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An Analytic Analysis of the Pion Decay Constant in Three-Flavoured Chiral Perturbation Theory

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Abstract

A representation of the two-loop contribution to the pion decay constant in $SU(3)$ chiral perturbation theory is presented. The result is analytic upto the contribution of the three (different) mass sunset integrals, for which an expansion in their external momentum has been taken. We also give an analytic expression for the two-loop contribution to the pion mass based on a renormalized representation and in terms of the physical eta mass. We find an expansion of F_π and M_π^2 in the strange quark mass in the isospin limit, and perform the matching of the chiral SU(2) and SU(3) low energy constants. A numerical analysis demonstrates the high accuracy of our representation, and the strong dependence of the pion decay constant upon the values of the low energy constants, especially in the chiral limit.

1 Introduction

The mass and decay constants of the pions, kaons and the eta have been worked out to two-loop accuracy in three-flavoured chiral perturbation theory (ChPT) in [1] some time ago. The expressions for these at this order bring in a class of diagrams known as the sunsets. For the decay constants, in addition to the sunset integral, derivatives of the sunsets with respect to the square of the external momentum (also known as 'butterfly' diagrams), evaluated at a value equal to the square of the mass of the particle in question, are needed. The sunset diagrams themselves have been studied in field theory literature for many years now, and for particular mass configurations analytic expressions exist in Laurent series expansions in $\epsilon = (4-d)/2$. In general, however, the sunsets and their derivatives have to be evaluated numerically and a publicly available software [2] does this with user driven inputs.

There is, however, a need for an analytic study of the observables in ChPT since one would like to have an intuitive sense for the results appearing therein. More importantly, with recent advances allowing lattice simulations to tune the quark masses to near physical values, a combining of lattice and ChPT results has become possible. However, at next to next to leading order (NNLO), three flavoured ChPT amplitudes are available only numerically or take a complicated form, and thus have not been used much by the lattice community. With this in mind, [3, 4] has advocated a large N_c motivated approach to replace the two-loop integrals by effective one-loop integrals, and find it fruitful for the study of the ratio F_K/F_π as well as F_π. The analytic studies of SU(3) amplitudes in the strange quark mass expansion of [5, 6, 7] are also steps in that direction, but as the results presented there are in the chiral limit $m_u = m_d = 0$, there is a need for more general expressions.

Some years ago, Kaiser [8] studied the problem of the pion mass in the analytic framework, and was able to employ well known properties of sunset integrals to reduce a large number of expressions to analytic ones. One exception was the sunset integral with kaons and an eta propagating in the loops with the external momentum at $s = m_\pi^2$, for which an expansion around m_π^2 was used. Kaiser [8] also replaced the m_η in his work by the leading order Gell-Mann-Okubo (GMO) formula. In principle, therefore, one can get an expansion in m_π^2 to arbitrary accuracy, proving thereby the accessibility of an analytical approach to the full two-loop result. For practical purposes, we have used the expansion up to and including m_π^4 terms. These are more than sufficient for the numerical accuracy required.

The reason why it is possible to attain the objectives above is that for many purposes, the sunset integrals are accessible analytically for kinematic configurations known as threshold and pseudo-threshold configurations [9], as well as for the case when the square of the external momentum vanishes [10]. Indeed, this is the case for most of the sunset integrals appearing in the expressions for the mass and decay constants. These properties also allow one to isolate the divergent parts in closed form, while the finite part remains calculable in analytic form only for special cases. On the other hand, there is always an integral representation for the finite part which can be evaluated numerically. Furthermore, for the most general case, all sunsets can be reduced to a set of master integrals.

All other vector and tensor integrals, as well their derivatives with respect to the square of the external momentum, can also be reduced to master integrals. The work of [11] in developing this work is noteworthy, as is the automation of these relations with the publicly available Mathematica package Tarcer [12]. Application of these methods and tools to sunset diagrams in ChPT is elucidated in [13].

Inspired by the developments above, we now seek to extend the work of [8] for the case of the pion decay constant in an expansion around $s = 0$, which also brings in the butterfly diagrams. In contrast to the approach of [8], we will retain the mass of the eta without recourse to the GMO. This is the main objective of the present work. As a side result, we also give the expression for the two-loop pion mass with the full eta mass dependence.

In principle, this may be also extended to the mass and decay constant of the kaon and the eta, but the expansion about $s = 0$ for these particles when particles of unequal mass are running around in the loops is bound to converge poorly, and one would have to go to very high orders in the expansion, thereby losing the appeal of such a result. Thus we confine ourselves to the pion in this work. We present expressions for the kaon and eta masses and decay constants in a future publication [14].

As an application of the expressions given here, we give their expansion in the strange quark mass in the isospin limit and perform the 'matching' of the three flavoured low energy constants F_0 and B_0 with their two flavoured counterparts F and B, respectively. We compare our results with those given in [15] and the chiral limit results of [5]. The results given in this work, however, go beyond the chiral limit matching done in the aforementioned papers. Indeed, the full expressions presented here allow for an expansion up to an arbitrary order in the quark masses.

The scheme of this paper is as follows. In Section 2 we briefly review sunset diagrams and their evaluation. In Section 3 we give the expressions for the analytical results up to $\mathcal{O}(m_\pi^4$ for the pion decay constant at two loops. We repeat the analysis for the two-loop pion mass contribution in Section 4. In Section 5, we give the s-quark expansion for both the pion decay constant as well as the pion mass, and perform the matching of the two- and three- flavour low-energy constants (SU(2) and SU(3) LECs). We present a numerical analysis of our results in Section 6, and conclude in Section 7 with a discussion of possible future work in this area.

2 Sunset Diagrams and their Derivatives

The sunset diagram, shown in Figure 1, represents the two-loop Feynman integral:

$$H^d_{\{\alpha,\beta,\gamma\}}(m_1, m_2, m_3; s) = \frac{1}{i^2}\int \frac{d^dq}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{1}{[q^2-m_1^2]^\alpha[r^2-m_2^2]^\beta[(q+r-p)^2-m_3^2]^\gamma} \quad (1)$$

Aside from the basic scalar integral, there exist tensor varieties of the sunset integral with loop-momenta in the numerator. The two tensor integrals that are of relevance to this work are H_μ and $H_{\mu\nu}$, in which the momenta q_μ and $q_\mu q_\nu$, respectively, appear in the

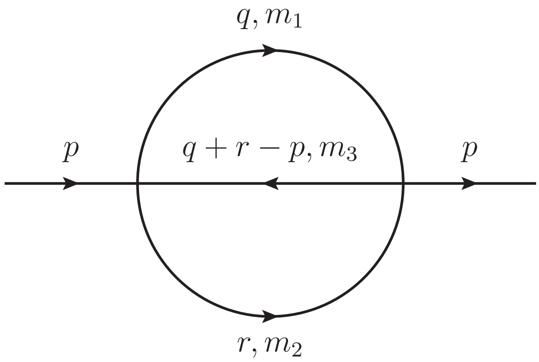

Figure 1: The two-loop self energy "sunset" diagram

numerator. These may be decomposed into linear combinations of scalar integrals via the Passarino-Veltman decomposition as:

$$\begin{aligned} H^d_\mu &= p_\mu H_1 \\ H^d_{\mu\nu} &= p_\mu p_\nu H_{21} + g_{\mu\nu} H_{22} \end{aligned} \tag{2}$$

The representation of the pion decay constants in [1] involves the scalar integrals H_1 and H_{21}. Taking the scalar product of H^d_μ with p^μ allows us to express the integral H_1 in terms of the sunset integral with the scalar numerator $q.p$. Similarly, we may express H_{21} in terms of sunset integrals with numerators $(q.p)^2$ and q^2:

$$\begin{aligned} H_1 &= \frac{\langle\langle q.p \rangle\rangle}{p^2} \\ H_{21} &= \frac{\langle\langle (q.p)^2 \rangle\rangle d - \langle\langle q^2 \rangle\rangle p^2}{p^4(d-1)} \end{aligned} \tag{3}$$

where $\langle\langle X \rangle\rangle$ represents a sunset integral with numerator X.

Another class of integrals that appear in the representation of [1] is the derivative of the sunset integrals and the H_1 and H_{21} with respect to the external momentum. In some places in the literature, these are sometimes known as 'butterfly' diagrams. These butterfly integrals may be expressed as sunset integrals of higher dimension by means of the following expression, which can be derived from the Feynman parameter representation of the sunset integrals, and a more general version of which is given in [8].

$$\left(\frac{\partial}{\partial s}\right)^n H^d_{\{\alpha,\beta,\gamma\}} = (-1)^n (4\pi)^{2n} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma+n)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} H^{d+2n}_{\{\alpha+n,\beta+n,\gamma+n\}} \tag{4}$$

Tarasov [11] has shown that by means of integration by parts relations, all sunset integrals may be expressed as linear combinations of four master integrals, namely $H^d_{\{1,1,1\}}$, $H^d_{\{2,1,1\}}$, $H^d_{\{1,2,1\}}$ and $H^d_{\{1,1,2\}}$, and the one-loop tadpole integral:

$$A^d(m) = \frac{1}{i}\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m^2} = -\frac{\Gamma\left(1 - d/2\right)}{(4\pi)^{d/2}} m^{d-2} \tag{5}$$

This includes sunset integrals of dimensions greater than d, permitting us to express the butterfly integrals in terms of the four master integrals and tadpoles. Scalar sunset integrals with non-unit numerators, such as those appearing in Eq.(2) may also be expressed in terms of the four master integrals and tadpoles. The Tarcer package [12], written in Mathematica, automates the application of Tarasov's relations, and we have made extensive use of it in this work. We have also made use of the package Ambre [16], which allows for a direct evaluation of many scalar and tensor Feynman integrals using a Mellin-Barnes approach, to numerically check our breakdown of the sunset and butterfly diagrams into master integrals. Recent analytic approaches to the evaluation of multi-fold Mellin-Barnes integrals in quantum field theory have been discussed in [17, 18] and applications in ChPT will appear in our future work [14].

As is the usual practice in ChPT, we use a modified version of the $\overline{MS}$ scheme to handle the divergences arising from the evaluation of the sunset diagrams. The subtraction procedure to two-loop order in ChPT is equivalent to multiplying Eq.(1) by $(\mu_\chi^2)^{4-d}$, where:

$$\mu_\chi^2 \equiv \mu^2 \frac{e^{\gamma_E - 1}}{4\pi} \tag{6}$$

and taking into consideration only the $\mathcal{O}(\epsilon^0)$ part of the result in a Laurent expansion about $\epsilon = 0$. We denote such renormalized sunset integrals by use of the subscript χ instead of d, i.e.

$$H^\chi_{\{a,b,c\}} \equiv (\mu_\chi^2)^{4-d} H^d_{\{a,b,c\}} \tag{7}$$

The inclusion of factor μ raised to a power of the dimension d introduces terms involving chiral logarithms, i.e.

$$l_P^r \equiv \frac{1}{2(4\pi)^2} \log\left[\frac{m_P^2}{\mu^2}\right] \qquad P = \pi, K, \eta \tag{8}$$

In the results presented in this paper, we group together all terms containing chiral logarithms, whether or not they arise from the renormalized sunset integrals. We therefore use the notation:

$$H^\chi_{\{a,b,c\}} \equiv \overline{H}^\chi_{\{a,b,c\}} + H^{\chi,\log}_{\{a,b,c\}} \tag{9}$$

where $H^{\chi,\log}$ are the terms of the sunset integral containing chiral logarithms, and $\overline{H}^\chi$ is the aggregation of the remainder. All results given hereafter have been renormalized using this subtraction scheme, and are presented using the notation above.

Analytic expressions for the master integrals themselves have been studied thoroughly, and several results exist in the literature [9, 10, 19, 20, 21, 22]. For sunset integrals with only one mass scale, there is a further reduction in the number of master integrals, and all sunsets can be expressed in terms of the tadpole integral, $A^\chi = \mu_\chi^{4-d} A^d$, and $H^\chi_{\{1,1,1\}}$, which is given in [9, 19], amongst others, as:

$$H^\chi_{\{1,1,1\}} = -\left(\mu^2 e^{\gamma_E - 1}\right)^{2\epsilon} \frac{(m^2)^{1-2\epsilon}}{(4\pi)^4} \frac{\Gamma^2(1+\epsilon)}{(1-\epsilon)(1-2\epsilon)} \left(-\frac{3}{2\epsilon^2} + \frac{1}{4\epsilon} + \frac{19}{8}\right) + \mathcal{O}(\epsilon) \tag{10}$$

Analytic expressions for the two mass scale integrals can be found by means of the pseudothreshold results of [9].

Expressions for the three mass sunset integrals are given in [22] in terms of elliptic dilogarithmic functions. However, as one of the principal reasons for the lack of use of ChPT results by the lattice community is the complicated form of many of the results, we wish to keep the expression derived here as simple and accessible as possible. To this end, and to stay true to the spirit of the method of [8], instead of using the results of [22] we take an expansion in the external momentum s upto order $\mathcal{O}(s^2)$:

$$H^{\chi}_{\{\alpha,\beta,\gamma\}} = K_{\{\alpha,\beta,\gamma\}} + s\, K'_{\{\alpha,\beta,\gamma\}} + \frac{s^2}{2!}\, K''_{\{\alpha,\beta,\gamma\}} + \mathcal{O}(s^3) \tag{11}$$

where $K_{\{\alpha,\beta,\gamma\}} \equiv H^{\chi}_{\{\alpha,\beta,\gamma\}}|_{s=0}$. In this special case of $s=0$, as in the case of the single mass scale sunsets, all sunset integrals may be expressed solely in terms of $K_{\{1,1,1\}}$ and tadpole integrals [11].

The pion mass and decay constant at two loops both involve a sunset integral with the following three mass scale configuration:

$$H^{\chi}_{\{\alpha,\beta,\gamma\}}\left(m_K, m_K, m_\eta; s = m_\pi^2\right)$$

This may be expanded in s by making use of the result [1, 8, 10]:

$$\begin{aligned}
&\frac{2\,(4\pi)^4}{M^2} H^{\chi}_{\{1,1,1\}}\{M,M,m;0\} \\
&\quad = \left(2 + \frac{m^2}{M^2}\right)\frac{1}{\epsilon^2} + \left(\frac{m^2}{M^2}\left(1 - 2\log\left[\frac{m^2}{\mu^2}\right]\right) + 2\left(1 - 2\log\left[\frac{M^2}{\mu^2}\right]\right)\right)\frac{1}{\epsilon} \\
&\quad - \frac{2}{(\mu^2)^{2\epsilon}}\left(\frac{m^2}{M^2}\log\left[\frac{m^2}{\mu^2}\right]\left(1 - \log\left[\frac{m^2}{\mu^2}\right]\right) + 2\log\left[\frac{M^2}{\mu^2}\right]\left(1 - \log\left[\frac{M^2}{\mu^2}\right]\right)\right) \\
&\quad - \frac{m^2}{M^2}\log^2\left[\frac{m^2}{M^2}\right] + \left(\frac{m^2}{M^2} - 4\right) F\left[\frac{m^2}{M^2}\right] + \left(2 + \frac{m^2}{M^2}\right)\left(\frac{\pi^2}{6} + 3\right) + \mathcal{O}(\epsilon)
\end{aligned} \tag{12}$$

where

$$F[x] = \frac{1}{\sigma}\left[4\mathrm{Li}_2\left(\frac{\sigma-1}{\sigma+1}\right) + \log^2\left(\frac{1-\sigma}{1+\sigma}\right) + \frac{\pi^2}{3}\right], \qquad \sigma = \sqrt{1 - \frac{4}{x}} \tag{13}$$

3 The Pion Decay Constant to Two Loops

The pion decay constant is given in [1] as:

$$F_\pi = F_0(1 + \overline{F}_\pi^{(4)} + \overline{F}_\pi^{(6)}) + \mathcal{O}(p^8) \tag{14}$$

where the $\mathcal{O}(p^6)$ contribution can be broken up into a piece that results from the model-dependent counterterms $(\overline{F}_\pi^{(6)})_{CT}$, and one that results from the chiral loop $(\overline{F}_\pi^{(6)})_{loop}$. For the pion, the explicit form of these terms are given by:

$$F_\pi^2 \overline{F}_\pi^{(4)} = 4(m_\pi^2 + 2m_K^2)L_4^r + 4m_\pi^2 L_5^r - 2m_\pi^2 l_\pi^r - m_K^2 l_K^r \tag{15}$$

$$F_\pi^4(\overline{F}_\pi)^{(6)}_{CT} = 8m_\pi^4 C_{14}^r + 8m_\pi^2\left(m_\pi^2+2m_K^2\right)C_{15}^r + 8\left(3m_\pi^4 - 4m_\pi^2 m_K^2 + 4m_K^4\right)C_{16}^r + 8m_\pi^4 C_{17}^r \tag{16}$$

where m_P with $P=\pi,K,\eta$ are the physical meson masses, and l_P^r are the chiral logarithms defined in Eq.(8). Note that the C_i used in this paper are dimensionless.

The loop contributions can be subdivided as follows:

$$F_\pi^4(\overline{F}_\pi)^{(6)}_{loop} = \overline{d}^\pi_{sunset} + d^\pi_{log\times log} + d^\pi_{log} + d^\pi_{log\times L_i} + d^\pi_{L_i} + d^\pi_{L_i\times L_j} \tag{17}$$

The terms containing the LECs L_i but no chiral logarithms are given by:

$$(16\pi^2)d^\pi_{L_i} = -2m_\pi^4 L_1^r - \frac{1}{9}\left(37m_\pi^4 - 8m_\pi^2 m_K^2 + 52m_K^4\right)L_2^r - \frac{1}{27}\left(28m_\pi^4 - 8m_\pi^2 m_K^2 + 43m_K^4\right)L_3^r \tag{18}$$

and the terms bilinear in the LECs are contained in:

$$\begin{aligned} d^\pi_{L_i\times L_j} =& 56\left(m_\pi^2+2m_K^2\right)^2 (L_4^r)^2 + 16\left(7m_\pi^4 + 10m_\pi^2 m_K^2 + 4m_K^4\right)L_4^r L_5^r \\ &- 64\left(m_\pi^2+2m_K^2\right)^2 L_4^r L_6^r - 64\left(m_\pi^4+2m_K^4\right)L_4^r L_8^r + 56m_\pi^4 (L_5^r)^2 \\ &- 64m_\pi^2\left(m_\pi^2+2m_K^2\right)L_5^r L_6^r - 64m_\pi^4 L_5^r L_8^r \end{aligned} \tag{19}$$

The remaining three terms of Eq.(17) give the terms containing the chiral logs. Explicitly, the following gives the terms linear in chiral logarithms:

$$\begin{aligned} (16\pi^2)d^\pi_{log} =& \left(\frac{9}{32}m_\eta^4 - \frac{3}{8}m_\eta^2 m_K^2 - \frac{7}{48}m_\eta^2 m_\pi^2 + \frac{9}{8}m_K^4 + \frac{9}{4}m_K^2 m_\pi^2 + \frac{679}{144}m_\pi^4\right)l_\pi^r \\ &+ \left(\frac{23}{8}m_K^4 - \frac{1}{2}m_K^2 m_\pi^2\right)l_K^r + \left(-\frac{9}{32}m_\eta^4 + \frac{7}{8}m_\eta^2 m_K^2 + \frac{1}{48}m_\eta^2 m_\pi^2\right)l_\eta^r \end{aligned} \tag{20}$$

while the terms bilinear in the l_P^r are contained in:

$$\begin{aligned} d^\pi_{log\times log} =& \left(\frac{15}{8}\frac{m_K^4 m_\eta^2}{m_\pi^2} - \frac{9}{8}\frac{m_K^2 m_\eta^4}{m_\pi^2} + \frac{1}{4}m_\pi^2 m_\eta^2 - \frac{17}{24}m_K^2 m_\eta^2 + \frac{3}{8}m_\eta^4\right)\left(l_\eta^r\right)^2 \\ &+ \left(\frac{25}{3}m_\pi^2 m_K^2 - \frac{3}{4}m_K^4\right)l_\pi^r l_K^r + \left(\frac{41}{8}m_\pi^4 - \frac{7}{6}m_\pi^2 m_K^2 + \frac{3}{8}m_K^4\right)(l_\pi^r)^2 \\ &+ \left(-\frac{15}{4}\frac{m_K^4 m_\eta^2}{m_\pi^2} + \frac{9}{4}\frac{m_K^2 m_\eta^4}{m_\pi^2} - \frac{7}{12}m_K^2 m_\eta^2\right)l_K^r l_\eta^r \\ &+ \left(\frac{15}{8}\frac{m_K^4 m_\eta^2}{m_\pi^2} - \frac{9}{8}\frac{m_K^2 m_\eta^4}{m_\pi^2} + \frac{1}{3}m_\pi^2 m_K^2 - \frac{5}{8}m_K^4 + \frac{7}{24}m_K^2 m_\eta^2\right)(l_K^r)^2 \end{aligned} \tag{21}$$

The contributions from terms involving products of chiral logarithms and the LECs are collected in:

$$\begin{aligned} d^\pi_{log\times L_i} =& 4m_\pi^2\left(14m_\pi^2 L_1^r + 8m_\pi^2 L_2^r + 7m_\pi^2 L_3^r - 13m_\pi^2 L_4^r - 12m_K^2 L_4^r - 10m_\pi^2 L_5^r\right)l_\pi^r \\ &+ 4m_K^2\left(16m_K^2 L_1^r + 4m_K^2 L_2^r + 5m_K^2 L_3^r - 3m_\pi^2 L_4^r - 14m_K^2 L_4^r - 5m_\pi^2 L_5^r\right)l_K^r \\ &- \frac{4}{3}m_\eta^2\left(m_\pi^2 - 4m_K^2\right)\left(4L_1^r + L_2^r + L_3^r - 3L_4^r\right)l_\eta^r \end{aligned} \tag{22}$$

And finally, the contributions from the sunset diagrams are given in:

$$
\begin{aligned}
d^{\pi}_{sunset} = & \frac{1}{(16\pi^2)^2}\left(\frac{35}{288}m_\pi^4\pi^2+\frac{41}{128}m_\pi^4+\frac{1}{144}m_\pi^2m_K^2\pi^2-\frac{5}{32}m_\pi^2m_K^2+\frac{11}{72}m_K^4\pi^2+\frac{15}{32}m_K^4\right)\\
& +\frac{5}{12}m_\pi^4\overline{H}'^{\chi}_{\pi\pi\pi}-\frac{1}{2}m_\pi^2\overline{H}^{\chi}_{\pi\pi\pi}-\frac{5}{16}m_\pi^4\overline{H}'^{\chi}_{\pi KK}+\frac{1}{16}m_\pi^2\overline{H}^{\chi}_{\pi KK}+\frac{1}{36}m_\pi^4\overline{H}'^{\chi}_{\pi\eta\eta}\\
& +\frac{1}{2}m_\pi^2m_K^2\overline{H}'^{\chi}_{K\pi K}-\frac{1}{2}m_K^2\overline{H}^{\chi}_{K\pi K}-\frac{5}{12}m_\pi^4H'^{\chi}_{KK\eta}-\frac{1}{16}m_\pi^4\overline{H}'^{\chi}_{\eta KK}+\frac{1}{4}m_\pi^2m_K^2\overline{H}'^{\chi}_{\eta KK}\\
& +\frac{1}{16}m_\pi^2\overline{H}^{\chi}_{\eta KK}-\frac{1}{4}m_K^2\overline{H}^{\chi}_{\eta KK}+\frac{1}{2}m_\pi^4\overline{H}'^{\chi}_{1\ \pi KK}+m_\pi^4\overline{H}'^{\chi}_{1\ KK\eta}+\frac{3}{2}m_\pi^4\overline{H}'^{\chi}_{21\pi\pi\pi}\\
& -\frac{3}{16}m_\pi^4\overline{H}'^{\ \chi}_{21\pi KK}+\frac{3}{2}m_\pi^4\overline{H}'^{\ \chi}_{21K\pi K}+\frac{9}{16}m_\pi^4\overline{H}'^{\ \chi}_{21\eta KK}
\end{aligned}
\tag{23}
$$

where we use the notation:

$$
\overline{H}^{\chi}_{aPbQcR}=\overline{H}^{\chi}_{\{a,b,c\}}\{m_P,m_Q,m_R;s=m_\pi^2\}
\tag{24}
$$

with $\overline{H}^{\chi}_{\{a,b,c\}}$ as defined in Eq. (9). a,b,c will be suppressed if equal to 1. The terms resulting from the sunset integrals which involving chiral logarithms have been included in d^{π}_{log} or $d^{\pi}_{log\times log}$ as appropriate.

Evaluating the sunset integrals as described in Section (2), d^{π}_{sunset} can be re-expressed as:

$$
\begin{aligned}
d^{\pi}_{sunset} = & \frac{1}{(16\pi^2)^2}\Bigg[-\frac{9}{32}\frac{m_\eta^6}{m_\pi^2}+\left(\frac{3}{8}+\frac{3\pi^2}{32}\right)\frac{m_\eta^4m_K^2}{m_\pi^2}+\frac{193}{768}m_\eta^4-\left(\frac{27}{16}+\frac{\pi^2}{4}\right)\frac{m_\eta^2m_K^4}{m_\pi^2}\\
& -\left(\frac{13}{64}+\frac{7\pi^2}{288}\right)m_\eta^2m_K^2+\left(\frac{49}{384}+\frac{\pi^2}{216}\right)m_\eta^2m_\pi^2+\left(\frac{3}{4}+\frac{\pi^2}{4}\right)\frac{m_K^6}{m_\pi^2}+\left(\frac{209}{192}+\frac{5\pi^2}{32}\right)m_K^4\\
& +\left(\frac{41}{192}+\frac{\pi^2}{36}\right)m_K^2m_\pi^2-\left(\frac{1}{1152}+\frac{\pi^2}{288}\right)m_\pi^4\Bigg]+d^{\pi}_{\pi KK}+d^{\pi}_{\pi\eta\eta}+d^{\pi}_{KK\eta}
\end{aligned}
\tag{25}
$$

where

$$
d^{\pi}_{\pi KK}=-\left(\frac{9}{16}\frac{m_K^4}{m_\pi^2}+\frac{3}{4}m_K^2+\frac{1}{48}m_\pi^2\right)\overline{H}^{\chi}_{\pi KK}+\left(\frac{3}{4}m_K^4+\frac{1}{6}m_K^2m_\pi^2+\frac{m_\pi^4}{12}\right)\overline{H}^{\chi}_{2\pi KK}
\tag{26}
$$

$$
d^{\pi}_{\pi\eta\eta}=\left(-\frac{1}{36}m_\pi^2\right)\overline{H}^{\chi}_{\pi\eta\eta}+\left(\frac{1}{36}m_\pi^4\right)\overline{H}^{\chi}_{2\pi\eta\eta}
\tag{27}
$$

$$
\begin{aligned}
d^{\pi}_{KK\eta} = & \left(\frac{1}{96}m_\pi^2-\frac{1}{24}m_K^2+\frac{15}{16}\frac{m_K^4}{m_\pi^2}-\frac{7}{48}m_\eta^2-\frac{3}{8}\frac{m_K^2m_\eta^2}{m_\pi^2}+\frac{9}{32}\frac{m_\eta^4}{m_\pi^2}\right)\overline{H}^{\chi}_{KK\eta}\\
& +\left(\frac{5}{48}m_\pi^2m_K^2-\frac{7}{48}m_K^4-\frac{15}{4}\frac{m_K^6}{m_\pi^2}+\frac{7}{16}m_K^2m_\eta^2+\frac{45}{16}\frac{m_K^4m_\eta^2}{m_\pi^2}-\frac{9}{8}\frac{m_K^2m_\eta^4}{m_\pi^2}\right)\overline{H}^{\chi}_{2KK\eta}\\
& +\left(\frac{5}{96}m_\pi^2m_\eta^2-\frac{1}{16}m_K^2m_\eta^2+\frac{7}{48}m_\eta^4-\frac{3}{16}\frac{m_K^2m_\eta^4}{m_\pi^2}-\frac{9}{32}\frac{m_\eta^6}{m_\pi^2}\right)\overline{H}^{\chi}_{KK2\eta}
\end{aligned}
\tag{28}
$$

Closed form expressions, at $\mathcal{O}(\epsilon^0)$, for the master integrals $\overline{H}^{\chi}$ appearing in $d_{\pi KK}$ and $d_{\pi\eta\eta}$ are given in Appendix A. The master integrals appearing in $d_{KK\eta}$ are of three mass scales, for which there exist no simple closed form expressions. For these, therefore, we take an expansion around $s = m_\pi^2 = 0$. Up to order $\mathcal{O}(m_\pi^4)$, we have:

$$(16\pi^2)^2\, d_{KK\eta} = d_{KK\eta}^{(-1)}(m_\pi^2)^{-1} + d_{KK\eta}^{(0)} + d_{KK\eta}^{(1)}(m_\pi^2) + d_{KK\eta}^{(2)}(m_\pi^2)^2 \tag{29}$$

where

$$\begin{aligned} d_{KK\eta}^{(-1)} = {} & \frac{9}{32}m_\eta^6 - \left(\frac{3}{8} + \frac{3\pi^2}{32}\right) m_\eta^4 m_K^2 + \left(\frac{27}{16} + \frac{\pi^2}{4}\right) m_\eta^2 m_K^4 + \left(\frac{15}{16} - \frac{5\pi^2}{32}\right) m_K^6 \\ & + \left(\frac{9}{32}m_\eta^4 m_K^2 - \frac{15}{32}m_\eta^2 m_K^4\right) \log^2[\tau] \end{aligned} \tag{30}$$

$$\begin{aligned} d_{KK\eta}^{(0)} = {} & -\frac{193}{768}m_\eta^4 - \left(\frac{11}{64} - \frac{\pi^2}{288}\right) m_\eta^2 m_K^2 - \left(\frac{211}{384} + \frac{11\pi^2}{576}\right) m_K^4 + \left(\frac{1}{2}m_K^4 - \frac{1}{8}m_\eta^2 m_K^2\right) F[\tau] \\ & + \left(\frac{5}{96}m_\eta^2 m_K^2\right) \log^2[\tau] + \left(\frac{9}{64}m_\eta^4 - \frac{3}{16}m_\eta^2 m_K^2\right) \log[\tau] \\ & + \left(-\frac{9}{64}m_\eta^4 + \frac{3}{16}m_\eta^2 m_K^2 - \frac{15}{32}m_K^4\right) \log[\rho] \end{aligned} \tag{31}$$

$$\begin{aligned} d_{KK\eta}^{(1)} = {} & \left(\frac{19}{384} + \frac{\pi^2}{192}\right) m_\eta^2 - \left(\frac{11}{192} - \frac{\pi^2}{96}\right) m_K^2 + F[\tau]\left(\frac{m_\eta^2}{32} - \frac{m_K^2}{8}\right) \\ & + \left(\frac{7}{96}m_\eta^2 + \frac{1}{48}m_K^2\right) \log[\rho] - \frac{1}{32}m_\eta^2 \log^2[\tau] - \frac{7}{96}\left(m_\eta^2\right) \log[\tau] \end{aligned} \tag{32}$$

$$\begin{aligned} d_{KK\eta}^{(2)} = {} & \frac{1}{\lambda^2}\left(\frac{23}{576}m_\eta^4 - \frac{1}{4}\frac{m_K^6}{m_\eta^2} - \frac{235}{576}m_\eta^2 m_K^2 + \frac{139}{288}m_K^4\right) \\ & + \frac{1}{\lambda^3}\left(-\frac{1}{2}\frac{m_K^{10}}{m_\eta^4} + \frac{17}{48}\frac{m_K^8}{m_\eta^2} - \frac{7}{48}m_\eta^2 m_K^4 - \frac{1}{3}m_K^6\right) F[\tau] \\ & + \frac{1}{\lambda^3}\left(\frac{1}{192}m_\eta^6 - \frac{1}{32}m_\eta^4 m_K^2 - \frac{1}{2}\frac{m_K^8}{m_\eta^2} + \frac{83}{96}m_\eta^2 m_K^4 + \frac{13}{48}m_K^6\right) \log[\tau] - \frac{1}{192}\log[\rho] \end{aligned} \tag{33}$$

In the above expressions, $\tau \equiv m_\eta^2/m_K^2$, $\rho \equiv m_\pi^2/m_K^2$, $\lambda \equiv m_\eta^2 - 4m_K^2$, and $F[x]$ is defined in Eq.(13). Note that in this expansion, divergences appear in the $m_\pi \to 0$ limit. The divergences from the $d_{KK\eta}^{(-1)}$ term cancel against the divergences in Eq.(25) and in Eq.(20), while those arising from the $\log[\rho]$ and $\log^2[\rho]$ in $d_{KK\eta}^{(0)}$ cancel against divergences in Eqs.(20),(21) and (26). Therefore the overall $\overline{F}_\pi^{(6)}$ remains non-divergent in the $m_\pi^2 \to 0$ limit.

4 The Pion Mass to Two Loops

We repeat the steps of the previous section for the pion mass. A representation for this is given in [1] as:

$$M_\pi^2 = m_{\pi 0}^2 + (m_\pi^2)^{(4)} + (m_\pi^2)^{(6)}_{CT} + (m_\pi^2)^{(6)}_{loop} + \mathcal{O}(p^8) \tag{34}$$

where $m_{\pi 0}^2 = 2B_0\hat{m}$ is the bare pion mass squared, and m_P are the physical meson masses.

$$\frac{F_\pi^2}{m_\pi^2}(m_\pi^2)^{(4)} = 8(m_\pi^2 + 2m_K^2)(2L_6^r - L_4^r) + 8m_\pi^2(2L_8^r - L_5^r) + m_\pi^2 l_\pi^r - \frac{1}{3}m_\eta^2 l_\eta^r \tag{35}$$

$$\begin{aligned}
-\frac{F_\pi^4}{16m_\pi^2}(m_\pi^2)^{(6)}_{CT} &= 2m_\pi^4 C_{12}^r + \left(2m_\pi^4 + 4m_\pi^2 m_K^2\right) C_{13}^r + m_\pi^4 C_{14}^r + \left(m_\pi^4 + 2m_\pi^2 m_K^2\right) C_{15}^r \\
&+ \left(3m_\pi^4 - 4m_\pi^2 m_K^2 + 4m_K^4\right) C_{16}^r + m_\pi^4 C_{17}^r - 3m_\pi^4 C_{19}^r - \left(5m_\pi^4 + 4m_K^4\right) C_{20}^r \\
&- \left(3m_\pi^4 + 12m_\pi^2 m_K^2 + 12m_K^4\right) C_{21}^r - 2m_\pi^4 C_{31}^r - \left(2m_\pi^4 + 4m_\pi^2 m_K^2\right) C_{32}^r
\end{aligned} \tag{36}$$

The $(m_\pi^2)^{(6)}_{loop}$ term can be subdivided into the following components:

$$F_\pi^4 (m_\pi^2)^{(6)}_{loop} = c^\pi_{sunset} + c^\pi_{log\times log} + c^\pi_{log} + c^\pi_{log\times L_i} + c^\pi_{L_i} + c^\pi_{L_i\times L_j} \tag{37}$$

where

$$\frac{16\pi^2}{m_\pi^2} c^\pi_{L_i} = 4m_\pi^4 L_1^r + \frac{1}{9}\left(74m_\pi^4 - 16m_\pi^2 m_K^2 + 104m_K^4\right) L_2^r + \frac{1}{27}\left(56m_\pi^4 - 16m_\pi^2 m_K^2 + 86m_K^4\right) L_3^r \tag{38}$$

$$\begin{aligned}
-\frac{c^\pi_{L_i\times L_j}}{128m_\pi^2} &= \left(4m_K^4 + 4m_K^2 m_\pi^2 + m_\pi^4\right)(L_4^r)^2 + \left(m_K^4 + 3m_K^2 m_\pi^2 + 2m_\pi^4\right) L_4^r L_5^r \\
&- 4\left(4m_K^4 + 4m_K^2 m_\pi^2 + m_\pi^4\right) L_4^r L_6^r - 2\left(m_K^4 + 3m_K^2 m_\pi^2 + 2m_\pi^4\right) L_4^r L_8^r + m_\pi^4 (L_5^r)^2 \\
&- 2\left(m_K^4 + 3m_K^2 m_\pi^2 + 2m_\pi^4\right) L_5^r L_6^r - 4m_\pi^4 L_5^r L_8^r + 4\left(4m_K^4 + 4m_K^2 m_\pi^2 + m_\pi^4\right)(L_6^r)^2 \\
&+ 4\left(m_K^4 + 3m_K^2 m_\pi^2 + 2m_\pi^4\right) L_6^r L_8^r + 4m_\pi^4 (L_8^r)^2
\end{aligned} \tag{39}$$

$$\begin{aligned}
(16\pi^2) c^\pi_{log} &= \left(-\frac{3}{16}m_\eta^4 m_\pi^2 + \frac{1}{4}m_\eta^2 m_K^2 m_\pi^2 + \frac{1}{3}m_\eta^2 m_\pi^4 - \frac{3}{4}m_K^4 m_\pi^2 - \frac{11}{6}m_K^2 m_\pi^4 - \frac{299}{36}m_\pi^6\right) l_\pi^r \\
&+ \left(-\frac{29}{4}m_K^4 m_\pi^2 - \frac{1}{3}m_K^2 m_\pi^4\right) l_K^r + \left(\frac{3}{16}m_\eta^4 m_\pi^2 - \frac{5}{4}m_\eta^2 m_K^2 m_\pi^2 - \frac{1}{72}m_\eta^2 m_\pi^4\right) l_\eta^r
\end{aligned} \tag{40}$$

$$
\begin{aligned}
c^{\pi}_{log\times log} &= \left(\frac{121}{36}m_\pi^6 + \frac{3}{2}m_\pi^4 m_K^2 - \frac{1}{4}m_\pi^2 m_K^4\right)(l^r_\pi)^2 + \left(\frac{1}{2}m_\pi^2 m_K^4 - 3m_\pi^4 m_K^2\right) l^r_\pi l^r_K \\
&+ \left(\frac{5}{3}m_\pi^4 m_\eta^2\right) l^r_\pi l^r_\eta + \left(\frac{5}{2}m_K^4 m_\eta^2 - \frac{3}{2}m_K^2 m_\eta^4 - \frac{3}{2}m_\pi^2 m_K^2 m_\eta^2\right) l^r_K l^r_\eta \\
&+ \left(\frac{1}{6}m_\pi^4 m_K^2 + \frac{19}{4}m_\pi^2 m_K^4 + \frac{1}{12}m_\pi^2 m_K^2 m_\eta^2 - \frac{5}{4}m_K^4 m_\eta^2 + \frac{3}{4}m_K^2 m_\eta^4\right)(l^r_K)^2 \\
&+ \left(\frac{1}{18}m_\pi^4 m_\eta^2 + \frac{25}{12}m_\pi^2 m_K^2 m_\eta^2 - \frac{5}{4}m_K^4 m_\eta^2 - \frac{29}{36}m_\pi^2 m_\eta^4 + \frac{3}{4}m_K^2 m_\eta^4\right)(l^r_\eta)^2 \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
\frac{c^{\pi}_{log\times L_i}}{m_\pi^2} &= -\Big(112m_\pi^2 L_1^r + 64m_\pi^2 L_2^r + 56m_\pi^2 L_3^r - (144m_\pi^2 + 80m_K^2)L_4^r - 96m_\pi^2 L_5^r \\
&\quad + \left(256m_\pi^2 + 160m_K^2\right) L_6^r + 176m_\pi^2 L_8^r\Big) m_\pi^2 l^r_\pi \\
&- \Big(128m_K^2 L_1^r + 32m_K^2 L_2^r + 40m_K^2 L_3^r - \left(16m_\pi^2 + 160m_K^2\right) L_4^r - \left(16m_\pi^2 + 32m_K^2\right) L_5^r \\
&\quad + \left(32m_\pi^2 + 192m_K^2\right) L_6^r + \left(32m_\pi^2 + 64m_K^2\right) L_8^r\Big) m_K^2 l^r_K \\
&+ \Big(\frac{8}{3}\left(m_\pi^2 - 4m_K^2\right)(4L_1^r + L_2^r + L_3^r) - 16\left(m_\pi^2 - 3m_K^2\right) L_4^r + \frac{32}{3}\left(2m_\pi^2 - 5m_K^2\right) L_6^r \\
&\quad - \frac{64}{9}\left(m_\pi^2 - m_K^2\right)(L_5^r + 6L_7^r) - \frac{16}{3}m_\pi^2 L_8^r\Big) m_\eta^2 l^r_\eta \qquad (42)
\end{aligned}
$$

The contribution from the sunset integrals is given by:

$$
\begin{aligned}
c^{\pi}_{sunset} &= \frac{1}{(16\pi^2)^2}\Bigg[\frac{3}{16}m_\eta^6 - \left(\frac{1}{4} + \frac{\pi^2}{16}\right) m_\eta^4 m_K^2 - \frac{155}{384}m_\eta^4 m_\pi^2 + \left(\frac{9}{8} + \frac{\pi^2}{6}\right) m_\eta^2 m_K^4 \\
&- \left(\frac{25}{32} + \frac{\pi^2}{144}\right) m_\eta^2 m_K^2 m_\pi^2 + \frac{25}{192}m_\eta^2 m_\pi^4 - \left(\frac{1}{2} + \frac{\pi^2}{6}\right) m_K^6 - \left(\frac{55}{96} + \frac{31\pi^2}{144}\right) m_K^4 m_\pi^2 \\
&+ \left(\frac{677}{864} - \frac{5\pi^2}{162}\right) m_K^2 m_\pi^4 + \left(\frac{2543}{1728} - \frac{41\pi^2}{1296}\right) m_\pi^6\Bigg] + c^{\pi}_{\pi KK} + c^{\pi}_{\pi\eta\eta} + c^{\pi}_{KK\eta} \qquad (43)
\end{aligned}
$$

where

$$
c^{\pi}_{\pi\eta\eta} = \left(\frac{m_\pi^4}{18}\right)\overline{H}^{\chi}_{\pi\eta\eta} \qquad (44)
$$

$$
c^{\pi}_{\pi KK} = \left(\frac{3}{8}m_K^4 + \frac{3}{4}m_\pi^2 m_K^2 - \frac{1}{8}m_\pi^4\right)\overline{H}^{\chi}_{\pi KK} + \left(\frac{1}{2}m_\pi^6 - \frac{1}{2}m_\pi^2 m_K^4\right)\overline{H}^{\chi}_{2\pi KK} \qquad (45)
$$

$$
\begin{aligned}
c^{\pi}_{KK\eta} = &\left(-\frac{5}{48}m_\pi^4 + \frac{2}{3}m_\pi^2 m_K^2 + \frac{1}{3}m_\pi^2 m_\eta^2 - \frac{5}{8}m_K^4 + \frac{1}{4}m_K^2 m_\eta^2 - \frac{3}{16}m_\eta^4\right)\overline{H}^{\chi}_{KK\eta} \\
&+\left(\frac{1}{24}m_\pi^4 m_K^2 - \frac{19}{24}m_\pi^2 m_K^4 - \frac{5}{8}m_\pi^2 m_K^2 m_\eta^2 + \frac{5}{2}m_K^6 - \frac{15}{8}m_K^4 m_\eta^2 + \frac{3}{4}m_K^2 m_\eta^4\right)\overline{H}^{\chi}_{2KK\eta} \\
&+\left(\frac{7}{48}m_\pi^4 m_\eta^2 - \frac{1}{8}m_\pi^2 m_K^2 m_\eta^2 - \frac{1}{3}m_\pi^2 m_\eta^4 + \frac{1}{8}m_K^2 m_\eta^4 + \frac{3}{16}m_\eta^6\right)\overline{H}^{\chi}_{KK2\eta} \qquad (46)
\end{aligned}
$$

With $\rho \equiv m_\pi^2/m_K^2$ and $\tau \equiv m_\eta^2/m_K^2$, expanding $c^{\pi}_{KK\eta}$ about $s = m_\pi^2 = 0$ gives:

$$
(16\pi^2)^2 c^{\pi}_{KK\eta} = c^{(0)}_{KK\eta} + c^{(1)}_{KK\eta}(m_\pi^2) + c^{(2)}_{KK\eta}(m_\pi^2)^2 + \mathcal{O}((m_\pi^2)^3) \qquad (47)
$$

where

$$
\begin{aligned}
c^{(0)}_{KK\eta} = &-\frac{3}{16}m_\eta^6 + \left(\frac{1}{4} + \frac{\pi^2}{16}\right)m_\eta^4 m_K^2 - \left(\frac{9}{8} + \frac{\pi^2}{6}\right)m_\eta^2 m_K^4 - \left(\frac{5}{8} - \frac{5\pi^2}{48}\right)m_K^6 \\
&+\left(\frac{5}{16}m_\eta^2 m_K^4 - \frac{3}{16}m_\eta^4 m_K^2\right)\log^2[\tau] \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
c^{(1)}_{KK\eta} = &\frac{155}{384}m_\eta^4 + \left(\frac{353}{192} + \frac{13\pi^2}{288}\right)m_K^4 + \left(\frac{49}{32} + \frac{7\pi^2}{144}\right)m_\eta^2 m_K^2 + \left(\frac{1}{4}m_\eta^2 m_K^2 - m_K^4\right)F[\tau] \\
&+\left(\frac{1}{8}m_\eta^2 m_K^2 - \frac{3}{32}m_\eta^4\right)\log[\tau] - \frac{13}{48}m_\eta^2 m_K^2\log^2[\tau] + \left(\frac{3}{32}m_\eta^4 - \frac{1}{8}m_\eta^2 m_K^2 + \frac{5}{16}m_K^4\right)\log[\rho] \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
c^{(2)}_{KK\eta} = &-\left(\frac{17}{96} - \frac{\pi^2}{288}\right)m_\eta^2 - \left(\frac{13}{48} + \frac{\pi^2}{72}\right)m_K^2 + \frac{1}{\lambda}\left(\frac{m_\eta^4}{48} - \frac{m_K^6}{2m_\eta^2} - \frac{m_\eta^2 m_K^2}{24} - \frac{13m_K^4}{24}\right)F[\tau] \\
&+\frac{1}{\lambda}\left(\frac{m_\eta^4}{6} - \frac{m_\eta^2 m_K^2}{24} - \frac{m_K^4}{2}\right)\log[\tau] - \frac{1}{48}m_\eta^2\log^2[\tau] - \left(\frac{m_\eta^2}{6} + \frac{m_K^2}{3}\right)\log[\rho] \qquad (50)
\end{aligned}
$$

The expressions of this section agree fully with those given in [8] when the eta masses here are expressed in terms of the pion and kaon masses by means of the Gell-Mann-Okubo formula. As with the expansion of the pion decay constant in m_π^2, here too divergences appear in the $m_\pi^2 \to 0$ limit. These are offset by the divergences appearing in Eqs.(40),(41),(43) and (45) in the same limit. In a similar way, the terms that do not vanish as $m_\pi^2 \to 0$ cancel.

5 Expansion in the Strange Quark Mass in the Isospin Limit

As an application of the expressions presented in the preceding sections, we present their expansion in the strange quark mass, m_s. More specifically, for the pion decay constant, we keep the physical kaon mass constant and expand in the small quark ratio $Q \equiv \hat{m}/m_s$ where $\hat{m} \equiv (m_u + m_d)/2$. Our choice of such an expansion, rather than one in which we keep m_s fixed and vary $\hat{m}$, is to facilitate comparison with the results given in [5]. For the pion mass we expand in m_s to compare with [15].

The isospin limit expansion of F_π is:

$$\frac{F_\pi}{F_0} = 1 + d_1 \left[\frac{M_K^2}{(4\pi F_0)^2}\right] + d_2 \left[\frac{M_K^2}{(4\pi F_0)^2}\right]^2 + \mathcal{O}(m_s^3) \tag{51}$$

where

$$\begin{aligned} d_1 =& 8(4\pi)^2 L_4^r - \frac{1}{2}\log\left[\frac{m_K^2}{\mu^2}\right] + \left\{8(4\pi)^2(L_4^r + L_5^r) - 2\log\left[\frac{m_K^2}{\mu^2}\right] - 2\log[2Q]\right\} Q \\ &+ \left\{2 - 8(4\pi)^2(L_4^r + L_5^r) + 2\log\left[\frac{m_K^2}{\mu^2}\right] + 2\log[2Q]\right\} Q^2 + \mathcal{O}(Q^3) \end{aligned} \tag{52}$$

$$d_2 = d_2^{\rm tree} + d_2^{\rm loop} \tag{53}$$

and

$$\begin{aligned} \frac{d_2^{\rm tree}}{32(4\pi)^4} &= C_{16}^r + L_4^r(3L_4^r + 2L_5^r - 8L_6^r - 4L_8^r) \\ &+ \left\{C_{15}^r - 2C_{16}^r + 6(L_4^r)^2 + 4L_4^rL_5^r - 16L_4^rL_6^r - 4L_4^rL_8^r + 2(L_5^r)^2 - 8L_5^rL_6^r - 4L_5^rL_8^r\right\} Q \\ &+ \left\{C_{14}^r + 5C_{16}^r + C_{17}^r - 3(L_4^r)^2 - 2L_4^rL_5^r + 8L_4^rL_6^r + 4L_4^rL_8^r - 3(L_5^r)^2 + 4L_5^rL_8^r\right\} Q^2 \\ &+ \mathcal{O}(Q^3) \end{aligned} \tag{54}$$

$$\begin{aligned} d_2^{\rm loop} =& -\frac{11}{12}\log^2\left[\frac{M_K^2}{\mu^2}\right] + \left(\frac{32}{9}\mathcal{D}_1^{(0)} + \frac{7}{3} - \frac{1}{3}\log\left[\frac{4}{3}\right]\right)\log\left[\frac{M_K^2}{\mu^2}\right] - \frac{73}{32} + \frac{1}{3}\log\left[\frac{4}{3}\right] \\ &- \frac{16}{9}\left(\mathcal{D}_2^{(0)} - 2\log\left[\frac{4}{3}\right]\mathcal{D}_3^{(0)}\right) + \frac{1}{3}F\left[\frac{4}{3}\right] \\ &+ \Bigg\{\frac{5}{4}\log^2\left[\frac{M_K^2}{\mu^2}\right] + \left(-\frac{16}{9}\mathcal{D}_1^{(1)} + \frac{35}{12} + \frac{5}{3}\log\left[\frac{4}{3}\right] + \frac{1}{3}\log[2Q]\right)\log\left[\frac{M_K^2}{\mu^2}\right] + \frac{157}{48} \\ &+ \frac{7}{6}\log\left[\frac{4}{3}\right] - \frac{8}{9}\left(\mathcal{D}_2^{(1)} + 2\mathcal{D}_3^{(1)}\log\left[\frac{4}{3}\right]\right) - \frac{5}{24}F\left[\frac{4}{3}\right] \\ &+ \left(\frac{4}{3}\log\left[\frac{4}{3}\right] + 16(4\pi)^2(L_4^r - L_5^r + 2L_8^r)\right)\log[2Q]\Bigg\}Q \\ &+ \Bigg\{ -\frac{41}{6}\log^2\left[\frac{M_K^2}{\mu^2}\right] + \left(\frac{2}{9}\mathcal{D}_1^{(2)} + \frac{101}{36} - \frac{29}{12}\log\left[\frac{4}{3}\right] - \frac{43}{4}\log[2Q]\right)\log\left[\frac{M_K^2}{\mu^2}\right] - \frac{8455}{1536} \end{aligned}$$

$$- \frac{61445}{18432} \log\left[\frac{4}{3}\right] + \frac{8}{9}\left(\mathcal{D}_2^{(2)} + \mathcal{D}_3^{(2)} \log\left[\frac{4}{3}\right]\right) + \frac{7873}{24576} F\left[\frac{4}{3}\right] - 5\log^2[2Q]$$
$$+ \left(8\mathcal{D}_4^{(2)} + \frac{29}{4} - 2\log\left[\frac{4}{3}\right]\right)\log[2Q] \Bigg\} Q^2 + \mathcal{O}(Q^3) \tag{55}$$

and

$$\begin{aligned}
\mathcal{D}_1^{(0)} &= (4\pi)^2\left(13L_1^r + \frac{13}{4}L_2^r + \frac{61}{16}L_3^r - \frac{51}{8}L_4^r\right) \\
\mathcal{D}_2^{(0)} &= (4\pi)^2\left(\frac{13}{4}L_2^r + \frac{43}{48}L_3^r\right) \\
\mathcal{D}_3^{(0)} &= (4\pi)^2\left(4L_1^r + L_2^r + L_3^r - 3L_4^r\right)
\end{aligned} \tag{56}$$

$$\begin{aligned}
\mathcal{D}_1^{(1)} &= (4\pi)^2\left(8L_1^r + 2L_2^r + 2L_3^r - \frac{57}{4}L_4^r + \frac{57}{4}L_5^r - 18L_8^r\right) \\
\mathcal{D}_2^{(1)} &= (4\pi)^2\left(8L_1^r + \frac{4}{3}L_3^r - 6L_4^r + 18L_5^r - 36L_8^r\right) \\
\mathcal{D}_3^{(1)} &= (4\pi)^2\left(8L_1^r + 2L_2^r + 2L_3^r - 3L_4^r + 3L_5^r\right)
\end{aligned} \tag{57}$$

$$\begin{aligned}
\mathcal{D}_1^{(2)} &= (4\pi)^2\left(584L_1^r + 308L_2^r + 272L_3^r - 258L_4^r + 234L_5^r - 432L_8^r\right) \\
\mathcal{D}_2^{(2)} &= (4\pi)^2\left(5L_1^r - 17L_2^r - \frac{11}{6}L_3^r - \frac{51}{2}L_4^r + 75L_5^r - 144L_8^r\right) \\
\mathcal{D}_3^{(2)} &= (4\pi)^2\left(20L_1^r + 5L_2^r + 5L_3^r - 6L_4^r + 9L_5^r\right) \\
\mathcal{D}_4^{(2)} &= (4\pi)^2\left(14L_1^r + 8L_2^r + 7L_3^r - 6L_4^r + 5L_5^r - 12L_8^r\right)
\end{aligned} \tag{58}$$

We can then connect the chiral SU(2) constant F in terms of the chiral SU(3) LECs as follows:

$$\frac{F}{F_0} = \lim_{m_u, m_d \to 0} \frac{F_\pi}{F_0} = 1 + d_1\left[\frac{M_K^2}{(4\pi F_0)^2}\right] + d_2\left[\frac{M_K^2}{(4\pi F_0)^2}\right]^2 + \mathcal{O}(m_s^3) \tag{59}$$

where d_1 and d_2 are understood to be in the limit $m_u = m_d = 0$. In this limit Eq.(51) agrees perfectly with the one-loop matching done in [5].

A similar expansion for the pion mass representation given in this paper is given below. In this case, we express the expansion in terms of the parameter $B_0 m_s$ rather than M_K^2 so as to facilitate comparison with the results of [15].

$$\frac{M_\pi^2}{(m_u + m_d)B_0} = 1 + c_1\left[\frac{m_s B_0}{(4\pi F_0)^2}\right] + c_2\left[\frac{m_s B_0}{(4\pi F_0)^2}\right]^2 + \mathcal{O}(m_s^3) \tag{60}$$

where

$$
\begin{aligned}
c_1 =& -16(4\pi)^2(L_4^r - 2L_6^r) - \frac{2}{9}\log\left[\frac{4B_0 m_s}{3\mu^2}\right] \\
& - \left\{16(4\pi)^2(2L_4^r + L_5^r - 4L_6^r - 2L_8^r) + \frac{1}{9} + \log\left[\frac{4}{3}\right] - \frac{8}{9}\log\left[\frac{4B_0 m_s}{3\mu^2}\right] - \log\left[2Q\right]\right\} Q \\
& - \left\{\frac{1}{36}\right\} Q^2 + \mathcal{O}(Q^3)
\end{aligned}
\tag{61}
$$

$$
c_2 = c_2^{\text{tree}} + c_2^{\text{loop}} \tag{62}
$$

and

$$
\begin{aligned}
\frac{c_2^{\text{tree}}}{64(4\pi)^4} =& -C_{16}^r + C_{20}^r + 3C_{21}^r + 4L_4^r(L_4^r - 2L_6^r) \\
& - \left\{2C_{13}^r + C_{15}^r - 2C_{20}^r - 12C_{21}^r - 2C_{32}^r - 8\left(L_4^r(2L_4^r + L_5^r - 4L_6^r - L_8^r) - L_5^r L_6^r\right)\right\} Q \\
& - \Big\{2C_{12}^r + 4C_{13}^r + C_{14}^r + 2C_{15}^r + 2C_{16}^r + C_{17}^r - 3C_{19}^r - 6C_{20}^r - 12C_{21}^r - 2C_{31}^r \\
& \quad - 4C_{32}^r - 4\left(2L_4^r + L_5^r\right)\left(2L_4^r + L_5^r - 4L_6^r - 2L_8^r\right)\Big\} Q^2 + \mathcal{O}(Q^3)
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
c_2^{\text{loop}} =& \frac{11}{12}\log^2\left[\frac{B_0 m_s}{\mu^2}\right] - \left(\frac{32}{9}\mathcal{C}_1^{(0)} + \frac{380}{81} - \frac{2}{9}\log\left[\frac{4}{3}\right]\right)\log\left[\frac{B_0 m_s}{\mu^2}\right] - \frac{38}{81}\log\left[\frac{4}{3}\right] \\
& + \frac{2}{9}\log^2\left[\frac{4}{3}\right] + \frac{16}{9}\left(\mathcal{C}_2^{(0)} - 2\log\left[\frac{4}{3}\right]\mathcal{C}_3^{(0)}\right) + \frac{73}{16} - \frac{2}{3}F\left[\frac{4}{3}\right] \\
& + \Big\{\frac{97}{54}\log^2\left[\frac{B_0 m_s}{\mu^2}\right] - \left(\frac{16}{9}\mathcal{C}_1^{(1)} + \frac{1549}{162} + \frac{5}{27}\log\left[\frac{4}{3}\right]\right)\log\left[\frac{B_0 m_s}{\mu^2}\right] - \frac{407}{324}\log\left[\frac{4}{3}\right] \\
& + \frac{8}{27}\log^2\left[\frac{4}{3}\right] - \frac{8}{9}\left(\mathcal{C}_2^{(1)} + 2\log\left[\frac{4}{3}\right]\mathcal{C}_3^{(1)}\right) + \frac{1075}{648} - \frac{79}{144}F\left[\frac{4}{3}\right] \\
& - \left(16\mathcal{C}_4^{(1)} + \frac{4}{9}\log\left[\frac{4}{3}\right] - \frac{5}{9}\log\left[\frac{B_0 m_s}{\mu^2}\right]\right)\log[2Q]\Big\} Q \\
& + \Big\{\frac{1165}{108}\log^2\left[\frac{B_0 m_s}{\mu^2}\right] - \left(\frac{8}{9}\mathcal{C}_1^{(2)} + \frac{6347}{324} - \frac{7}{54}\log\left[\frac{4}{3}\right]\right)\log\left[\frac{B_0 m_s}{\mu^2}\right] - \frac{11663}{6912} \\
& - \frac{71117}{82944}\log\left[\frac{4}{3}\right] - \frac{1}{54}\log^2\left[\frac{4}{3}\right] + \frac{4}{9}\left(\mathcal{C}_2^{(2)} - 4\log\left[\frac{4}{3}\right]\mathcal{C}_3^{(2)}\right) - \frac{1373}{36864}F\left[\frac{4}{3}\right] \\
& - \left(\frac{8}{9}\mathcal{C}_4^{(2)} + \frac{27}{2} - \frac{1}{3}\log\left[\frac{4}{3}\right] - \frac{119}{6}\log\left[\frac{B_0 m_s}{\mu^2}\right]\right)\log[2Q] + \frac{17}{2}\log^2[2Q]\Big\} Q^2 + \mathcal{O}(Q^3)
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
\mathcal{C}_1^{(0)} &= (4\pi)^2 \left(26L_1^r + \frac{13}{2}L_2^r + \frac{61}{8}L_3^r - 29L_4^r - \frac{13}{2}L_5^r + 30L_6^r - 6L_7^r + 11L_8^r\right) \\
\mathcal{C}_2^{(0)} &= (4\pi)^2 \left(\frac{13}{2}L_2^r + \frac{43}{24}L_3^r + 2L_4^r + \frac{4}{3}L_5^r - 4(L_6^r + L_7^r + L_8^r)\right) \\
\mathcal{C}_3^{(0)} &= (4\pi)^2 \left(8L_1^r + 2(L_2^r + L_3^r) - 11L_4^r - 2L_5^r + 12L_6^r - 6L_7^r + 2L_8^r\right)
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
\mathcal{C}_1^{(1)} &= (4\pi)^2 \left(88L_1^r + 22L_2^r + \frac{53}{2}L_3^r - 76L_4^r - 26L_5^r + 72L_6^r + 52L_8^r\right) \\
\mathcal{C}_2^{(1)} &= (4\pi)^2 \left(88L_1^r + \frac{62}{3}L_3^r - 86L_4^r - \frac{74}{3}L_5^r + 80L_6^r - 28L_7^r + 40L_8^r\right) \\
\mathcal{C}_3^{(1)} &= (4\pi)^2 \left(16L_1^r + 4(L_2^r + L_3^r) - 31L_4^r - 8L_5^r + 36L_6^r + 16L_8^r\right) \\
\mathcal{C}_4^{(1)} &= (4\pi)^2 \left(3L_4^r - 4L_6^r\right)
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
\mathcal{C}_1^{(2)} &= (4\pi)^2 \left(332L_1^r + 164L_2^r + \frac{301}{2}L_3^r - 200L_4^r - 78L_5^r + 312L_6^r + 24L_7^r + 164L_8^r\right) \\
\mathcal{C}_2^{(2)} &= (4\pi)^2 \left(-204L_1^r + 32L_2^r - \frac{151}{3}L_3^r + 203L_4^r + \frac{100}{3}L_5^r - 148L_6^r - 22L_7^r - 74L_8^r\right) \\
\mathcal{C}_3^{(2)} &= (4\pi)^2 \left(4L_1^r + L_2^r + L_3^r - 10L_4^r - 3L_5^r + 12L_6^r + 12L_7^r + 10L_8^r\right) \\
\mathcal{C}_4^{(2)} &= (4\pi)^2 \left(252L_1^r + 144L_2^r + 126L_3^r - 108L_4^r - 54L_5^r + 216L_6^r + 108L_8^r\right)
\end{aligned} \tag{67}
$$

From Eq.(60) we obtain the matching for B, which agrees completely with [15] in the chiral limit:

$$\frac{B}{B_0} = 1 + c_1 \left[\frac{m_s B_0}{(4\pi F_0)^2}\right] + c_2 \left[\frac{m_s B_0}{(4\pi F_0)^2}\right]^2 + \mathcal{O}(m_s^3) \tag{68}$$

6 Numerical Analysis

We present in this section a numerical analysis of the expressions given in the preceding sections, and discuss some of their implications.

6.1 F_π

We begin by giving a breakdown of the relative numerical contributions of the different terms constituting the $\mathcal{O}(p^6)$ term of F_π. As the expressions used in sections 3 and 4 of

$d^{\pi}_{\pi KK}$	$d^{\pi}_{\pi\eta\eta}$	$d^{\pi}_{KK\eta}$	d^{π}_{sunset}	$d^{\pi}_{log\times log}$	d^{π}_{log}	Sum
−93.227	−0.028	100.891	−0.381	1.825	−8.891	−7.447

Table 1: Numerical contributions (in units of 10^{-6} GeV^4) of different terms to $\left(\overline{F}_{\pi}\right)^{(6)}_{\text{loop}}$, the parts not depending on LECs. The inputs to these were $m_{\pi} = m_{\pi^0} = 0.1350$ GeV, $m_K = m_K^{\text{avg}} = 0.4955$ GeV, $m_{\eta} = 0.5479$ GeV and $F_{\pi} = F_{\pi\ \text{phys}} = 0.0922$ GeV. The renormalization scale $\mu = 0.77$ GeV.

Fit	$d^{\pi}_{log\times L_i}$	$d^{\pi}_{L_i}$	$d^{\pi}_{L_i\times L_i}$	Sum L_i	Sum
BE14exact	7.475	0.064	0.817	8.356	0.909
BE14paper	7.456	0.072	0.841	8.372	0.925
free-fit	12.052	0.391	2.817	15.260	7.813
CQMfit	12.851	0.461	−0.702	12.611	5.164

Table 2: Numerical contributions (in units of 10^{-6} GeV^4) of different terms to $\left(\overline{F}_{\pi}\right)^{(6)}_{\text{loop}}$, the part depening on the LECs. The inputs are the same as in Table 1.

[1] correspond to those expressed in physical meson masses, we use the physical values of the masses. The caption of Table 2 gives the numerical input values we used. Our expressions are exact except for the approximation used for $d^{\pi}_{KK\eta}$. The value calculated using the approximate expression Eq. 29) agrees with using precise numerical expressions for the sunset integrals in Eq. (28) to 8 significant digits. The parts that do not depend on the LECs are given in Table 1. The large cancellations are due to the terms that diverge for $m_{\pi} \to 0$.

The most recent fit of LECs with a number of different assumptions are given in Ref. [23]. Their main fit is called BE14 and can be found in Table 3 [23]. We show results both for the exact fit results (BE14exact) and with the two digit precision given in the reference (BE14paper). The free fit in Table 2 in [23] was done with L_4^r free and a slightly different choice of p^6 LECs, this fit we call free-fit and finally we take the fit with the p^6 LECs estimated with a chiral quark model of Table 2 in [23], labelled CQMfit. The results for the three L_i^r-dependent contribution, their sum and the sum including the contributions from Table 1 are given in Table 2.

We examine the contributions calculated using the BE14exact LECs. The largest contribution arises from the d_{log} term, followed by the $d_{log\times L_i}$ term. The sign of these two terms being opposite, however, reduces the overall contribution of the explicitly μ-scale dependent terms to the decay constant. In absolute value terms, the bilinear chiral log terms $d_{log\times log}$ provide the next largest contribution. The bilinear L_i terms are of an order of magnitude smaller. The sunsets have a relatively small contribution in absolute value terms, but due to cancellations of the other contributions, the value of d_{sunset} is little over a third of the total contribution to the sum.

The sum of the contributions calculated using BE14exact (free-fit) LECs yields:

$$\begin{aligned}\frac{F_\pi}{F_0} &= 1 + \overline{F}_\pi^{(4)} + \left(\overline{F}_\pi^{(6)}\right)_{\text{loop}} + \left(\overline{F}_\pi^{(6)}\right)_{\text{CT}} \\ &= 1 + 0.2085(0.3143) + 0.0126(0.1081) + 0.0755(0.0193) \\ &= 1 + 0.2085(0.3143) + 0.0881(0.1274) \\ &= 1.2966(1.4414) \end{aligned} \tag{69}$$

The value given in [23] is:

$$\frac{F_\pi}{F_0} = 1 + 0.208(0.313) + 0.088(0.127) \tag{70}$$

which agrees excellently with our representation. Note that the last term has been calculated with exact p^6 LECs as used in [23].

The numerical values calculated using the free-fit LECs demonstrate the sensitivity of the two-loop contribution to F_π to the values of the LECs. In particular, it is to be noted that L_4^r and L_6^r are difficult to determine precisely, and the free fit values for these two low energy constants have relatively large uncertainties. The variation of $(\overline{F}_\pi^{(6)})_{\text{loop}}$ with L_4^r and L_6^r over their possible range in the free fit is shown in Figures 2 and 3. The trend is of a progressively smaller value of $(\overline{F}_\pi^{(6)})_{\text{loop}}$ for increasing L_6^r and decreasing L_4^r. A more thorough fit and detailed analysis of the LECs with the F_π representation is planned for the future aftre a similar representation for the kaon and eta have been obtained.

The dependence of F_π/F_0 on M_K^2 given in Eq.(59), with $M_K = 0.4955$ GeV and F_0 on the r.h.s. replaced by the physical $F_{\pi\ \text{phys}}$, has the following numerical form in the chiral limit:

$$\frac{F}{F_0} = 1 + 0.1499(0.2562) + 0.0157(-0.0516) + ... \tag{71}$$

The first set of numbers correspond to the use of the BE14exact LECs, while the numbers in parentheses are calculated using the free fit. Figure 4 shows the M_K dependence of F/F_0 using these inputs, keeping $F_0 = F_\pi$ fixed on the. A significant divergence in the two sets of values is observed as M_K^2 increases.

The largest contribution to F/F_0 at $\mathcal{O}(m_s^2)$ comes from the d_2^{tree} term, followed by the term proportional to $\log(B_0 m_s/\mu^2)$. In absolute terms, the pure number contribution to d_2 is greater than that of the $(-11/12)\log(B_0 m_s/\mu^2)$ term, but its sign being negative, the pure number serves to decrease the numerical size of d_2, as do all the remaining terms as well. Ignoring the terms proportional to the L_i in d_2^{loop}, one gets a value of -1.4244 for d_2, in contrast to 0.4698 when the L_i proportional terms are retained. The L_i therefore contribute significantly to the $\mathcal{O}(M_K^2)$ contribution to F_π.

The effect of the higher order terms in Q can be seen by comparing comparing Eq.(71) with Eq.(75) below, which gives numerical values for F_π/F_0. We use a value of $Q =$

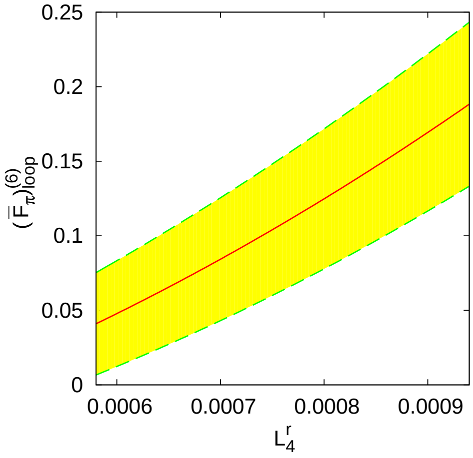

Figure 2: L_4^r dependence of $(\overline{F}_\pi^{(6)})_{\text{loop}}$. The full line is the value for $L_6^r = 0.49 \times 10^{-3}$, while the shaded area indicates the range of possible values corresponding to the ± 0.25 uncertainty of L_6^r in the free fit.

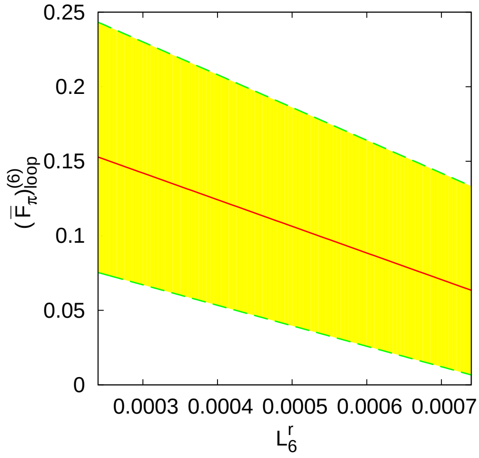

Figure 3: L_6^r dependence of $(\overline{F}_\pi^{(6)})_{\text{loop}}$. The dashed line is the value for $L_4^r = 0.76 \times 10^{-3}$, while the shaded area indicates the range of possible values corresponding to the ± 0.18 uncertainty of L_4^r in the free fit.

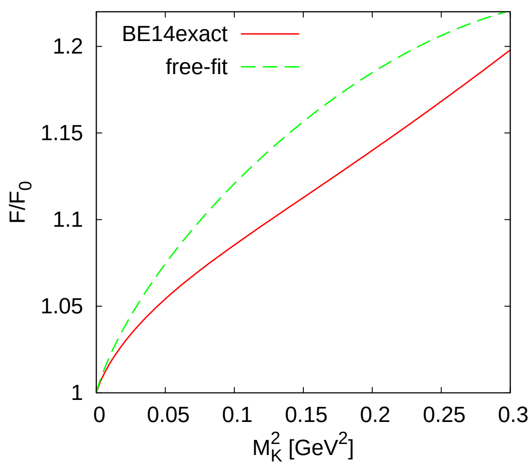

Figure 4: M_K^2 dependence of F/F_0 in the chiral limit.

$C^{\pi}_{\pi KK}$	$C^{\pi}_{\pi\eta\eta}$	$C^{\pi}_{KK\eta}$	C^{π}_{sunset}	$C^{\pi}_{log\times log}$	C^{π}_{log}	Sum
11.721	0.009	−10.781	0.774	0.312	2.272	3.358

Table 3: Numerical contributions (in units of 10^{-7} GeV^6) of different terms to $(m_\pi^2)^{(6)}_{\text{loop}}$, the parts not depending on LECs. The inputs are the same as in Table 1.

$\hat{m}/m_s = 1/24.4$ obtained from [24], the numerical value of d_1, Eq.(52), with corrections upto $\mathcal{O}(Q^2)$ is:

$$\begin{aligned} d_1 &= 0.8198(1.4009) + 0.3454(0.3425) - 0.0108(-0.0107) \\ &= 1.1544(1.7327) \end{aligned} \tag{72}$$

Similarly,

$$\begin{aligned} d_2^{tree} &= 2.5022(-0.0863) - 0.3229(-0.2641) + 0.0170(0.0129) \\ &= 2.1963(-0.3375) \end{aligned} \tag{73}$$

$$\begin{aligned} d_2^{loop} &= -2.0324(-1.4574) - 0.0180(-0.1834) - 0.0729(-0.0718) \\ &= -2.1233(-1.7126) \end{aligned} \tag{74}$$

Note that the $\mathcal{O}(Q)$ contribution of d_2^{loop} evaluated using the BE14exact LECs is numerically smaller than the $\mathcal{O}(Q^2)$. Note too that the $\mathcal{O}(Q)$ value calculated using the free fit value differs from the one calculated using BE14exact by an order of magnitude. Putting it all together we obtain up to $\mathcal{O}(Q^2, s^2)$ the following expansion:

$$\frac{F_\pi}{F_0} = 1 + 0.2111(0.3169) + 0.0024(-0.0686) + \cdots \tag{75}$$

gives a more accurate numerical representation of the effect on F_π of integrating the strange-quark mass out. The effect of the correction due to $\hat{m}$ to the chiral limit is particular pronounced at $\mathcal{O}(Q^2)$, with the value of the chiral limit number at this order given in Eq.(71) calculated using the BE14 fit differs from its analogous value in Eq.(75) by one order of magnitude, due to cancelations between the different parts.

6.2 m_π^2

An analysis of the expression for the pion mass produces the numerical results given in Table 3 and 4. The large cancellations in the sunset contributions follow from the fact that the separate parts do not vanish in the limit $m_\pi \to 0$ but their sum does. Except for CQMfit which was not a good fit in [23], the largest contribution comes from the pure logarithmic terms, the contribution of which, however, is cancelled to a large degree by the $\log \times L_i$ term of similar magnitude but opposite sign. The bulk of the net contribution to

Fit	$c^{\pi}_{log\times L_i}$	$c^{\pi}_{L_i}$	$c^{\pi}_{L_i\times L_i}$	Sum L_i	Sum
BE14exact	−1.681	−0.023	−0.002	−1.707	1.652
BE14paper	−1.717	−0.026	−0.005	−1.748	1.610
free-fit	−1.283	−0.142	−0.231	−1.657	1.701
CQMfit	1.570	−0.168	−3.844	−2.442	0.916

Table 4: Numerical contributions (in units of 10^{-6} GeV^4) of different terms to $(m^2_\pi)^{(6)}_{\text{loop}}$, the part depening on the LECs. The inputs are the same as in Table 1.

$(M^{(6)}_\pi)_{\text{loop}}$ therefore comes from the sunsets diagrams and the bilinears in the chiral logs. The c_{L_i} and $c_{L_i\times L_j}$ contribute very little. Using the BE14exact (free-fit) LECs, we get:

$$\begin{aligned}\frac{M^2_\pi}{m^2_\pi} &= 1.057(0.940) + (m^2_\pi)^{(4)} + (m^2_\pi)^{(6)}_{\text{loop}} + (m^2_\pi)^{(6)}_{\text{CT}} \\ &= 1.057(0.940) - 0.0051(0.1044) + 0.1254(0.1292) - 0.1769(-0.1732) \\ &= 1.057(0.940) - 0.0051(0.1044) - 0.0515(-0.0440)\,. \end{aligned} \tag{76}$$

The lowest order term is determined by having the right hand side sum to 1. This agrees well with the numerical values given in [23].

Numerically, with $\sqrt{m_s B_0} = 0.484$ GeV, $F_0 = 0.0922$ GeV and BE14exact (free-fit) LECs, we have for the expansion given in Eq.(68) in the chiral limit:

$$\frac{B}{B_0} = 1 + 0.0197(0.1219) - 0.0586(-0.1027) + ... \tag{77}$$

Figure 5 shows the m_s dependence of B/B_0 for two sets of LECs, BE14exact and free-fit. Both sets of LECs produce the same general behaviour, but are different numerically.

7 Conclusions

In this work, we have used the explicit representations of the two loop contribution to the pion decay constant and mass in three flavour ChPT [1] to derive (semi-)analytic expressions for them. That it is semi-analytic and not fully analytic stems from the fact that we treated the three mass configuration sunset integrals appearing in them as an expansion in the square of the external momentum and have retained only the first few terms. This semi-analytic representation is nonetheless very accurate and numerically reproduces the full result to a high degree [1, 2].

We have used these expressions to expand F_π and M_π in the strange quark mass to $\mathcal{O}(m_s^2)$ and to perform the matching of two flavour low energy constants B and F with their three flavour counterparts in the chiral limit. The results obtained fully agree with those previously derived in [5, 15, 25].

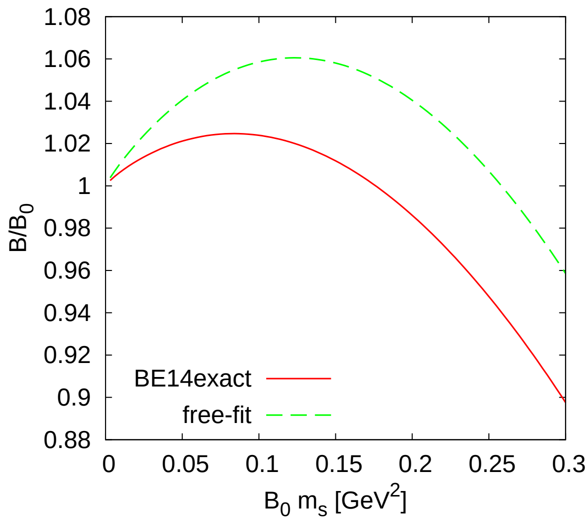

Figure 5: m_s dependence of M_π^2/m_π^2 in the chiral limit

Aside from an investigation of the numerical implications of the strange quark expansion of both F_π and B_0, we have also done a preliminary study of the dependence of F_π on the low energy constants L_4^r and L_6^r. These show trends that are possibly in contradiction with the large N_c analysis of these LECs, and a more detailed study needs to be done. The breakdown of the relative numerical contributions to the pion decay constant at two loops shows that the contribution from the terms involving the L_i^r and C_i^r, although not large, is not insignificant. Their contribution is amplified partially due to the cancellation of other terms that have a larger absolute value. Furthermore, in the chiral limit m_s expansion, the terms proportional to the low energy constants contribute greatly to the $\mathcal{O}(m_s^2)$ term. All these point to the need for a thorough study into the dependence of such quantities on the LECs for a better understanding of the chiral perturbation series.

In forthcoming work, we will present similar semi-analytic expressions for the three flavour two-loop contributions to the kaon and eta mass and decay constants, and use those results and the ones presented in this work to do a preliminary fit of lattice data to obtain new values for some low energy constants. That exercise, along with the results and analyses presented in this work, are indicative of the usefulness of such analytic representations of ChPT amplitudes and other quantities, and will hopefully encourage and facilitate the lattice community in making use of full NNLO results from ChPT.

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A Two Mass Sunset Master Integrals

The finite parts of the master integrals appearing in the expressions for $d_{\pi KK}$ and $d_{\pi\eta\eta}$ are presented here. The chiral logarithms arising from these integrals do not appear in the expressions below, having been removed and included in the c_{log}, $c_{log\times log}$, d_{log} or $d_{log\times log}$ term as appropriate.

$$\begin{aligned}\overline{H}^{\chi}_{\pi KK} = \frac{m_K^2}{(16\pi^2)^2}\Bigg(& 2 + \frac{\pi^2}{6} + \frac{m_\pi^2}{m_K^2}\left(\frac{\pi^2}{12} - \frac{1}{8}\right) - \frac{m_\pi^2}{2m_K^2}\log^2\left[\frac{m_\pi^2}{m_K^2}\right] + \log\left[\frac{m_\pi^2}{m_K^2}\right] \\ &+ \left(\frac{m_K^2}{m_\pi^2} + \frac{m_\pi^2}{m_K^2} - 2\right)\left(\mathrm{Li}_2\left[\frac{m_\pi^2}{m_K^2}\right] + \log\left[1 - \frac{m_\pi^2}{m_K^2}\right]\log\left[\frac{m_\pi^2}{m_K^2}\right]\right)\Bigg)\end{aligned} \tag{78}$$

$$\begin{aligned}\overline{H}^{\chi}_{2\pi KK} = \frac{1}{(16\pi^2)^2}\Bigg(& \frac{\pi^2}{12} - \frac{1}{2} - \frac{1}{2}\log^2\left[\frac{m_\pi^2}{m_K^2}\right] \\ &+ \left(1 - \frac{m_K^2}{m_\pi^2}\right)\left(\mathrm{Li}_2\left[\frac{m_\pi^2}{m_K^2}\right] + \log\left[\frac{m_\pi^2}{m_K^2}\right]\log\left[1 - \frac{m_\pi^2}{m_K^2}\right]\right)\Bigg)\end{aligned} \tag{79}$$

The expressions for $\overline{H}^{\chi}_{\pi\eta\eta}$ and $\overline{H}^{\chi}_{2\pi\eta\eta}$ can be obtained from the above by making the replacement $m_K \to m_\eta$.

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