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**A BLACK HOLE RADIATING COLOR CHARGED  
PARTICLES**  
Bachelor thesis

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## Abstract

This thesis makes an attempt to extend Hawking radiation, which has previously been studied for photons and QED charged leptons, to emission of particles with color charge. A sufficiently hot black hole should be able to radiate asymptotically free quarks and gluons in addition to leptons and photons. The black holes in consideration in this thesis are those hot enough to radiate color charged particles which can be described by perturbative quantum chromodynamics (QCD) but macroscopic enough to have a static metric during the emission so that the process can be treated semiclassically. For the purpose of this thesis the kinematics of the emission of color charged particles will not be significant since the goal is to compare QCD to QED. Therefore the color structure is studied separate from the kinematics. The calculations are performed using the birdtrack notation which is a pictorial representation of  $SU(N)$  tensor calculations. The radiation is assumed to be gluons, emitted one at a time, and from a black hole of low dimensional representation of the color group. A methodology for how to answer these questions is supplied and in addition an outline to a solution.

## Populärvetenskaplig beskrivning

I Einsteins allmänna relativitetsteori uttrycks gravitation som rumtidens krökning. Källan till gravitation är energitäthet; ju tätare materia är koncentrerat desto starkare gravitation uppstår. I extrema fall kan materia bli så tät att rummet och tiden kröks så att ett svart hål bildas. Vid ett visst avstånd från ett svart hål finns en händelsehorisont. När något passerar händelsehorisonten är rumtidens krökning så stor att inte ens ljus från det kan nå ut. Precis utanför horisonten kan kvantmekaniska fluktuationer av partiklar och antipartiklar ske ur vakuum. Sådana fluktuationer sker i vakuum överallt under korta tider, inte bara kring svarta hål. Just i situationer nära händelsehorisonter kan dessa fluktuationer ge upphov till påtagliga fysikaliska effekter.

När en av partiklarna passerar händelsehorisonten förlorar denna och dess partner kontakten. Detta hindrar paret från att återgå till vakuum och den partikel som är kvar utanför horisonten kan undkomma det svarta hålet. Denna ström av partiklar är en värmestrålning och är lik det ljus som glödheta kroppar strålar. Svarta hål strålar alltså ut fotoner, ljuspartiklar, men också partiklar med elektrisk laddning och dessutom partiklar med andra liknande laddningar som förekommer i partikelfysik. En sådan liknande laddning är färgladdning (som bara till namnet har med färger i vardagen att göra). Denna uppsats behandlar hur strålning av sådana färgladdade partiklar skulle gå till.

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# 1 Introduction

The idea that black holes should have thermodynamic properties such as temperature and entropy was first proposed by Jacob Bekenstein in 1972. Stephen Hawking soon followed up with a calculation that black holes should radiate particles due to quantum fluctuations and that the spectrum of this radiation should be thermal. Those ideas gave rise to various calculations for the emission of photons and leptons.

Due to the non-Abelian nature of QCD the emission of quarks and gluons is more complicated. The aim of this report is to study if the QCD emission case can be, by studying the  $SU(3)$  group structure separately, reduced to the QED case. In QED there is a tendency for the emissions of charged particles to bring the black hole to less spin and less electric charge. The aim of this thesis is to verify if a QCD charged black hole has the tendency to emit particles in such a way that it probably is in a lower dimensional representation of the color group after emission. A naive comparison between the coupling constants suggest that QCD has this same property as QED but this needs not at all hold true under careful scrutiny. This is due to the fact that QCD is non-Abelian, which will make it so qualitatively different that the comparison of coupling strengths is little more than initial speculation.

Since the Hawking temperature of the black hole scales as its inverse mass and is proportional to  $\hbar$ , macroscopic black holes will be very cold. This thesis uses QCD in the perturbative limit and thus the black hole emitting the quarks and gluons, should have a temperature such that the emitted particles will likely have energies where they are asymptotically free. Since the QCD scale parameter is  $\Lambda_{QCD} = 220\text{MeV}$ , the Planck mass (times  $c^2$ ) is  $M_{\text{P}}c^2 = 2.4 \cdot 10^{18}\text{GeV}$ , and Newton's constant is  $G = \frac{\hbar c}{M_{\text{P}}^2}$  then for a Hawking temperature of  $k_{\text{B}}T = \frac{\hbar c^3}{8\pi GM} = \Lambda_{QCD}$  the mass would be  $\frac{M_{\text{P}}^2 c^2}{8\pi \Lambda_{QCD}} \approx 10^{18}M_{\text{P}}$ . This means that there are black holes hot enough to radiate perturbative quarks and gluons. Also, at such high masses the change in mass due to radiation of the black hole can be neglected and the metric is to good approximation static. The black hole cannot be comparable to the Planck scale either, which is where the Schwarzschild radius and Compton wavelength are comparable, since this would require Quantum Gravity to be taken into consideration.

This thesis starts by giving a background to groups and representations, with QCD in particular focus, as well as the notation called birdtracks. An introduction to Hawking radiation of both neutral and QED charged particles is also given and used to set up a formalism for simulating QED emissions from black holes that reproduces known results. Finally the birdtrack for-

malism is used to take the first steps to extending the emission model to QCD.

## 2 Theory

### 2.1 Units and conventions

In this thesis so called natural units are used. In those units  $\hbar = c = k_B = 1$  where  $\hbar$  is Planck's constant,  $c$  is the speed of light, and  $k_B$  is Boltzmann's constant. All dimensional quantities are measured in some power of electron volts (eV). Newton's constant,  $G$ , in natural units has dimension of inverse squared mass, i.e.  $G = 1/M_P^2$ . The mass  $M_P$  is called the Planck mass.

This thesis will employ the Einstein summation convention. This is the notion that whenever a specific index appears on one side of an equation exactly twice then there is an implicit summation over all allowed values of that index e.g.

$$f^{abc}t^c \equiv \sum_{c=1}^N f^{abc}t^c. \quad (2.1)$$

This expression will turn up later in the report where  $f^{abc}$  and  $t^c$  are the so-called structure constants, and generators, respectively, of a structure called a Lie group.

### 2.2 Groups and representations

The treatment here follows the book on groups and Lie algebra by Georgi [1], which contains a much more thorough introduction to the subject. The idea of a group is central to the current understanding of fundamental physics. Mathematically a group  $\mathcal{G}$  is a set of elements and a composition, the operation "·", of ordered pairs of elements fulfilling the axioms:

- (i) **Closure:** For every pair of elements  $a, b \in \mathcal{G} \Rightarrow a \cdot b \in \mathcal{G}$
- (ii) **Associativity:**  $a, b, c \in \mathcal{G} \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- (iii) **Identity:** There is an element  $e \in \mathcal{G}$  so that  $\forall g \in \mathcal{G}: e \cdot g = g \cdot e = g$
- (iv) **Inverse:** For every  $g \in \mathcal{G}$  there is a  $g^{-1}$  such that  $g^{-1} \cdot g = g \cdot g^{-1} = e$ .

A group with the property that  $a \cdot b = b \cdot a \quad \forall a, b \in \mathcal{G}$  is called an Abelian group. A group without this property is called non-Abelian.

A representation  $\Gamma$  of a group is a map from the elements of the group to some set of objects that preserves the group structure. This means that  $\Gamma(g \cdot g') = \Gamma(g) \cdot \Gamma(g')$  and  $\Gamma(e) = \mathbf{1}$  where  $\mathbf{1}$  is the identity of the space i.e.  $\mathbf{1} \cdot \Gamma(g) = \Gamma(g)$ . For practical purposes the representations of a group are often taken to be matrices.

### 2.3 Lie groups, gauge groups and SU(3)

A Lie group is a group containing a continuous infinity of group elements that can be characterized by smooth, differentiable functions of a finite number of parameters. By going to a matrix representation,  $\Gamma$ , and considering group elements close to the identity (defining  $g(0, \dots, 0) = e$ ), one can characterize the group elements by Taylor expansion i.e.

$$\Gamma[g(\delta\theta_1, \dots, \delta\theta_N)] \equiv \Gamma(\underline{\theta}) = \mathbf{1} + \left. \frac{\partial \Gamma(\underline{\theta})}{\partial \theta_a} \right|_{\underline{\theta}=0} \delta\theta_a + \dots \equiv \mathbf{1} + i\delta\theta_a t^a + \dots \quad (2.2)$$

where the so-called generators  $t^a$  are defined. The generators are a closed set under commutation; they form a Lie algebra

$$[t^a, t^b] = i f^{abc} t^c \quad (2.3)$$

where the  $f^{abc}$  are called the structure constants of the group. The Lie algebra is derived from the fact that  $\Gamma(g \cdot g') = \Gamma(g)\Gamma(g')$  and the Taylor expansion of the group elements. Given a representation for the Lie algebra, the representation of any element in the group (that is continuously connected to the identity element) can be found by taking infinitely many infinitesimal steps from the identity so that

$$g(\theta_1, \dots, \theta_N) = \lim_{n \rightarrow \infty} \left( \mathbf{1} + \frac{i\theta_a t^a}{n} \right)^n = e^{i\theta_a t^a}. \quad (2.4)$$

A gauge group is a group of local transformations, i.e. the coefficients in the linear combination of generators are functions of spacetime. QCD is a non-Abelian gauge theory based on  $SU(3)$ , called the color force. The  $U$  stands for unitary;  $U(N)$  is the group defined by that it can faithfully be represented by  $N \times N$  matrices,  $u$ , with the property  $u^\dagger u = uu^\dagger = \mathbf{1}$ . The  $S$  stands for special and by this it is meant that  $SU(3)$  is the subgroup of  $U(3)$  that fulfills the additional condition that  $\det(u) = 1$ .

Familiar from quantum mechanics is the addition of angular momentum. The different possible angular momenta in quantum mechanics are determined by the representation theory for the group  $SU(2)$ . For instance, the angular momentum commutation relations are  $[J_i, J_j] = i\epsilon_{ijk} J_k$ . This is the

Lie algebra of  $SU(2)$  so in the language of group theory the angular momentum operators are the generators of the rotations on the quantum states. A system consisting of two spins,  $j_1$  and  $j_2$ , is a so-called direct product space of two spin spaces written as  $j_1 \otimes j_2$ . It is also known that this system can be written in terms of a basis which has definite total angular momentum. For instance, for two spin  $\frac{1}{2}$  particles one can choose a basis that decomposes the system into a spin 0 and a spin 1 system that transform independently under rotations, written as

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1.$$

This procedure of choosing a basis in which the total angular momentum operator is diagonal is called Clebsch-Gordan decomposition and carries directly over to group theory. Here the representations are characterized by dimensionality (i.e.  $2j + 1$ ) rather than spin so the tensor product of two spin  $\frac{1}{2}$  systems is written as

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}.$$

In QCD each quark has a color quantum number. This means that the quark field  $\psi_i$  for  $i = 1, 2, 3$  comes in three different color kinds. A quark can then be associated with a three entry column matrix in the so-called color space

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_{\text{red}} \\ \psi_{\text{green}} \\ \psi_{\text{blue}} \end{pmatrix} \quad (2.5)$$

where the  $\psi_i$ :s are Dirac spinor fields. The color symmetry is assumed to be a local symmetry, i.e. a gauge symmetry, so that the theory is invariant under rotations in color space where the rotations can be different at different points in spacetime i.e.  $\psi(x) \rightarrow U(x)\psi(x)$  while  $\bar{\psi}(x) \rightarrow \bar{\psi}(x)U^\dagger(x)$  for the antiquark field  $\bar{\psi}$ . The generators for  $U(x)$ ,  $T^a$ , is any basis for Hermitian, traceless  $3 \times 3$  matrices (e.g. the so-called Gell-Mann matrices).

The Lagrangian (density) for QCD is

$$\mathcal{L} = \sum_f \bar{\psi}_i^f (\delta_j^i i\gamma^\mu \partial_\mu + g\gamma^\mu G_\mu^a (T^a)_{ij} - m_f \delta_{ij}) \psi_j^f - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \quad (2.6)$$

where  $f$  is the sum over quark species (flavors),  $i$  and  $j$  are color space indices,  $G_\mu^a$  for  $a = 1, 2, \dots, 8$  are the gluon fields,  $\gamma^\mu$  are the usual Dirac matrices and  $g$  is the strong gauge coupling parameter.  $G_{\mu\nu}^a$  are the gluon field strengths defined by

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + gf^{abc} G_\mu^b G_\nu^c. \quad (2.7)$$



These also transform under gauge transformations. Under an infinitesimal transformation, i.e.  $U(x) = e^{iT^a\chi^a(x)}$  for some set  $\chi^a$  of infinitesimal parameters, the fields transform as

$$\psi_i \rightarrow \psi_i + i\chi^a(T^a)_{ij}\psi_j \quad (2.8)$$

$$\bar{\psi}_i \rightarrow \bar{\psi}_i - i\chi^a\bar{\psi}_j(T^a)^*_{ji} \quad (2.9)$$

$$G_\mu^a \rightarrow G_\mu^a + \frac{1}{g}\partial_\mu\chi^a - f^{abc}\chi^b G_\mu^c. \quad (2.10)$$

These equations define the most important representations of  $SU(3)$  in nature called  $\mathbf{3}$ ,  $\bar{\mathbf{3}}$  and  $\mathbf{8}$  in which the matrix elements of the generator  $t^a$  are  $(T^a)_{ij}$ ,  $-(T^a)^\dagger_{ij}$  and  $if^{abc}$  respectively.

## 2.4 Black holes, Hawking temperature, and particle emission

Black holes need general relativity for an accurate description. An introduction to the general theory of relativity can be found in [5]. In general relativity the metric  $g_{\mu\nu}$  in the relativistically invariant spacetime distance  $ds^2 = dx^\mu g_{\mu\nu} dx^\nu$  is promoted to a field, i.e. it is a function of spacetime,  $g_{\mu\nu}(x)$ , and is interpreted as the gravitational field. The metric outside a static, spherically symmetric gravitating object is in general relativity given by the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{1}{1 - 2GM/r} dr^2 - r^2 d\Omega^2 \quad (2.11)$$

where  $t$  is the coordinate time,  $r$  is a coordinate representing radii of the spheres of the spherical symmetry (the generalization of a radial coordinate) and  $\Omega$  is the solid angle. If the gravitating object is localized within the Schwarzschild radius  $r_S = 2GM$  the metric has a coordinate singularity at the surface  $r = r_S$ . This singularity is called the event horizon and the gravitating body is called a black hole.

The name black hole arises from the fact that there are no timelike or lightlike trajectories from spacetime points within the event horizon to outside it i.e. no particle can travel out of the event horizon once it is inside, thus making the body black. Combining this generally relativistic notion of a black hole with the principles of quantum mechanics makes the prediction of a phenomenon called Hawking radiation. Due to quantum fluctuations near the horizon a thermal distribution of particles are detected by a faraway observer; the black hole seems to radiate. An argument for this statement is given in the next section where the Unruh temperature is outlined and then generalized to the Hawking temperature.

## 2.5 The Unruh effect and Hawking temperature

The Unruh effect is a special relativistic effect saying that a uniformly accelerated observer in Minkowski spacetime will see a thermal distribution of particles where an inertial observer would see the vacuum. The reason for this temperature is that the field equation describing the particles will have different orthogonal function expansions for the inertial and the accelerated observer and therefore the Fock creation and annihilation operators will be different. Different annihilation operators will result in different vacuum states and particle number in the same state for the different observers. Thus the Minkowski ground state will be a superposition of so-called Rindler states, which are states perceived by a uniformly accelerated observer, of thermal particle number distributions.

To see how this arises consider the toy theory of a massless scalar field in 1+1 dimensional Minkowski spacetime. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (2.12)$$

(with Minkowski metric  $\eta_{00} = 1; \eta_{01} = \eta_{10} = 0; \eta_{11} = -1$ ). The equation of motion is  $\square\phi = 0$  and a basis for solutions is the plane wave Fourier basis  $e^{\pm ik \cdot x}$  with  $k^0 = |k^1| \equiv \omega_k$ . The quantum field is then

$$\phi(x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} \left( a_k e^{-ik \cdot x} + a_k^\dagger e^{ik \cdot x} \right) \Big|_{k^0 = \omega_k}, \quad (2.13)$$

where  $a_k^\dagger$  and  $a_k$  are the usual creation and annihilation operators for particles in momentum state  $k$  and the (Minkowski) vacuum is defined by  $a_k |0_M\rangle = 0; \forall k$ .

The worldline of a particle experiencing uniform proper acceleration,  $\alpha$ , can be written as

$$t(\tau) = \frac{1}{\alpha} \sinh(\alpha\tau) \quad \text{and} \quad x(\tau) = \frac{1}{\alpha} \cosh(\alpha\tau). \quad (2.14)$$

A suitable coordinate system, called Rindler coordinates, for an accelerated frame is then given by  $\tau$  and  $\xi$  defined by

$$t = \frac{1}{\alpha} e^{\alpha\xi} \sinh(\alpha\tau) \quad \text{and} \quad x = \frac{1}{\alpha} e^{\alpha\xi} \cosh(\alpha\tau). \quad (2.15)$$

In this frame the metric is given by  $ds^2 = e^{2\alpha\xi} (d\tau^2 - d\xi^2)$  and the equation of motion becomes  $0 = \nabla_\mu \nabla^\mu \phi = e^{-2\alpha\xi} (\partial_\tau^2 - \partial_\xi^2) \phi$  where  $\nabla_\mu$  is the coordinate covariant derivative. In this coordinate system one can find new Fourier

modes of the field equation. Associated with these modes come creation and annihilation operators  $a_{\text{R},k}$ ,  $a_{\text{R},k}^\dagger$  and a vacuum  $|0_{\text{R}}\rangle$ . Now it can be seen that the mode expansions in the inertial and Rindler frames do not coincide and they will thus have different notions of particle number in the given states and what the vacuum is. The details are supplied by Carroll [3]. The expectation value of particle number of a Rindler observer in the inertial vacuum is shown to be

$$\langle 0_M | a_{\text{R},k}^\dagger a_{\text{R},k} | 0_M \rangle = \frac{1}{e^{2\pi\omega_k/\alpha} - 1} 2\pi\delta(0) = \frac{\langle 0_M | a_k a_k^\dagger | 0_M \rangle}{e^{2\pi\omega_k/\alpha} - 1}. \quad (2.16)$$

The  $\delta(0)$  factor arises from the non-proper norm of the momentum states. The expectation number of particles detected by an observer with uniform acceleration then has the thermal Bose-Einstein occupation number i.e.

$$\langle N(\omega) \rangle = \frac{1}{e^{\omega/T} - 1} \quad (2.17)$$

with the Unruh temperature

$$T = \frac{\alpha}{2\pi}. \quad (2.18)$$

The thermal factor should be a bosonic occupation number, which can be traced to the so-called spin-statistics theorem which states that particles of integer spin (like scalars) are bosons.

The equivalence principle of general relativity can be used to argue that the Hawking effect should arise from the Unruh effect since gravity and uniform acceleration are equivalent locally. If an observer falling freely near the horizon sees the Minkowski vacuum then an observer stationary with respect to the horizon will behave as an accelerated observer in special relativity. This observer will see a thermal spectrum of particles emitted from the black hole horizon. This thermal distribution of particles can escape and reach a distant observer at a finitely redshifted temperature. The redshifted acceleration that the distant observer measures is equal to the surface gravity. For a Schwarzschild black hole this is given by  $\alpha = \frac{1}{4GM}$  so the Hawking temperature is

$$T_H = \frac{1}{8\pi GM}. \quad (2.19)$$

## 2.6 Emission of electrically charged leptons from black holes

Page [2] presents the following equation for the emission of QED charged leptons from non-rotating black holes

$$\frac{\partial N_{\pm}}{\partial t \partial \omega} = \frac{1}{2\pi} \sum_i \frac{\Gamma_i(\omega)}{e^{8\pi GM\omega \mp 4\pi\alpha Z} + 1}. \quad (2.20)$$

Here  $N_{\pm}$  is the number of charged leptons of charge  $\pm|e|$  emitted,  $\omega$  is the energy of the lepton,  $\Gamma_i$  is a so-called greybody factor (its frequency dependence prevents the radiation from being purely blackbody),  $\alpha = \frac{e^2}{4\pi}$ , the sum over  $i$  is a sum over angular momentum modes of the lepton, and  $Z$  is defined by  $Q = Z|e|$  where  $Q$  is the charge of the black hole after emission. The thermal occupation is now  $\frac{1}{e^{\omega/T} + 1}$  from Fermi-Dirac statistics in a direct generalization of equation (2.16). The origin of  $4\pi\alpha Z$  is the electrostatic potential energy at the surface of magnitude  $\frac{eQ/4\pi r_s}{T} = \frac{eQ/8\pi GM}{1/8\pi GM} = eQ = Ze^2 = 4\pi Z\alpha$ .

The peak of the distribution of the product  $GM\omega$  is shown by Page [2] to be much larger than zero so that the  $+1$  in the denominator can be ignored as long as  $\alpha Z \ll 1$ . Carroll [3] states that for a scalar field the greybody factor is

$$\Gamma(\omega) \simeq 1; \quad \omega \gg \frac{1}{GM} \quad (2.21)$$

$$\Gamma(\omega) \simeq \frac{A\omega^2}{4\pi}; \quad \omega \ll \frac{1}{GM} \quad (2.22)$$

where the factors arise from backscattering of waves and  $A$  is the surface area of the black hole. Since fermions and gauge bosons both obey the Klein-Gordon equation (in addition to their own equations of motion) these expressions for grey-body factors can give an estimate for other particles.

If the greybody factor can be treated as constant then

$$\frac{\partial N_{\pm}}{\partial t \partial \omega} \approx \frac{1}{2\pi} \Gamma_{\text{effective}} e^{-8\pi GM\omega \pm 4\pi\alpha Z} \quad (2.23)$$

where  $\Gamma_{\text{effective}}$  is the effective greybody factor. Furthermore, this can be averaged over all energies of the emitted lepton yielding

$$\frac{dN_{\pm}}{dt} = \text{constant} \cdot e^{\pm 4\pi\alpha Z}. \quad (2.24)$$

This equation allows for a simple interpretation. The product in the exponent is the product of the charge of the black hole (after emission) and the charge

of the emitted lepton. The black hole repels virtual leptons of the same charge as itself and is therefore more likely to radiate leptons of the same charge.

Since the model assumes  $Z$  to be the charge after emission of the charged particle and it remains  $Z$  until the charged particle can be detected by a far away observer one has assumed that the black hole emits charged particles one at the time. The same assumption is assumed to carry over to the study of gluon emission in QCD.

## 2.7 Birdtracks

Birdtracks are diagrammatic ways to perform group theoretic calculations. In appearance birdtrack resemble Feynman diagrams but in spirit they are different. While Feynman diagrams symbolize numbers or integrals that have to be written down using Feynman rules, birdtracks, on the other hand, is a way to perform group theoretic calculations in a diagrammatic fashion. The birdtrack notation is thoroughly developed in [4].

Representations of a group element will have matrix indices (in various vector spaces) which in birdtrack notation will be symbolized by external lines in the diagram. The generators of the group has a notation analogue to Feynman diagram vertices. To exemplify, in the birdtrack notation the generator in the  $\mathbf{3}$  representation,  $T^a$ , is written

$$(T^a)_{ij} = \begin{array}{c} i \longrightarrow \bullet \longrightarrow j \\ \downarrow \\ \text{---} \\ \downarrow \\ a \end{array} \quad (2.25)$$

and the so-called adjoint representation of the same generator (i.e. the generator for the  $\mathbf{8}$  representation) is represented by

$$(T^{(\text{adj})})_{bc} = if^{abc} = \begin{array}{c} a \text{---} \bullet \text{---} c \\ \downarrow \\ \text{---} \\ \downarrow \\ b \end{array} . \quad (2.26)$$

Matrix multiplication in the color indices is represented by connecting the vertices by internal lines so for instance

$$(T^a)_{ik}(T^b)_{kj} = \begin{array}{c} i \longrightarrow \bullet \longrightarrow \bullet \longrightarrow j \\ \downarrow \quad \downarrow \\ \text{---} \quad \text{---} \\ \downarrow \quad \downarrow \\ a \quad b \end{array} . \quad (2.27)$$

Thus internal lines represent sums over the states in color space. Closing the identity (2.27) (i.e. contracting by inserting the tensor  $\delta_{ij}\delta^{ab}$ ) yields a color

invariant known as a Wigner  $3j$ -coefficient. Those are structures of the form

$$\begin{array}{c} \mu \\ \circlearrowleft \\ \lambda \\ \circlearrowright \\ \nu \end{array} \quad (2.28)$$

where  $\mu$ ,  $\nu$  and  $\lambda$  label representations. In this thesis the normalization of the vertices is chosen such that the non-zero Wigner  $3j$ -coefficients are all equal to one. Another important contraction of vertices is a trace over four of them, called the Wigner  $6j$ -coefficient

$$\begin{array}{c} \mu \\ \circlearrowleft \\ \beta \quad \alpha \\ \circlearrowright \\ \lambda \\ \circlearrowleft \\ \sigma \quad \nu \end{array} \quad (2.29)$$

One reason to choose this normalization of the vertices is that the basis for color space become orthonormal and another is the availability of Wigner coefficients in this normalization. In the Wigner coefficients as well as in the remainder of the thesis a general representation of  $SU(3)$  is depicted by a double line while a gluon is represented by a single line (replacing the usual curly line used above). This is illustrated in the birdtrack below

$$\begin{array}{c} \text{---} \\ \text{=} \\ \text{=} \end{array} \begin{array}{c} \mathbf{8} \\ \mu \end{array} \quad (2.30)$$

Two equations that are of great importance to this report are the completeness relation

$$\begin{array}{c} \mu \\ \text{=} \\ \text{=} \\ \nu \\ \text{=} \\ \text{=} \end{array} = \sum_{\lambda} \frac{d_{\lambda}}{d_{\mu}} \begin{array}{c} \mu \\ \circlearrowleft \\ \lambda \\ \circlearrowright \\ \nu \end{array} \begin{array}{c} \mu \\ \text{=} \\ \text{=} \\ \nu \end{array} \quad (2.31)$$

and an identity based on Schur's lemma

$$\begin{array}{c} \mu \\ \text{=} \\ \text{=} \\ \nu \end{array} = \frac{1}{d_{\mu}} \delta_{\mu, \nu} \begin{array}{c} \mu \\ \text{=} \\ \text{=} \\ \mu \end{array} \quad (2.32)$$

The completeness relation is founded in the statement that the Clebsch-Gordan decomposition of a direct product into a direct sum is complete. In both cases  $d_\alpha$  is the dimension of the representation  $\alpha$ . The proof of these relations and more on the birdtrack notation can be found in [4].

### 3 Method

#### 3.1 Simulation of QED emission from Schwarzschild black holes

From equation (2.24) one can see that the emission rate is unchanged while the system is between emissions. Thus the average number of emitted particles of charge  $\pm|e|$  in time  $t$  is

$$N_\pm = \text{constant} \cdot t e^{\pm 4\pi\alpha Z}. \quad (3.1)$$

The probability that the next particle emitted from the black hole has a charge of  $\pm|e|$  will then be  $P_\pm = N_\pm / (N_+ + N_-)$  i.e.

$$P_\pm = \frac{e^{\pm 4\pi\alpha Z}}{e^{4\pi\alpha Z} + e^{-4\pi\alpha Z}}. \quad (3.2)$$

The emission process can be simulated numerically by drawing random numbers and iterate over many emitted particles. From this one can get values for the root-mean-square and the average charge of the black hole. For simplicity the time between emission of charges is assumed to be constant. This is not completely correct since higher values of charge of the black hole will lead to faster emissions but will hold as long as  $\alpha Z \ll 1$  so that the emission probabilities are not radically different.

#### 3.2 Gluon exchange and diagonalization of the exchange matrix

If the black hole in QED only emits leptons one at the time then the charge of the black hole is unchanged by any exchange of virtual photons as can be seen in (3.3) (where the charge  $q = Q - e$ ). This is, however, not the case in QCD since the virtual exchange of gluons (like all gluons) carry color charge. This means that the black hole can change not only color charge state within a given representation of the color  $SU(3)$  but also change representation by the exchange of virtual gluons. This is illustrated in (3.4)

$$(3.3)$$

$$(3.4)$$

where  $\mathbf{M}$  labels the  $SU(3)$  representation of the black hole.

Below the emission of gluons from black holes is studied under the assumption that the gluons are emitted one at a time. That is, the emitted gluon reaches a distant observer and the state collapses before a new gluon is emitted. The gluons are also exchanged one at the time meaning that processes where two virtual gluons exist at the same time (i.e. another exchange gluon is emitted before the first exchange gluon is absorbed) are disregarded. Thus the exchange is of the form (3.4).

After emitting a gluon the black hole will have some color charge and be in some representation of  $SU(3)$ . Since nothing collapses until the gluon reaches a distant observer the black hole will be in a superposition of color representations. If the black hole was initially in  $\mathbf{8}$  then it is immediately after emission in a linear combination of states in  $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$ . The color structure immediately after the emission of a gluon by the black hole is thus

$$\mathbf{A} = \sum_i c_i \mathbf{M} \text{---} \text{---} \mathbf{M}_i$$

for some  $c_i$  and where  $\mathbf{M} = \mathbf{8}$ . This color structure is a vector in a vector space and a natural first choice of basis is

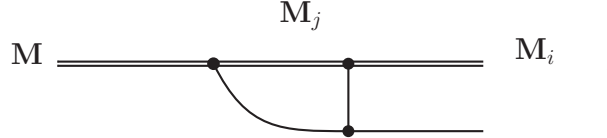
$$\mathbf{e}^i = \mathbf{M} \text{---} \text{---} \mathbf{M}_i \quad (3.5)$$

were  $i = 1, 2, 3, 4, 5, 6$  for  $\mathbf{M}_i = \mathbf{1}, \mathbf{8}(\mathbf{f}), \mathbf{8}(\mathbf{d}), \mathbf{10}, \overline{\mathbf{10}}, \mathbf{27}$ . The representations  $\mathbf{8}(\mathbf{f})$  and  $\mathbf{8}(\mathbf{d})$  are the states coupling with a completely antisymmetric and completely symmetric triple gluon vertex, respectively. The above basis is referred to as the multiplet basis.

In the following section the summation convention is no longer in effect, that is expressions like  $G_{ij}\mathbf{e}^i$  do not have an implicit summation attached.



This is in order to be able to have quantities of twice (or more) repeated indices without explicitly stating that no summation is intended. The implicit summations inherent to the birdtrack notation are still performed. After the gluon is emitted a virtual gluon can be exchanged between the black hole and the emitted gluon



The new state is an element of the same vector space since it has the same external lines (tensor indices). Therefore, given some state,  $\mathbf{A} = \sum_i A_i \mathbf{e}^i$ , before exchange the state after,  $\mathbf{A}' = \sum_i A'_i \mathbf{e}^i$ , is given by some linear operator acting on the original state  $\mathbf{A}$ . This can be written

$$\mathbf{A}' = \sum_i A'_i \mathbf{e}^i = \sum_i \sum_j \mathbf{e}^i G_{ij} A_j \quad (3.6)$$

where  $G_{ij}$  are the elements of a matrix which will be referred to as the gluon exchange matrix. One way to find  $G_{ij}$  is to study the action of gluon exchange on  $\mathbf{e}^j$ , i.e. to treat the projections of  $\mathbf{A}$  onto the basis vectors one at a time.

$$\mathbf{e}^i G_{ij} = \text{M} \begin{array}{c} \text{M}_j \\ \text{---} \end{array} \text{M}_i \quad (3.7)$$

Inserting the completeness relation, equation (2.31), on the right yields

$$\mathbf{e}^i G_{ij} = \sum_{\alpha} \frac{d_{\alpha}}{\text{M}_i} \text{M} \begin{array}{c} \text{M}_j \quad \text{M}_i \quad \alpha \quad \text{M}_i \\ \text{---} \end{array} \quad (3.8)$$

By connecting the  $\mathbf{M}$  and the  $\alpha$  lines with each other by using equation (2.32) one arrives at

$$\mathbf{e}^i G_{ij} = \sum_{\alpha} \frac{d_{\alpha}}{\text{M}_i} \frac{\delta_{\mathbf{M},\alpha}}{d_{\alpha}} \begin{array}{c} \text{M} \\ \text{---} \\ \text{M}_j \quad \text{M}_i \end{array} \text{M} \begin{array}{c} \text{M}_i \\ \text{---} \end{array} \quad (3.9)$$

Letting the sum collapse using the Kronecker delta

$$\mathbf{e}^i G_{ij} = \frac{1}{\text{Diagram 1}} \cdot \text{Diagram 2} \cdot \text{Diagram 3} \quad (3.10)$$

and recognizing the basis vector  $\mathbf{e}^i$  yields

$$\mathbf{e}^i G_{ij} = \frac{\mathbf{e}^i}{\text{Diagram 1}} \cdot \text{Diagram 2} \quad (3.11)$$

Since this must be true for any of the basis vectors  $i$  the identity must be an equation for the matrix element

$$G_{ij} = \frac{1}{\text{Diagram 1}} \cdot \text{Diagram 2} \quad (3.12)$$

This expression for the gluon exchange matrix is the primary result of this thesis. In what follows an outline for how gluon emissions affects the (final) color representations of black holes is given.

Due to symmetries of the Wigner 6j-coefficients the matrix  $G_{ij} = G_{ji}$  is symmetric and can thus be diagonalized by a orthogonal matrix,  $R_{\tilde{m}i}$ . The basis vectors of the new basis are  $\mathbf{e}^{\tilde{m}}$  and  $\mathbf{A} = \sum_{\tilde{m}} A_{\tilde{m}} \mathbf{e}^{\tilde{m}}$ . In this basis the gluon exchange matrix is diagonal  $G_{\tilde{m},\tilde{n}} = \sum_i \sum_j R_{\tilde{m}i} G_{ij} (R^T)_{j\tilde{n}} = g_{\tilde{m}} \delta_{\tilde{m},\tilde{n}}$  where  $g_{\tilde{m}}$  is the eigenvalue of the exchange matrix belonging to  $\mathbf{e}^{\tilde{m}}$ . In this basis the gluon exchange matrix acts as trivially as possible on the states, i.e.

$$A'_{\tilde{m}} = \sum_{\tilde{n}} G_{\tilde{m},\tilde{n}} A_{\tilde{n}} = g_{\tilde{m}} A_{\tilde{m}}. \quad (3.13)$$

Upon emission of a gluon, the probability of being in state  $\mathbf{e}^{\tilde{m}}$  is proportional to  $|A_{\tilde{m}}|^2$  and the analogue of the QED charge weight is  $e^{4\pi g_{\tilde{m}} \alpha_s}$  so the probability of the state being  $\mathbf{e}^{\tilde{m}}$  is

$$P(\tilde{m}) = \frac{|A_{\tilde{m}}|^2 e^{4\pi g_{\tilde{m}} \alpha_s}}{\sum_{\tilde{n}} |A_{\tilde{n}}|^2 e^{4\pi g_{\tilde{n}} \alpha_s}}. \quad (3.14)$$

However, the expression in equation (3.14) is not complete. No compensation has been made for the choice of normalizing the  $3j$ -coefficients to one. This choice affects the vertex factors and makes them different from the usual QCD normalization. The vertices in the  $6j$ -coefficients in the gluon exchange matrix will change value and the matrix elements and eigenvalues will be different. This can, in principle, be compensated for but it is left for future work.

The amplitude for the different states is proportional to the square root  $\sqrt{P(\tilde{m})}$ . It is possible to imagine a phase factor on this amplitude but since the origin of such a phase must be kinematic it is the same for all states. Thus it is possible to make the choice to be unity.

Since the state is not collapsed between the initial emission of a gluon and the exchange of gluons between the emitted gluon and the black hole, the probability amplitudes must be summed before they are norm-squared. Just before the state collapses, by the initially emitted gluon reaching a distant observer, it is

$$\sum_{\tilde{m}} A_{\tilde{m}} \sqrt{P(\tilde{m})} \mathbf{e}^{\tilde{m}}. \quad (3.15)$$

When the gluon reaches a distant observer and the state collapses it does so into a color representation eigenstate. The probability amplitude for going from the state in (3.15) to the black hole being in representation  $\mathbf{M}_i$  is the scalar product with  $\mathbf{e}^i$ . Thus the probability is

$$\begin{aligned} P(\mathbf{M} \rightarrow \mathbf{M}_i) &= \left| \mathbf{e}^i \cdot \sum_{\tilde{m}} A_{\tilde{m}} \sqrt{P(\tilde{m})} \mathbf{e}^{\tilde{m}} \right|^2 \\ &= \left| \sum_{\tilde{n}} R_{\tilde{n}i} \mathbf{e}^{\tilde{n}} \cdot \sum_{\tilde{m}} A_{\tilde{m}} \sqrt{P(\tilde{m})} \mathbf{e}^{\tilde{m}} \right|^2 = \left| \sum_{\tilde{m}} \sum_{\tilde{n}} R_{\tilde{n}i} A_{\tilde{m}} \sqrt{P(\tilde{m})} \delta_{\tilde{m},\tilde{n}} \right|^2 \\ &= \left| \sum_{\tilde{m}} R_{\tilde{m}i} A_{\tilde{m}} \sqrt{P(\tilde{m})} \right|^2. \end{aligned}$$

Here it has been used that due to the choice of normalization of  $3j$ -coefficients the basis  $\mathbf{e}^i$  is orthonormal and thereby the basis  $\mathbf{e}^{\tilde{m}}$  also has this property since they are related by an orthogonal transformation.

## 4 Results

### 4.1 Emission of QED charged particles from Schwarzschild black holes

The method described in section 3.1 was used to calculate the root mean square electric charge of a black hole. A simulation using  $10^8$  emitted particles (corresponding to  $10^8$  iterations in the program) yielded a root-mean-square of electric charge on the black hole being 2.382 in units of the elementary charge. The average charge was  $2.89 \cdot 10^{-4}$  in the same units, consistent with the zero charge plus the statistical  $1/\sqrt{N}$  sample spread.

The results of the simulation of the QED emission seem to be consistent with the results by Page [2] (at 2.33 elementary charges). The reproduction of the result in the QED case was performed in order to verify that the method gave the correct prediction in a domain where the predictions are well-known. This becomes important when the simulation will be generalized to QCD where the predictions are unknown.

### 4.2 Gluon exchange matrix and vertex normalization

The choice of normalizing all the QCD vertices in such a way that the  $3j$ -coefficients are equal to one will affect the elements of the gluon exchange matrix. This is essentially due to the fact that setting the  $3j$ -coefficients to one obscures how strongly the different multiplet representations couples to the gluon. All these normalization issues will affect the probability of finding the black hole in a given multiplet representation of the color group. Even so, the method for finding the gluon exchange matrix is schematically correct and can be used as a foundation for upcoming attempts to figure out the QCD properties of black holes.

## 5 Conclusions and Outlook

Many steps remain before anything conclusive can be said about gluon emission from black holes. For instance, the normalization chosen here, setting the Wigner  $3j$ -coefficients to unity, is not the standard choice in QCD and in particular not one that is consistent with a universal strong coupling constant. The trouble with this choice is determining how the components of the vector  $\mathbf{A}$  in the  $e^{\tilde{n}}$  basis would look and how the vertices has to be compensated for. Since the amplitudes for finding the black hole in different representations is greatly dependent on the structure of this vector these

issues need to be sorted out in order to know the color properties of black holes.

This thesis has covered gluon emissions only. For completeness quark emissions must also be studied. These emissions are not treated in this thesis; the most obvious problem is that the Wigner  $6j$ -coefficients for the  $\mathbf{3}$  and  $\bar{\mathbf{3}}$  contracted with higher multiplet representations were not available. The gluon exchange matrix for quark emissions could be obtained by only a small modification of the analogous derivation for gluons.

## References

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