm L}\right)^f_l + {\rm i}\mathcal{B}^\dagger_{\rm L}\overline{\sigma}_\mu\boldsymbol{D}^\mu\mathcal{B}_{\rm L} + {\rm i}D^\dagger_{{\rm L}\,f}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu D_{\rm L}\right)^f\right] \\
&+ \sum_{\rm A=L,R}\left[{\rm i}\mathcal{S}^\dagger_A\overline{\sigma}_\mu\partial^\mu\mathcal{S}_A + {\rm i}\mathcal{T}^{i\dagger}_A\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\mathcal{T}_A\right)^i\right] + {\rm i}\widetilde{g}^{a\,\dagger}\overline{\sigma}_\mu\left(\boldsymbol{D}^\mu\widetilde{g}\right)^a + {\rm c.c.} \\
\mathcal{L}_{\rm pg} =& -\frac{1}{4}\left[\sum_{\rm A=L,R}\left(B^{\mu\nu}_{\rm A}B_{{\rm A}\mu\nu} + F^{\mu\nu\,i}_{\rm A}F^i_{{\rm A}\mu\nu}\right) + G^{\mu\nu\,a}G^a_{\mu\nu} + B^{\mu\nu}_{\rm L}B_{{\rm R}\mu\nu}\right].
\end{aligned} \tag{C.27}$$

C.3.1 Covariant derivatives and field strengths

The covariant derivatives of the LR-symmetric effective model can be written in a compact matrix form as follows

$$\begin{aligned}
\boldsymbol{D}^\mu\left(H,h\right) =& \left(\mathbb{1}_L{\otimes}\mathbb{1}_R\partial^\mu - {\rm i}g_{\rm L}A^{\mu\,i}_{\rm L}\boldsymbol{\tau}^i{\otimes}\mathbb{1}_R - {\rm i}g_{\rm R}A^{\mu\,i}_{\rm R}\boldsymbol{\tau}^i{\otimes}\mathbb{1}_L + {\rm i}g'_{\rm L}Y_{\rm L}B^\mu_{\rm L}\mathbb{1}_L{\otimes}\mathbb{1}_R \right.\\
&\left.+ {\rm i}g'_{\rm R}Y_{\rm R}B^\mu_{\rm R}\mathbb{1}_L{\otimes}\mathbb{1}_R\right)\left(H,h\right) \\
\widehat{\mathcal{P}}_{\rm LR}\left[\boldsymbol{D}^\mu\left(E_L,\mathcal{E}_L\right)\right] =& \widehat{\mathcal{P}}_{\rm LR}\left[\left(\mathbb{1}_L\partial^\mu - {\rm i}g_{\rm L}A^{\mu\,i}_{\rm L}\boldsymbol{\tau}^i + {\rm i}g'_{\rm L}Y_{\rm L}B^\mu_{\rm L}\mathbb{1}_L + {\rm i}g'_{\rm R}Y_{\rm R}B^\mu_{\rm R}\mathbb{1}_L\right)\left(E_L,\mathcal{E}_L\right)\right] \\
\boldsymbol{D}^\mu\left(\phi,\varphi\right) =& \left(\partial^\mu + {\rm i}g'_{\rm L}Y_{\rm L}B^\mu_{\rm L} + {\rm i}g'_{\rm R}Y_{\rm R}B^\mu_{\rm R}\right)\left(\phi,\varphi\right) \\
\widehat{\mathcal{P}}_{\rm LR}\left[\boldsymbol{D}^\mu\left(Q_L,q_L\right)\right] =& \widehat{\mathcal{P}}_{\rm LR}\left[\left(\mathbb{1}_C{\otimes}\mathbb{1}_L\partial^\mu - {\rm i}g_{\rm C}G^{\mu\,a}_{\rm C}\boldsymbol{T}^a{\otimes}\mathbb{1}_L - {\rm i}g_{\rm L}A^{\mu\,i}_{\rm L}\boldsymbol{\tau}^i{\otimes}\mathbb{1}_C \right.\right.\\
&\left.\left.+{\rm i}g'_{\rm L}Y_{\rm L}B^\mu_{\rm L}\mathbb{1}_C{\otimes}\mathbb{1}_L\right)\left(Q_L,q_L\right)\right] \\
\widehat{\mathcal{P}}_{\rm LR}\left[\boldsymbol{D}^\mu\left(D_L,\mathcal{B}_L\right)\right] =& \widehat{\mathcal{P}}_{\rm LR}\left[\left(\mathbb{1}_C\partial^\mu - {\rm i}g_{\rm C}G^{\mu\,a}_{\rm C}\boldsymbol{T}^a + {\rm i}g'_{\rm L}Y_{\rm L}B^\mu_{\rm L}\mathbb{1}_C\right)\left(D_L,\mathcal{B}_L\right)\right] \\
\boldsymbol{D}^\mu\,\mathcal{T}_A =& \left(\mathbb{1}^{\rm adj}_{L,R}\partial^\mu - {\rm i}g_{\rm L,R}A^{\mu\,i}_{\rm L,R}\boldsymbol{\tau}^i_{\rm adj}\right)\mathcal{T}_A \\
\boldsymbol{D}^\mu\,\widetilde{g} =& \left(\mathbb{1}^{\rm adj}_C\partial^\mu - {\rm i}g_{\rm C}G^{\mu\,a}_{\rm C}\boldsymbol{T}^a_{\rm adj}\right)\widetilde{g}
\end{aligned} \tag{C.28}$$

where summation is assumed over each pair of repeated indices, $Y_{\rm A}$ is the $\mathrm{U}(1)_{\rm A}$ hypercharge and $\mathbb{1}_A$ and $\mathbb{1}_A^{\rm adj}$ are the identity matrices with the same dimensions of the fundamental and adjoint representations, respectively. The field strength tensors of the $\mathrm{U}(1)_{\rm A}$, $\mathrm{SU}(2)_{\rm A}$ and $\mathrm{SU}(3)_{\rm C}$ gauge symmetries are given by

$$\begin{aligned} B_{\rm A}^{\mu\nu} &= \partial^\mu B_{\rm A}^\nu - \partial^\nu B_{\rm A}^\mu \\ F_{\rm A}^{\mu\nu\, i} &= \partial^\mu A_{\rm A}^{\nu\, i} - \partial^\nu A_{\rm A}^{\mu\, i} + g_{\rm A} \varepsilon^{ijk} A_{\rm A}^{\mu\, j} A_{\rm A}^{\nu\, k} \\ G^{\mu\nu\, a} &= \partial^\mu G_{\rm C}^{\nu\, a} - \partial^\nu G_{\rm C}^{\mu\, a} + g_{\rm C} f^{abc} G_{\rm C}^{\mu\, b} G_{\rm C}^{\nu\, c} \,. \end{aligned} \tag{C.29}$$

C.3.2 Abelian D-terms

The $\mathrm{U}(1)_{\rm L,R}$ D-terms of the LR-symmetric SUSY theory read

$$\begin{aligned} \mathcal{D}_{\rm L} &= \frac{1}{\left(1-\frac{\chi^2}{4}\right)} \left[-\frac{1}{2}\chi \left(X_{\rm R} - \kappa\right) + X_{\rm L} + \kappa \right] \,, \\ \mathcal{D}_{\rm R} &= \frac{1}{\left(1-\frac{\chi^2}{4}\right)} \left[-\frac{1}{2}\chi \left(X_{\rm L} + \kappa\right) + X_{\rm R} - \kappa \right] \,, \\ X_{\rm L} &= H^{*\, r}_{f\, l} H^{l\, f}_{r} - 2\widetilde{\phi}^*_f \widetilde{\phi}^f + \widetilde{E}^*_{{\rm L}\, f\, l} \widetilde{E}^{f\, l}_{\rm L} - 2\widetilde{E}^{*\, r}_{{\rm R}\, f} \widetilde{E}^{f}_{{\rm R}\, r} \\ &\quad - \widetilde{\mathcal{Q}}^{*\, l}_{{\rm L}\, f} \widetilde{\mathcal{Q}}^{f}_{{\rm L}\, l} + 2\widetilde{D}^*_{{\rm L}\, f} \widetilde{D}^f_{\rm L} \,, \\ X_{\rm R} &= -H^{*\, r}_{f\, l} H^{l,f}_{r} + 2\widetilde{\phi}^*_f \widetilde{\phi}^f + 2\widetilde{E}^*_{{\rm L}\, f\, l} \widetilde{E}^{f\, l}_{\rm L} - \widetilde{E}^{*\, r}_{{\rm R}\, f} \widetilde{E}^{f}_{{\rm R}\, r} \\ &\quad + \widetilde{\mathcal{Q}}^*_{{\rm R}\, f\, r} \widetilde{\mathcal{Q}}^{f\, r}_{\rm R} - 2\widetilde{D}^*_{{\rm R}\, f} \widetilde{D}^f_{\rm R} \,, \end{aligned} \tag{C.30}$$

with $f = 1,2,3$.

References

[1] H. Georgi and S. L. Glashow, *Unity of All Elementary Particle Forces*, *Phys. Rev. Lett.* **32** (1974) 438–441.

[2] H. Fritzsch and P. Minkowski, *Unified Interactions of Leptons and Hadrons*, *Annals Phys.* **93** (1975) 193–266.

[3] M. S. Chanowitz, J. R. Ellis, and M. K. Gaillard, *The Price of Natural Flavor Conservation in Neutral Weak Interactions*, *Nucl. Phys.* **B128** (1977) 506–536.

[4] H. Georgi and D. V. Nanopoulos, *T Quark Mass in a Superunified Theory*, *Phys. Lett.* **B82** (1979) 392–394.

[5] H. Georgi and D. V. Nanopoulos, *Ordinary Predictions from Grand Principles: T Quark Mass in O(10)*, *Nucl. Phys.* **B155** (1979) 52–74.

[6] H. Georgi and D. V. Nanopoulos, *Masses and Mixing in Unified Theories*, *Nucl. Phys.* **B159** (1979) 16–28.

[7] H. Georgi, *LIE ALGEBRAS IN PARTICLE PHYSICS. FROM ISOSPIN TO UNIFIED THEORIES*, *Front. Phys.* **54** (1982) 1–255.

[8] F. Gursey, P. Ramond, and P. Sikivie, *A Universal Gauge Theory Model Based on E6*, *Phys. Lett.* **B60** (1976) 177–180.

[9] F. Gursey and M. Serdaroglu, *E6 GAUGE FIELD THEORY MODEL REVISITED*, *Nuovo Cim.* **A65** (1981) 337. [,271(1981)].

[10] Y. Achiman and B. Stech, *Quark Lepton Symmetry and Mass Scales in an E6 Unified Gauge Model*, *Phys. Lett.* **B77** (1978) 389–393.

[11] F. Gursey and P. Sikivie, *E(7) as a Universal Gauge Group*, *Phys. Rev. Lett.* **36** (1976) 775.

[12] S. Glashow *Fifth Workshop on Grand Unification edited by K. Kang, H.Fried, and F.Frampton* (1984).

[13] A. G. Dias, C. A. de S. Pires, and P. S. Rodrigues da Silva, *The Left-Right SU(3)(L)xSU(3)(R)xU(1)(X) Model with Light, keV and Heavy Neutrinos*, *Phys. Rev.* **D82** (2010) 035013, [1003.3260].

[14] M. Reig, J. W. F. Valle, and C. A. Vaquera-Araujo, *Unifying left-right symmetry and 331 electroweak theories*, 1611.02066.

[15] K. S. Babu, X.-G. He, and S. Pakvasa, *Neutrino Masses and Proton Decay Modes in SU(3) X SU(3) X SU(3) Trinification*, *Phys. Rev.* **D33** (1986) 763.

[16] X.-G. He and S. Pakvasa, *Baryon Asymmetry in SU(3)**3 X Z(3) Trinification Model*, *Phys. Lett.* **B173** (1986) 159.

[17] B. R. Greene, K. H. Kirklin, P. J. Miron, and G. G. Ross, *A Three Generation Superstring Model. 2. Symmetry Breaking and the Low-Energy Theory*, *Nucl. Phys.* **B292** (1987) 606–652.

[18] M. Y. Wang and E. D. Carlson, *The Breaking of the SU(3)**3 gauge group*, hep-ph/9302215.

[19] G. Lazarides and C. Panagiotakopoulos, *MSSM from SUSY trinification*, *Phys. Lett.* **B336** (1994) 190–193, [hep-ph/9403317].

[20] G. R. Dvali and Q. Shafi, *Gauge hierarchy in SU(3)(C) x SU(3)(L) x SU(3)-R and low-energy implications*, *Phys. Lett.* **B326** (1994) 258–263, [hep-ph/9401337].

[21] N. Maekawa and Q. Shafi, *Supersymmetric SU(3)**3 unification with anomalous U(1)(a) gauge symmetry*, *Prog. Theor. Phys.* **109** (2003) 279–293, [hep-ph/0204030].

[22] J. E. Kim, *Z(3) orbifold construction of SU(3)**3 GUT with sin**2 (theta0(W)) = 3/8*, *Phys. Lett.* **B564** (2003) 35–41, [hep-th/0301177].

[23] J. E. Kim, *Trinification with sin**2 theta(W) = 3/8 and seesaw neutrino mass*, *Phys. Lett.* **B591** (2004) 119–126, [hep-ph/0403196].

[24] C. D. Carone and J. M. Conroy, *Higgsless GUT breaking and trinification*, *Phys. Rev.* **D70** (2004) 075013, [hep-ph/0407116].

[25] C. D. Carone, *Tri-N-ification*, *Phys. Rev.* **D71** (2005) 075013, [hep-ph/0503069].

[26] J. Sayre, S. Wiesenfeldt, and S. Willenbrock, *Minimal trinification*, *Phys. Rev.* **D73** (2006) 035013, [hep-ph/0601040].

[27] E. Di Napoli, D. George, M. Hertzberg, F. Metzler, and E. Siegel, *Dark Matter In Minimal Trinification*, in *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime Les Houches, France, July 31-August 25, 2006*, 2006. hep-ph/0611012.

[28] C. Cauet, H. Pas, and S. Wiesenfeldt, *Trinification, the Hierarchy Problem and Inverse Seesaw Neutrino Masses*, *Phys. Rev.* **D83** (2011) 093008, [1012.4083].

[29] B. Stech, *The mass of the Higgs boson in the trinification subgroup of E6*, *Phys. Rev.* **D86** (2012) 055003, [1206.4233].

[30] B. Stech, *Trinification Phenomenology and the structure of Higgs Bosons*, *JHEP* **08** (2014) 139, [1403.2714].

[31] J. Hetzel and B. Stech, *Low-energy phenomenology of trinification: an effective left-right-symmetric model*, *Phys. Rev.* **D91** (2015) 055026, [1502.00919].

[32] J. Hetzel, *Phenomenology of a left-right-symmetric model inspired by the trinification model.* PhD thesis, Inst. Appl. Math., Heidelberg, 2015. 1504.06739.

[33] G. M. Pelaggi, A. Strumia, and S. Vignali, *Totally asymptotically free trinification*, *JHEP* **08** (2015) 130, [1507.06848].

[34] G. M. Pelaggi, A. Strumia, and E. Vigiani, *Trinification can explain the di-photon and di-boson LHC anomalies*, *JHEP* **03** (2016) 025, [1512.07225].

[35] O. Rodríguez, R. H. Benavides, W. A. Ponce, and E. Rojas, *Flipped models in Trinification: A Comprehensive Approach*, 1605.00575.

[36] J. E. Camargo-Molina, A. P. Morais, R. Pasechnik, and J. Wessén, *On a radiative origin of the Standard Model from Trinification*, *JHEP* **09** (2016) 129, [1606.03492].

[37] S. Willenbrock, *Triplicated trinification*, *Phys. Lett.* **B561** (2003) 130–134, [hep-ph/0302168].

[38] P. Nath and R. L. Arnowitt, *Symmetry Breaking in Three Generation Calabi-yau Manifolds*, *Phys. Rev.* **D39** (1989) 2006.

[39] J. E. Camargo-Molina, A. P. Morais, A. Ordell, R. Pasechnik, M. O. Sampaio, and J. Wessén, *Reviving trinification models through an E6 -extended supersymmetric GUT*, *Phys. Rev.* **D95** (2017), no. 7 075031, [1610.03642].

[40] T. Blazek, R. Dermisek, and S. Raby, *Predictions for Higgs and supersymmetry spectra from*

SO(10) Yukawa unification with mu greater than 0, *Phys. Rev. Lett.* **88** (2002) 111804, [hep-ph/0107097].

[41] H. Baer and J. Ferrandis, *Supersymmetric SO(10) GUT models with Yukawa unification and a positive mu term*, *Phys. Rev. Lett.* **87** (2001) 211803, [hep-ph/0106352].

[42] A. Anandakrishnan and S. Raby, *Yukawa Unification Predictions with effective "Mirage" Mediation*, *Phys. Rev. Lett.* **111** (2013), no. 21 211801, [1303.5125].

[43] T. Blazek, R. Dermisek, and S. Raby, *Yukawa unification in SO(10)*, *Phys. Rev.* **D65** (2002) 115004, [hep-ph/0201081].

[44] K. Tobe and J. D. Wells, *Revisiting top bottom tau Yukawa unification in supersymmetric grand unified theories*, *Nucl. Phys.* **B663** (2003) 123–140, [hep-ph/0301015].

[45] H. Baer, S. Kraml, and S. Sekmen, *Is 'just-so' Higgs splitting needed for t - b - tau Yukawa unified SUSY GUTs?*, *JHEP* **09** (2009) 005, [0908.0134].

[46] M. Badziak, M. Olechowski, and S. Pokorski, *Yukawa unification in SO(10) with light sparticle spectrum*, *JHEP* **08** (2011) 147, [1107.2764].

[47] A. Anandakrishnan, S. Raby, and A. Wingerter, *Yukawa Unification Predictions for the LHC*, *Phys. Rev.* **D87** (2013), no. 5 055005, [1212.0542].

[48] A. S. Joshipura and K. M. Patel, *Yukawa coupling unification in SO(10) with positive μ and a heavier gluino*, *Phys. Rev.* **D86** (2012) 035019, [1206.3910].

[49] A. Anandakrishnan, B. C. Bryant, S. Raby, and A. Wingerter, *LHC Phenomenology of SO(10) Models with Yukawa Unification*, *Phys. Rev.* **D88** (2013) 075002, [1307.7723].

[50] A. Anandakrishnan, B. C. Bryant, and S. Raby, *LHC Phenomenology of SO(10) Models with Yukawa Unification II*, *Phys. Rev.* **D90** (2014), no. 1 015030, [1404.5628].

[51] M. Badziak, M. Olechowski, and S. Pokorski, *Light staus and enhanced Higgs diphoton rate with non-universal gaugino masses and SO(10) Yukawa unification*, *JHEP* **10** (2013) 088, [1307.7999].

[52] M. Adeel Ajaib, I. Gogoladze, Q. Shafi, and C. S. Un, *A Predictive Yukawa Unified SO(10) Model: Higgs and Sparticle Masses*, *JHEP* **07** (2013) 139, [1303.6964].

[53] M. T. Grisaru, W. Siegel, and M. Rocek, *Improved Methods for Supergraphs*, *Nucl. Phys.* **B159** (1979) 429.

[54] K. Benakli, M. Goodsell, F. Staub, and W. Porod, *Constrained minimal Dirac gaugino supersymmetric standard model*, *Phys. Rev.* **D90** (2014), no. 4 045017, [1403.5122].

[55] K. Benakli, L. Darmé, M. D. Goodsell, and J. Harz, *The Di-Photon Excess in a Perturbative SUSY Model*, 1605.05313.

[56] D. I. Kazakov, *Renormalization properties of softly broken SUSY gauge theories*, in *Continuous advances in QCD. Proceedings, Conference, Minneapolis, USA, May 17-23, 2002*, pp. 421–448, 2002. hep-ph/0208200.