

LU TP 14-13  
June 4, 2014

# SUPERSYMMETRY IN QUANTUM MECHANICS

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## Abstract

This thesis gives an introduction to the basic formalism of one-dimensional supersymmetric quantum mechanics. The factorization of a Hamiltonian is used to create a supersymmetric partner Hamiltonian. The connections between the energy spectra and wave functions of these partner Hamiltonians are deduced and examined for the case of broken and unbroken supersymmetry. An extension to hierarchies of Hamiltonians is made and used to describe shape invariant potentials.

The formalism is used to solve some textbook examples like the infinite square well and the harmonic oscillator potential in a new way and to determine the wave functions and energy levels of the hydrogen atom in a nonrelativistic and a relativistic treatment.

A two-dimensional extension of the formalism is introduced and applied to find a way to solve the eigenvalue problem for a matrix Pauli Hamiltonian through its scalar partner Hamiltonians. The two-dimensional formalism is further used to examine a chain of two-dimensional real singular Morse potentials and to determine the wave functions and energy spectra based on the solution of the one-dimensional Morse potential.

## Populärvetenskapligt sammanfattning

A quantum mechanics course belongs to the main parts of undergraduate physics studies and the content is necessary as a basis in all fields of modern physics. However the treatment of supersymmetric quantum mechanics as described in this thesis does not in general belong to the curriculum although it offers different and sometimes easier solutions to problems that are solved in a quantum mechanics course.

The basis of supersymmetric quantum mechanics was set in theoretical particle physics. Many of the properties of the universe can today be described by the standard model of particle physics that includes the known particles and forces that build up our universe. The study of their properties is a very active field and news from the Large Hadron Collider at CERN like the confirmation of the existence of the Higgs particle cause a lot of interest. Despite its great success, the standard model is not able to describe all processes in the world of particle physics and theoretical physicists try to extend the model to be able to predict and explain these processes. A theory that arose was supersymmetry. It states that every one of the basic particles, which can be distinguished between fermions and bosons, has a partner with the exact mass that is of the other kind. This model allows it to explain some of the processes that are not included in the standard model. Unfortunately the concept has not been confirmed experimentally, today none of the predicted superparticles have been observed which would have been the case if they had the same mass as their already known counterparts. The only explanation that can save the concept of supersymmetry is a spontaneous symmetry breaking. The search for possible breaking mechanisms and their mathematical description led to the development of supersymmetric quantum mechanics.

The formalism of this concept is used in this thesis to solve well-known problems in quantum mechanics in a new and elegant way and to find solutions for problems that cannot be handled with other methods. This thesis starts with presenting the easiest application, the one-dimensional treatment. The formalism is introduced and its use is shown by solving some examples. The second part of the thesis handles two-dimensional problems which are the first step to a general higher-dimensional description of supersymmetric quantum mechanics. For example it is shown that it is possible to solve certain two-dimensional problems just by knowing the solution of a simpler one-dimensional problem and the formalism of supersymmetric quantum mechanics.

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# 1 Introduction

Supersymmetry (SUSY) is a concept that was developed in particle physics. The advantage of this model is its ability to give answers to questions that cannot be explained with the standard model of particle physics. It postulates a symmetry between half-integer spin fermions and integer spin bosons where each boson and fermion is supposed to have a superpartner with the same mass. This symmetry has not been observed yet in nature so it has to be spontaneously broken. In 1981 Edward Witten proposed a simple quantum mechanical model to study a possible breaking mechanism for SUSY in his article *Dynamical breaking of supersymmetry* [1] and the new idea of supersymmetric quantum mechanics has grown to a research field on its own.

This thesis gives an insight into the basic formalism of supersymmetric quantum mechanics and shows the application of the formalism for some exemplary problems. Based on the work in [2], [3], [4], [5] and [6] it starts with the factorization of the one-dimensional Hamiltonian as a first step to the creation of partner Hamiltonians and potentials. The properties of these partner potentials are evaluated and discussed and the special case of broken supersymmetry is described. The next step is the extension of the formalism for two partner Hamiltonians to a whole hierarchy of Hamiltonians which are connected via supersymmetry. This allows to introduce the concept of shape invariant potentials (SIP) which can be used to determine the properties of all members of a chain of Hamiltonians and to algebraically solve their spectrum just based on the properties of the first one .

The possibility of constructing a partner potential with an energy spectrum that is nearly identical to the original one is first applied to the well-known infinite square well. The wave functions and energy levels of this standard text book example are used to determine the properties of the new potential. As a next step the harmonic oscillator potential is used to present the application of the SIP-formalism. A sequence of partner potentials is constructed and the energy levels of all potentials are determined.

Afterwards the properties of the radial Coulomb potential in a hydrogen atom are determined. In the first case this is done in a nonrelativistic manner and the formalism of supersymmetric quantum mechanics allows the determination of the energy levels and wave functions in accordance with the solutions which are obtained via the normal textbook calculations. The next step is the relativistic treatment of the same potential. This is shown based on the work done in [3], [7] and [8]. The problem can be treated just like the far simpler potentials and is a good example for the possibility to use the introduced formalism to find a simpler solution than the ones in textbook examples.

In the last part of the thesis a higher dimensional treatment of supersymmetric quantum mechanics as described in [9] is introduced. The extension of the formalism to two dimensions is shown and the relations between the new partner Hamiltonians are derived. An analogy of the matrix Hamiltonian that appears in the two-dimensional treatment can be found in the Pauli Hamiltonian for the movement of a fermion in two dimensions. It is shown that the scalar partner potentials can be used to solve the eigenvalue problem for the Pauli Hamiltonian. The formalism is also used to handle a two-dimensional model of a Morse potential which is introduced in [10] and [11]. This model contains two partner po-

tentials that depend on two variables. The solution of one of the potentials via separation of variables allows the determination of the wave functions and energy levels of the partner potential. Afterwards the potentials are examined for shape invariance and this property is used to determine the wave functions and energy levels of a whole chain of Hamiltonians.

## 2 Formalism of Supersymmetric Quantum Mechanics

### 2.1 Factorization of the One Dimensional Hamiltonian

The exact solution of one-dimensional potential problems is a basic task in quantum mechanics. The eigenvalue of the Hamiltonian  $H^{(1)}$  acting on a known ground state wave function  $\psi_0(x)$  is the ground state energy  $E_0^{(1)}$ . If  $\psi_0(x)$  is assumed to be nodeless and normalizable and if  $E_0^{(1)}$  is shifted to zero, applying of  $H^{(1)}$  yields

$$H^{(1)}\psi_0(x) = -\frac{\hbar^2}{2m} \frac{d^2\psi_0}{dx^2} + V^{(1)}(x)\psi_0(x) = 0, \quad (2.1)$$

from which it is possible to reconstruct the potential  $V^{(1)}(x)$ :

$$V^{(1)}(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0(x)}. \quad (2.2)$$

The Hamiltonian  $H^{(1)}$  can be factorized into two operators

$$H^{(1)} = A^\dagger A \quad (2.3)$$

where

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \quad \text{and} \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad (2.4)$$

with the superpotential  $W(x)$  which is connected with the potential  $V^{(1)}(x)$ . The relation can be found by comparing the two different forms of  $H^{(1)}$ :

$$\begin{aligned} H^{(1)}\psi(x) = A^\dagger A\psi(x) &= \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}} W'(x) + W(x)^2 \right) \psi_0(x) \\ &= \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x) \right) \psi(x) \end{aligned} \quad (2.5)$$

The resulting differential equation

$$V^{(1)}(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x). \quad (2.6)$$

is known as the Riccati equation. The general solution of this equation can be found when a special solution is known. Knowing that  $H^{(1)}\psi_0 = A^\dagger A\psi_0 = 0$  is fulfilled when  $A\psi_0 = 0$ , leads with (2.4) to a solution for  $W(x)$ :

$$W(x) = \frac{-\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)} = \frac{-\hbar}{\sqrt{2m}} \frac{d(\ln \psi_0(x))}{dx}. \quad (2.7)$$

The ground state wave function  $\psi_0$  can be denoted as zero mode of  $A$ . The differential equation (2.7) can be used to calculate  $\psi_0$  with a known superpotential

$$\psi_0^{(1)}(x) = N \exp\left(-\frac{\sqrt{2m}}{\hbar} \int^x W(x') dx'\right), \quad (2.8)$$

where  $N$  is a normalization constant.

## 2.2 Partner Hamiltonians and Potentials

Exchanging the order of the two operators in the factorized Hamiltonian generates the supersymmetric partner Hamiltonian

$$\begin{aligned} H^{(2)} = AA^\dagger &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W(x)^2 + \frac{\hbar}{\sqrt{2m}} W'(x) \\ &\equiv -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(2)}(x) \end{aligned} \quad (2.9)$$

with the partner potential

$$V^{(2)}(x) = W(x)^2 + \frac{\hbar}{\sqrt{2m}} W'(x). \quad (2.10)$$

These partner Hamiltonians are related not only by the superpotential but also by their energy eigenvalues and wave functions. The energy eigenvalues of the Hamiltonians  $H^{(1)}$  and  $H^{(2)}$  are both positive-semidefinite ( $E_n^{(1,2)} \geq 0$ ). Starting with  $n > 0$  the Schrödinger equation for  $H^{(1)}$

$$H^{(1)}\psi_n^{(1)} = A^\dagger A\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)} \quad (2.11)$$

helps to find an energy eigenvalue equation for  $H^{(2)}$  which connects the eigenvalues of both Hamiltonians:

$$H^{(2)}(A\psi_n^{(1)}) = AA^\dagger A\psi_n^{(1)} = E_n^{(1)}(A\psi_n^{(1)}). \quad (2.12)$$

The same can be done for the Schrödinger equation for  $H^{(2)}$

$$H^{(2)}\psi_n^{(2)} = AA^\dagger\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)} \quad (2.13)$$



implies

$$H^{(1)}(A^\dagger\psi_n^{(2)}) = A^\dagger AA^\dagger\psi_n^{(2)} = E_n^{(2)}(A^\dagger\psi_n^{(2)}) \quad (2.14)$$

Equations (2.12) and (2.14) show that  $A^\dagger\psi_n^{(2)}$  is an eigenstate of  $H^{(1)}$  and  $A\psi_n^{(1)}$  is an eigenstate of  $H^{(2)}$  respectively. Using this result and equations (2.11)-(2.14) it is now possible to include the case  $n = 0$  with  $E_n^{(1)} = 0$  and formulate the relations between the two Hamiltonians for  $n = 0, 1, 2, \dots$

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad E_0^{(1)} = 0, \quad (2.15)$$

$$\psi_n^{(2)} = \left(E_{n+1}^{(1)}\right)^{-\frac{1}{2}} A\psi_{n+1}^{(1)}, \quad (2.16)$$

$$\psi_{n+1}^{(1)} = \left(E_n^{(2)}\right)^{-\frac{1}{2}} A^\dagger\psi_n^{(2)}, \quad (2.17)$$

where the normalization constants can be gained from the norm of the eigenfunctions, for example

$$\|A\psi_{n+1}^{(1)}\|^2 = \langle\psi_{n+1}^{(1)}|A^\dagger A|\psi_{n+1}^{(1)}\rangle = \langle\psi_{n+1}^{(1)}|H^{(1)}|\psi_{n+1}^{(1)}\rangle = E_{n+1}^{(1)}. \quad (2.18)$$

The relations

$$AH^{(1)} = AA^\dagger A = H^{(2)}A \quad \text{and} \quad H^{(1)}A^\dagger = A^\dagger AA^\dagger = A^\dagger H^{(2)} \quad (2.19)$$

which are used for the calculations of the eigenvalues above are called intertwining relations. This way of expressing the connection between the Hamiltonians is used in the two-dimensional treatment which is described later.

Equation (2.15) shows that the energy spectra of both Hamiltonians are nearly exactly the same, the only difference is an additional ground state of  $H^{(1)}$ . The operators  $A$  and  $A^\dagger$  are used to switch between the two systems.  $A$  maps the eigenfunction of  $H^{(1)}$  to the partner system and destroys one node of the eigenfunction. Vice versa  $A^\dagger$  maps an eigenfunction of  $H^{(2)}$  to the system of  $H^{(1)}$  and creates an additional node. This makes it possible to determine all eigenfunctions and energy eigenvalues of the second system when the first system is known. This works vice versa except for the ground state wave function of  $H^{(1)}$ . The energy spectra and the mapping are illustrated in figure 1

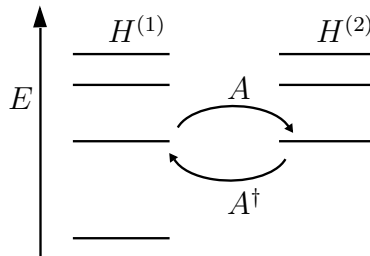


Figure 1: Scheme of the energy spectra of two supersymmetric partner Hamiltonians and their connection via the operators  $A$  and  $A^\dagger$ .

This coincidence of the spectra has its origin in the algebra of supersymmetry. The Hamiltonians and the factorization operators can be presented as elements of matrix  $2 \times 2$  operators, the superhamiltonian  $\mathbf{H}$  and the supercharges  $Q^\pm$

$$\mathbf{H} = \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(2)} \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \quad (2.20)$$

In this form the elements describe the supersymmetric algebra [2] and obey the commutation and anticommutation relations

$$[\mathbf{H}, Q] = [\mathbf{H}, Q^\dagger] = 0, \quad \{Q, Q^\dagger\} = \mathbf{H}, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \quad (2.21)$$

As described in the introduction supersymmetry creates a connection between bosons and fermions or bosonic and fermionic states and the algebra mirrors this. The two Hamiltonians in the superhamiltonian are the Hamiltonians of the bosonic state and the fermionic states respectively. The total wave function

$$\psi_n = \begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix}; \quad \begin{array}{l} \psi_n^{(1)} : \text{ bosonic} \\ \psi_n^{(2)} : \text{ fermionic} \end{array} \quad (2.22)$$

contains the wave functions of both states. The supercharges are the symmetry operators which change bosonic in fermionic states and vice versa

$$Q|\text{boson}\rangle \propto |\text{fermion}\rangle \quad \text{and} \quad Q^\dagger|\text{fermion}\rangle \propto |\text{boson}\rangle. \quad (2.23)$$

Apparently the left spectrum in figure 1 belongs to the bosonics states and the right one to the fermionic. The ground state is bosonic.

## 2.3 Broken Supersymmetry

Although SUSY is considered to be a good explanation for many unsolved problems in the standard model, it has not been observed in nature. A possible explanation is that SUSY is spontaneously broken. This symmetry breaking can also be described in supersymmetric quantum mechanics.

In the former sections it was assumed that the ground state energy  $E_0^{(1)}$  is zero. This assumption leads to the equation for the normalizable ground state wave function  $\psi_0^{(1)}$  in (2.8) and the relations between the eigenfunctions of the partner Hamiltonians in (2.16) and (2.17). The normalization of  $\psi_0^{(1)}$  determines some properties of the superpotential; the exponential function has to vanish at the boundaries, the exponent has to go to minus infinity. This means that the integral of the superpotential has to converge to infinity for  $x \rightarrow \pm\infty$ , so

$$\int_{-\infty}^0 W(x')dx' = \infty \quad \text{and} \quad \int_0^\infty W(x')dx' = \infty. \quad (2.24)$$

Therefore  $W(x)$  has to converge to infinity for large positive values of  $x$  and to converge to minus infinity for large negative values of  $x$ . If these criteria are fulfilled SUSY is unbroken.

If the properties of the two partner Hamiltonians are just switched, which means that  $E_0^{(2)} = 0$ ,  $E_0^{(1)} \neq 0$  and  $\psi_0^{(2)}(x)$  is normalizable, SUSY is unbroken, too. This case does not have to be considered specially because exchanging the definitions of  $H^{(1)}$  and  $H^{(2)}$  restores the former case.

If neither  $E_0^{(1)}$  nor  $E_0^{(2)}$  are zero, the application of the ladder operators  $A$  and  $A^\dagger$  to the ground state wave functions does not annihilate them. This implies that the operators do not destroy nodes. In this case the energy spectra of both partner potentials are exactly the same and all energy eigenvalues are positive

$$E_n^{(1)} = E_n^{(2)} > 0. \quad (2.25)$$

The operators  $A$  and  $A^\dagger$  still switch between the eigenfunctions on the same energy levels

$$\psi_n^{(2)} = (E_n^{(1)})^{-\frac{1}{2}} A \psi_n^{(1)} \quad \text{and} \quad \psi_n^{(1)} = (E_n^{(2)})^{-\frac{1}{2}} A^\dagger \psi_n^{(2)}. \quad (2.26)$$

In this case SUSY is broken.

This breaking can also be described with the algebra of supersymmetry. In general the action of a symmetry operator on one state gives a symmetric one. The exception from this is the ground state which is unique. So the action of a symmetry operator on the ground state has to result in zero, the symmetry operators must annihilate the vacuum. Because of  $\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = \mathbf{H}$  from equation (2.21), the ground state energy as eigenvalue of  $\mathbf{H}$  has to be zero. If the ground state energy is nonzero and the ground state is not unique anymore supersymmetry is broken.

## 2.4 Hierarchy of Hamiltonians

The previous sections show that it is possible to factorize a Hamiltonian  $H^{(1)}$  into two operators  $A$  and  $A^\dagger$  that are dependent on the superpotential  $W(x)$ . It is further shown that these operators can be used to create a new Hamiltonian  $H^{(2)}$ . The introduction of a new superpotential  $W_2(x)$  allows a refactorization of  $H^{(2)}$  in two new operators  $A_2$  and  $A_2^\dagger$ . The partner Hamiltonian that can be created by the commutation of  $A_2$  and  $A_2^\dagger$  is called  $H^{(3)}$  and similar to the calculations above it is possible to determine the energy levels and ground state wave functions of this new Hamiltonian.  $H^{(3)}$  has the same energy spectrum as  $H^{(2)}$  except for the ground state  $E_0^{(2)}$ . The repeated application of this procedure leads to the creation of a chain of Hamiltonians, each with an energy level less than the previous one. The number of Hamiltonians is limited by the number of bound states in the first potential. The knowledge of the wave functions and the energy levels of  $H^{(1)}$  makes it possible to calculate the energy spectrum and the wave functions of all Hamiltonians.

Although the previous treatment of the first Hamiltonian assumed the ground state to be zero, this is not in general the case so the Hamiltonian  $H^{(1)}$  is rewritten as

$$H^{(1)} = A_1^\dagger A_1 + E_0^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x), \quad (2.27)$$

with the rewritten potential

$$V^{(1)} = W_1(x)^2 - \frac{\hbar}{\sqrt{2m}}W_1'(x) + E_0^{(1)}, \quad (2.28)$$

where the operators  $A$  and  $A^\dagger$  from section 2.1 are written with the index 1 for clarity and  $E_0^{(1)}$  is the ground state energy of  $H^{(1)}$ .

Similar to the steps in section 2.2 the partner Hamiltonian  $H^{(2)}$  can be calculated by an exchange of the operators

$$H^{(2)} = A_1A_1^\dagger + E_0^{(1)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(2)}(x) \quad (2.29)$$

with the potential

$$\begin{aligned} V^{(2)} &= W_1(x)^2 + \frac{\hbar}{\sqrt{2m}}W_1'(x) + E_0^{(1)} \\ &= V^{(1)}(x) + \frac{2\hbar}{\sqrt{2m}}W_1'(x) = V^{(1)}(x) - \frac{2\hbar}{\sqrt{2m}}\frac{d^2}{dx^2}\ln\left(\psi_0^{(1)}\right). \end{aligned} \quad (2.30)$$

Similar to the former calculations the associated energy levels and wave functions are calculated by

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad (2.31)$$

$$\psi_n^{(2)} = \left(E_{n+1}^{(1)} - E_0^{(1)}\right)^{-\frac{1}{2}} A_1\psi_{n+1}^{(1)}. \quad (2.32)$$

The ground state energy of  $H^{(2)}$  is  $E_0^{(2)} = E_1^{(1)}$ . The Hamiltonian can be refactorized as  $H^{(1)}$  before,

$$H^{(2)} = A_1A_1^\dagger + E_0^{(1)} = A_2^\dagger A_2 + E_0^{(2)} = A_2^\dagger A_2 + E_1^{(1)} \quad (2.33)$$

with the new factorization operators similar to the former calculations

$$A_2 = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_2(x), \quad A_2^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_2(x) \quad (2.34)$$

and the new superpotential

$$W_2(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'^{(2)}(x)}{\psi_0^{(2)}(x)} = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ln\left(\psi_0^{(2)}\right). \quad (2.35)$$

It is now possible to obtain the third Hamiltonian in the same manner

$$H^{(3)} = A_2A_2^\dagger + E_1^{(1)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(3)}(x) \quad (2.36)$$

with the potential

$$\begin{aligned} V^{(3)}(x) &= W_2(x)^2 + \frac{\hbar}{\sqrt{2m}} W_2'(x) + E_1^{(1)} \\ &= V^{(2)}(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left( \psi_0^{(2)} \right) = V^{(1)}(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left( \psi_0^{(1)} \psi_0^{(2)} \right) \end{aligned} \quad (2.37)$$

and the corresponding energy levels and wave functions

$$E_n^{(3)} = E_{n+1}^{(2)} = E_{n+2}^{(1)} \quad (2.38)$$

$$\begin{aligned} \psi_n^{(3)} &= \left( E_{n+1}^{(2)} - E_0^{(2)} \right)^{-\frac{1}{2}} A_2 \psi_{n+1}^{(2)} \\ &= \left( E_{n+2}^{(1)} - E_1^{(1)} \right)^{-\frac{1}{2}} \left( E_{n+2}^{(1)} - E_0^{(1)} \right)^{-\frac{1}{2}} A_2 A_1 \psi_{n+1}^{(1)}. \end{aligned} \quad (2.39)$$

It can be seen that it is possible to express the energy levels and wave functions in terms of the system of the first Hamiltonian. This allows to create a whole chain of Hamiltonians like it was stated in the introduction of this section. Equation (2.38) implies that every Hamiltonian has the same energy spectrum as the former one except for the ground state. The cut off of the ground states leads to the limitation of Hamiltonian numbers based on the number of energy levels in the first potential. Figure 1 could be increased by additional energy ladders, which are connected by the operators  $A_n$  and  $A_n^\dagger$ . The procedure of creating a chain of Hamiltonians is used in the treatment of Shape Invariant Potentials (SIP).

## 2.5 Shape Invariant Potentials

The analytical determination of energy eigenvalues and eigenfunctions is possible for a number of potentials and a common property of these potentials is called shape invariance. If two supersymmetric partner potentials are shape invariant, they have a similar shape and differ only in the set of parameters they are defined in and a remainder. This can be expressed with

$$V^{(2)}(x, a_1) = V^{(1)}(x, a_2) + R(a_1) \quad (2.40)$$

where  $a_1$  is a set of parameters and  $a_2$  is a different set of parameters that can be expressed as a function  $f$  of  $a_1$ ;  $a_2 = f(a_1)$ . The remainder  $R(a_1)$  is independent of  $x$ .

In order to study the properties of SIP, a chain of Hamiltonians can be constructed in the way it is shown in the previous section. Starting with a Hamiltonian  $H^{(1)}$  with  $E_0^{(1)} = 0$  and

$$H^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_1), \quad (2.41)$$

the chain of Hamiltonians  $H^{(s)}$  with  $s = 1, 2, 3, \dots$  contains as many members as there are bound states in  $V^{(1)}(x, a_1)$ . A repeated use of the shape invariance condition (2.40) allows a comparison between two sequent shape invariant Hamiltonians  $H^{(s)}$  and  $H^{(s+1)}$

$$H^{(s)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_s) + \sum_{k=1}^{s-1} R(a_k), \quad (2.42)$$

$$\begin{aligned} H^{(s+1)} &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_{s+1}) + \sum_{k=1}^s R(a_k) \\ &\stackrel{(2.40)}{=} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(2)}(x, a_s) + \sum_{k=1}^{s-1} R(a_k) \end{aligned} \quad (2.43)$$

where  $a_k = f^{(k)}(a_1)$  means that  $f$  is applied  $k$  times. The comparison shows that the Hamiltonians are according to (2.40) partner Hamiltonians and therefore have identical energy levels except for the ground state of  $H^{(s)}$  which is

$$E_0^{(s)} = \sum_{k=1}^{s-1} R(a_k) \quad (2.44)$$

because  $E_0^{(1)} = 0$ .

It is possible to go back from  $H^{(s)}$  to  $H^{(1)}$  where an energy level is added for every step until the level  $E = 0$  is reached. Thus the complete energy spectrum of  $H^{(1)}$  is described by

$$E_n^{(1)}(a_1) = \sum_{k=1}^n R(a_k); \quad E_0^{(1)} = 0. \quad (2.45)$$

Because the Operators  $A_n$  and  $A_n^\dagger$  link the eigenfunctions at the same energy levels for different partner Hamiltonians, they can be used to determine the bound state wave functions  $\psi_n^{(1)}(x, a_1)$  for a shape invariant potential based on the known ground state wave function  $\psi_0^{(1)}(x, a_1)$ . Equation (2.42) shows that each Hamiltonian depends on the potential  $V^{(1)}(x, a)$  and a set of parameters  $a_s$ . Thus the ground state wave function of  $H^{(s)}$  is  $\psi_0^{(1)}(x, a_s)$ , which is a consequence of the shape invariance condition (2.40) and

$$\psi_n^{(s)}(x, a_1) = \psi_n^{(1)}(x, a_s). \quad (2.46)$$

The relation between wave functions of the two partner Hamiltonians at the same energy level (2.17) can now be used to express the first excited wave function of a Hamiltonian  $H^{(s)}$

$$\psi_1^{(1)}(x, a_s) \propto A^\dagger(x, a_s) \psi_0^{(2)}(x, a_s) = A^\dagger(x, a_s) \psi_0^{(1)}(x, a_{s+1}). \quad (2.47)$$

Repeating this step makes it possible to determine the  $n$ 'th unnormalized wave function of the first Hamiltonian  $H^{(1)}$

$$\psi_n^{(1)}(x, a_1) \propto A^\dagger(x, a_1) \psi_{n-1}^{(1)}(x, a_2), \quad (2.48)$$

$$\psi_n^{(1)}(x, a_1) \propto A^\dagger(x, a_1) A^\dagger(x, a_2) \dots A^\dagger(x, a_n) \psi_0^{(1)}(x, a_{n+1}). \quad (2.49)$$

The explicit relation (2.17) leads to the formula

$$\psi_n^{(1)}(x, a_1) = (E_n^{(1)})^{-\frac{1}{2}} A^\dagger(x, a_1) \psi_{n-1}^{(1)}(x, a_2) \quad (2.50)$$

and the possibility to determine all wave functions and energy levels of Hamiltonians obeying the SIP condition just by knowing the first ground state wave function the remainder  $R(a)$  and the relation between two subsequent sets of parameters  $f(a)$ .

Hence the property of shape invariance simplifies the treatment of partner potentials and their wave functions considerably. Table I of [4] shows all shape invariant potentials that were known at the time of its publication and their properties.

## 3 Examples

### 3.1 Infinite Square Well

Calculating the energy levels and wave functions of the bound states in a square well of infinite depth is a basic task in a quantum mechanics lecture. The formalism of supersymmetric quantum mechanics makes it possible to use the well-known solutions to create a supersymmetric partner potential with the same energy levels and its eigenfunctions.

Starting with an symmetric infinite square well with the potential

$$V(x) = \begin{cases} 0 & \text{if } |x| \leq \frac{L}{2} \\ \infty & \text{else} \end{cases} \quad (3.1)$$

the Schrödinger equation

$$H\psi_n(x) = \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi_n(x) = E_n \psi_n(x) \quad (3.2)$$

reduces to

$$\psi_n''(x) = -\frac{2m}{\hbar^2} E_n \psi_n(x) \quad (3.3)$$

for  $|x| \leq \frac{L}{2}$  with  $E_n \geq 0$ . This symmetrical potential has symmetric  $\psi_n^{(s)}$  and antisymmetric  $\psi_n^{(a)}$  wave functions as eigenfunctions. With  $n$  equivalent to the number of nodes the eigenfunction can be written as

$$\psi_n(x) = \begin{cases} \psi_n^{(s)}(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{(n+1)\pi}{L}x\right) & \text{for } n = 0, 2, 4, \dots \\ \psi_n^{(a)}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{(n+1)\pi}{L}x\right) & \text{for } n = 1, 3, 5, \dots \end{cases} \quad (3.4)$$

and the energy levels are

$$E_n = \frac{\pi^2 \hbar^2}{2mL^2} (n+1)^2. \quad (3.5)$$

The factorization of the Hamiltonian requires the ground state energy to be zero, which leads to a necessary shift of the potential and the connected energy levels

$$V^{(1)}(x) = V(x) - E_0 \quad \text{and} \quad E_n^{(1)} = \frac{\pi^2 \hbar^2}{2mL^2} n(n+2). \quad (3.6)$$

The superpotential  $W(x)$  is calculated with the ground state wave function  $\psi_0(x)$  and equation (2.7)

$$W(x) = \frac{\hbar\pi}{\sqrt{2mL}} \tan\left(\frac{\pi}{L}x\right) \quad (3.7)$$

and is used to determine the partner potential  $V^{(2)}$  with equation (2.10)

$$V^{(2)} = \frac{\hbar^2\pi^2}{2mL^2} \left(1 + 2 \tan^2\left(\frac{\pi}{L}x\right)\right), \quad (3.8)$$

$$E_n^{(2)} = \frac{\pi^2 \hbar^2}{2mL^2} (n+1)(n+3) \quad (3.9)$$

and its energy levels according to the relations given in (2.15). The eigenfunctions are

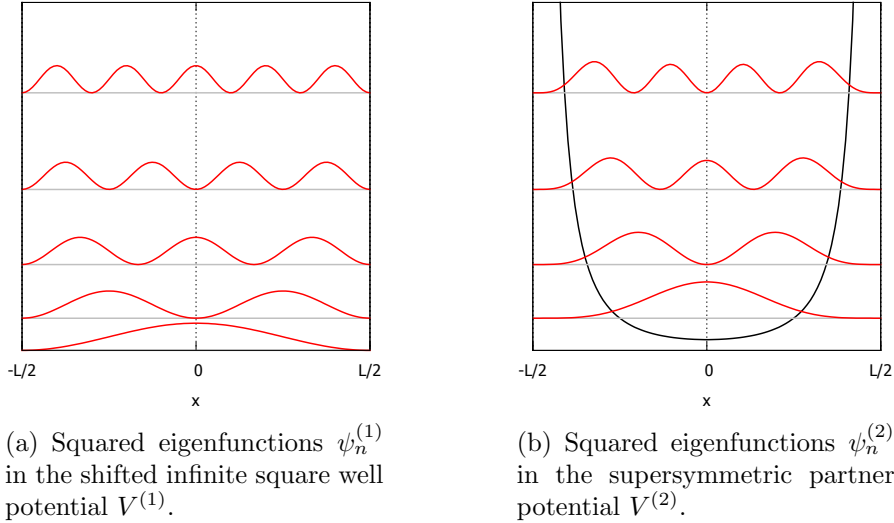


Figure 2: Comparison between the eigenfunctions and energy levels of the infinite square well potential  $V^{(1)}$  and its partner potential  $V^{(2)}$ .



determined with the definition of  $A$  in (2.4) and equation (2.16)

$$\begin{aligned}\psi_n^{(2)} &= \left(E_{n+1}^{(1)}\right)^{-\frac{1}{2}} A\psi_{n+1}^{(1)} \\ &= \begin{cases} \left(\frac{2}{L(n+1)(n+3)}\right)^{\frac{1}{2}} \left((n+2) \cos\left(\frac{(n+2)\pi}{L}x\right) + \tan\left(\frac{\pi}{L}x\right) \sin\left(\frac{(n+2)\pi}{L}x\right)\right) & \text{for } n \text{ even} \\ \left(\frac{2}{L(n+1)(n+3)}\right)^{\frac{1}{2}} \left(-(n+2) \sin\left(\frac{(n+2)\pi}{L}x\right) + \tan\left(\frac{\pi}{L}x\right) \cos\left(\frac{(n+2)\pi}{L}x\right)\right) & \text{for } n \text{ odd.} \end{cases}\end{aligned}\quad (3.10)$$

The squared eigenfunctions at their corresponding energy levels in the two potentials are shown in figure 2. The comparison between the figures 2(a) and 2(b) shows that the conclusions of section 2.2 fit to the calculated functions and energy levels. As expected,  $V^{(2)}$  has the exact same energy levels as  $V^{(1)}$  except for the missing ground state  $E_0^{(1)}$ . Switching from  $\psi_{n+1}^{(1)}$  to  $\psi_n^{(2)}$  with the operator  $A$  destroys one node.

### 3.2 The Harmonic Oscillator

One of the known shape invariant potentials is the three-dimensional oscillator potential. The treatment of the radial equation is a good example for the use of the properties of shape invariant potentials. The superpotential can be taken from [4], for simplicity  $\hbar$  and  $2m$  are set to 1:

$$W(r,l) = \frac{1}{2}\omega r - \frac{(l+1)}{r}, \quad (3.11)$$

$$W'(r,l) = \frac{1}{2}\omega + \frac{(l+1)}{r^2} \quad (3.12)$$

with the azimuthal quantum number  $l$  and the frequency  $\omega$ . The partner potentials are calculated with the help of equations (2.6) and (2.10)

$$V^{(1)}(r,l) = \frac{1}{4}\omega^2 r^2 - \omega \left(l + \frac{3}{2}\right) + \frac{(l+1)l}{r^2}, \quad (3.13)$$

$$V^{(2)}(r,l) = \frac{1}{4}\omega^2 r^2 - \omega \left(l + \frac{1}{2}\right) + \frac{(l+1)(l+2)}{r^2} \quad (3.14)$$

In order to use the SIP condition and the hierarchy of Hamiltonians it has to be checked, if SUSY is unbroken, which includes that the ground state eigenfunction

$$\begin{aligned}\psi_0^{(1)}(r,l) &= N \exp\left(-\frac{\sqrt{2m}}{\hbar} \int^r W(r',l) dr'\right) = N \exp\left(-\frac{1}{4}\omega r^2 + \ln(r)(l+1)\right) \\ &= N r^{l+1} e^{-\frac{1}{4}\omega r^2}\end{aligned}\quad (3.15)$$

is normalizable. The integral for the superpotential converges to infinity for the integration limits 0 and infinity, so the ground state wave function  $\psi_0^{(1)}$  is normalizable. The application of the Hamiltonian  $H^{(1)} = -\frac{d^2}{dr^2} + V^{(1)}(r,l)$  on  $\psi_0^{(1)}$  shows that the ground state energy

is zero. Thus SUSY is unbroken. The potential is taken from the table of shape invariant potentials which already contains all properties of this example. Nevertheless the check of the SIP condition (2.40) is done here to show the use of the formalism.

$$V^{(2)}(r, l_1) = V^{(1)}(r, l_2) + R(l_1) \quad (3.16)$$

$$\frac{1}{4}\omega^2 r^2 - \omega \left( l_1 + \frac{1}{2} \right) + \frac{(l_1 + 1)(l_1 + 2)}{r^2} = \frac{1}{4}\omega^2 r^2 - \omega \left( l_2 + \frac{3}{2} \right) + \frac{(l_2 + 1)l_2}{r^2} + R(l_1) \quad (3.17)$$

$R$  is independent of  $r$ , so the  $\frac{1}{r^2}$  terms can be equated in order to get a relation between  $l_2$  and  $l_1$ . The solution is

$$l_2 = f(l_1) = l_1 + 1 \quad (3.18)$$

because the other possible solution results in negative values for the quantum number  $l_2$  which are not allowed. Inserting this in the  $r$ -independent terms gives the result that  $R$  has no dependency on  $l_i$ ;

$$R(l_i) = 2\omega. \quad (3.19)$$

The determined equations are sufficient to construct a series of shape invariant potentials  $V^{(n)}(r, l)$ . The first four and the superpotential are shown in figure 3 for  $l = 2$ . The comparison shows clearly the potentials' property of shape invariance.

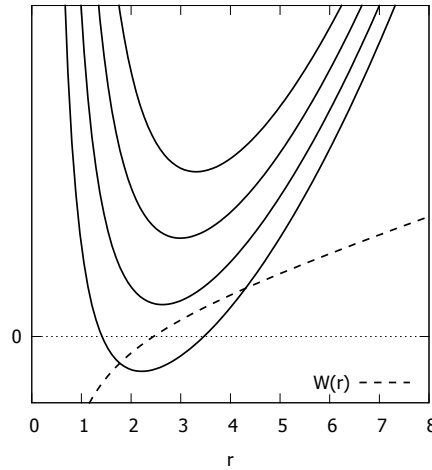


Figure 3: Shape invariant harmonic oscillator potentials  $V^{(n)}$  for  $n = 1, 2, 3, 4$  and the first superpotential  $W(r)$  for  $l = 2$ .

The property of shape invariance allows the determination of the energy levels according to equation (2.45) via

$$E_n^{(1)} = \sum_{i=1}^n R(l_i) = 2n\omega \quad (3.20)$$

It is shown in section 2.5 that the ground state eigenfunction  $\psi_0^{(1)}$  can be used to calculate all other eigenfunctions of every partner potential by the application of  $A^\dagger$  which is here given by

$$A^\dagger(r,l) = -\frac{d}{dr} + W(r,l) = -\frac{d}{dr} + \frac{1}{2}\omega r - \frac{(l+1)}{r}. \quad (3.21)$$

The first potential  $V^{(1)}$  and the first three eigenfunctions  $\psi_n^{(1)}$  at their energy levels  $E_n^{(1)}$  are shown in figure 4.

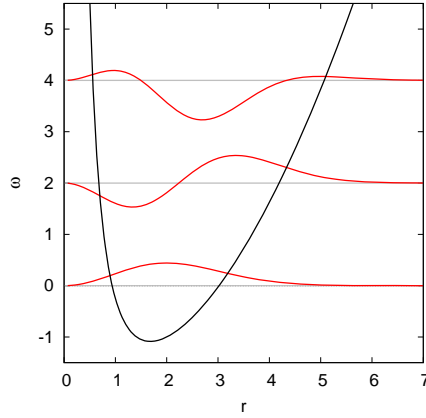


Figure 4: Eigenfunctions of the harmonic oscillator  $\psi_n^{(1)}(r,l)$  for  $n = 0,1,2$  at their energy levels  $E_n^{(1)}$  and the corresponding potential  $V^{(1)}(r,l)$  for  $l = 1$ .

### 3.3 The Nonrelativistic Hydrogen Atom

The hydrogen atom is a system that consists of a proton and an electron. The potential between these particles is the spherical symmetric Coulomb potential. This symmetry allows a separation between the radial and the angular parts of the eigenstates. This section deals with the determination of the radial eigenstates in the Coulomb potential.

The potential for an electron-proton-system is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}. \quad (3.22)$$

The related radial Schrödinger equation can be taken from textbooks, for example from [12],

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} + V(r) \right) \psi(r) = E\psi(r) \quad (3.23)$$

with the boundary conditions

$$\psi(0) = 0 \quad \text{and} \quad \int_0^\infty dr |\psi(r)|^2 = 1 \quad (3.24)$$

for the radial wave function  $\psi(r)$ . In order to solve the equation the effective potential  $V^{(1)}(r)$  is introduced and the SUSY formalism additionally demands a shift by the ground state energy  $E_0$ , so

$$\begin{aligned} V^{(1)}(r) &= -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - E_0 \\ &= W(r)^2 - \frac{\hbar}{\sqrt{2m}} W'(r). \end{aligned} \quad (3.25)$$

In order to determine the superpotential  $W(r)$ , an educated guess has to be made. Knowing the form of the shifted effective potential  $V^{(1)}(r)$ , this guess is

$$W(r) = C - \frac{D}{r}. \quad (3.26)$$

Inserting  $W(r)$  in (3.25) yields

$$V^{(1)} = -\frac{2CD}{r} + \left( D - \frac{\hbar}{\sqrt{2m}} \right) \frac{D}{r^2} + C^2. \quad (3.27)$$

The comparison shows that the r-independence can be used to gain the relation

$$-C^2 = E_0, \quad (3.28)$$

furthermore the r-dependent terms yield the equations

$$\frac{e^2}{4\pi\epsilon_0} = 2CD \quad \text{and} \quad D^2 - \frac{\hbar}{\sqrt{2m}} D = \frac{\hbar^2}{2m} l(l+1) \quad (3.29)$$

which can be solved to

$$C = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{8\pi\epsilon_0(l+1)} \quad \text{and} \quad D = -\frac{\hbar}{\sqrt{2m}} l. \quad (3.30)$$

This leads directly to the ground state energy  $E_0$

$$E_0 = -C^2 = -\frac{m}{2\hbar^2} \left( \frac{e^4}{4\pi\epsilon_0(l+1)} \right)^2 = -\frac{\hbar^2}{2ma_0^2} \frac{1}{(l+1)^2} \quad (3.31)$$

with the Bohr radius  $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ . The determined ground state energy is equal to the literature values (cf. [12]). Furthermore the now also determined superpotential

$$W(r) = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{8\pi\epsilon_0(l+1)} - \frac{\hbar}{\sqrt{2m}} \frac{(l+1)}{r} \quad (3.32)$$

can be used to calculate the partner potential with equation (2.10),

$$V^{(2)}(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} + \frac{\hbar^2}{2m} \frac{(l+1)(l+2)}{r^2} - E_0. \quad (3.33)$$

The potentials are shown in figure 5 for  $l = 1$ , they are nonnegative because of the shift by  $E_0$ .

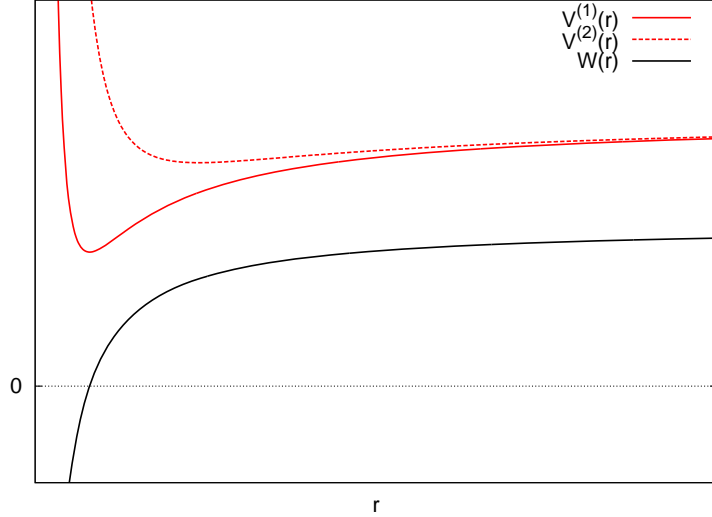


Figure 5: The effective radial Coulomb potential  $V^{(1)}(r)$ , the partner potential  $V^{(2)}(r)$  and the superpotential  $W(r)$  for  $l = 1$ .

To determine the energy spectrum of the hydrogen atom the SIP formalism is used. Thus the remainder  $R(l)$  has to be determined. The SIP condition (2.40) and comparing the terms that depend on  $r$  and  $l$  lead to the equation

$$l_2(l_2 + 1) = (l_1 + 2)(l_1 + 1) \quad (3.34)$$

which is solved by

$$l_2 = f(l_1) = l_1 + 1. \quad (3.35)$$

The remainder  $R(l)$  is then

$$R(l) = \frac{\hbar^2}{2ma_0^2} \frac{2l + 3}{(l + 1)^2(l + 2)^2}. \quad (3.36)$$

According to the results of section 2.5, the remainder and the relation between two sequent sets of parameters are sufficient to determine all energy values. With  $l = l_1$ ,  $l_k = (l + k - 1)$  and equation (2.45), the energy spectrum of  $V^{(1)}(r)$  can be calculated with

$$\begin{aligned} E_n^{(1)}(l) &= E_0 + \sum_{k=1}^n R(l_k) \\ &= \frac{\hbar^2}{2ma_0^2} \left( -\frac{1}{(l + 1)^2} + \sum_{k=1}^n \frac{2(l + k) + 1}{(l + k)^2(l + k + 1)^2} \right) \end{aligned} \quad (3.37)$$

where the former shift by  $E_0$  is reversed in order to get the real spectrum.

The formalism for shape invariant potentials allows also to calculate the eigenfunctions of an electron in the hydrogen atom, which is nontrivial in the textbook examples. Starting with the formula for the ground state wave function  $\psi_0^{(1)}(r)$  (2.8) and the superpotential  $W(r)$ , one gets

$$\psi_{0,l}^{(1)}(r) = Nr^{l+1}e^{-\frac{r}{a_0(l+1)}} \quad (3.38)$$

with the normalization constant  $N$ . The normalization for the only possible value of the angular momentum quantum number,  $l = 0$  yields

$$\psi_{0,0}^{(1)}(r) = 2a_0^{-\frac{3}{2}}re^{-\frac{r}{a_0}} \quad (3.39)$$

which is equal to the textbook solution (cf. [12]). Equation (2.50) can be used to determine the next eigenfunction of  $H^{(1)}$

$$\psi_{1,l}^{(1)} = N\sqrt{a_0}\sqrt{2l+3} \left( (l+1)(l+2)r^{l+1} - \frac{r^{l+2}}{a_0} \right) e^{-\frac{r}{2a_0}} \quad (3.40)$$

and normalizing the function for  $l = 0$  reproduces the textbook equation

$$\psi_{1,0}^{(1)}(r) = 2(2a_0)^{-\frac{3}{2}} \left( r - \frac{r^2}{2a_0} \right) e^{-\frac{r}{2a_0}}. \quad (3.41)$$

The deduced properties can be used to calculate all wave functions of the radial Coulomb potential and the most difficult part of the calculation is the normalization.

### 3.4 The Dirac Equation

The Dirac equation makes it possible to describe quantum mechanical problems relativistically. In the previous section the exact solution of the nonrelativistic Schrödinger equation for a Coulomb potential is described. The relativistic Dirac equation of this potential can also be solved exactly, the solutions for bound states are presented in this section. Starting with the formulas and notations given in [7] a SUSY quantum mechanical treatment is possible, this is further described in [3] and [8].

With the parameters

$$\gamma = \frac{ze^2}{ch} \quad , \quad \alpha_1 = m + E \quad , \quad \alpha_2 = m - E \quad (3.42)$$

the two coupled radial equations (4.13) in [7] which are satisfied by the two-component eigenfunction  $(G_k, F_k)$  can be written as

$$\begin{pmatrix} \frac{dG_k}{dr} \\ \frac{dF_k}{dr} \end{pmatrix} + \frac{1}{r} \begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix} \begin{pmatrix} G_k \\ F_k \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} G_k \\ F_k \end{pmatrix}. \quad (3.43)$$

This is the radial eigenfunction after the separation of the angular part of the Dirac equation. The variable  $k$  is an eigenvalue of the operator  $-(\sigma \cdot \mathbf{L} + 1)$  where  $\mathbf{L}$  is the angular momentum operator and can take the values  $k = \pm 1, \pm 2, \pm 3, \dots$ .  $k$  also satisfies  $|k| = J + \frac{1}{2}$ , where  $J$  is the quantum number for the total spin. In order to solve the coupled equations, the system has to be multiplied with an appropriate matrix  $\mathbf{D}$  from the left and its inverse  $\mathbf{D}^{-1}$  from the right.

$$\mathbf{D} = \begin{pmatrix} k + s & -\gamma \\ -\gamma & k + s \end{pmatrix} \quad \text{with} \quad s = \sqrt{k^2 - \gamma^2} \quad (3.44)$$

and its inverse

$$\mathbf{D}^{-1} = \frac{1}{2s(k + s)} \begin{pmatrix} k + s & \gamma \\ \gamma & k + s \end{pmatrix} \quad (3.45)$$

diagonalize  $\begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix}$  to  $\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$  and with the definitions

$$\begin{pmatrix} G \\ F \end{pmatrix} = \mathbf{D} \begin{pmatrix} G_k \\ F_k \end{pmatrix} \quad (3.46)$$

and  $\rho = Er$  the equation (3.43) can be written as

$$\begin{aligned} -\left(\frac{m}{E} + \frac{k}{s}\right) F &= A_0^\dagger G \\ \left(\frac{m}{E} - \frac{k}{s}\right) G &= A_0 F \end{aligned} \quad (3.47)$$

with the operators

$$A_0 = \frac{d}{d\rho} - \frac{s}{\rho} + \frac{\gamma}{s}, \quad A_0^\dagger = -\frac{d}{d\rho} - \frac{s}{\rho} + \frac{\gamma}{s}. \quad (3.48)$$

Like it is done in the previous sections, these adjoint operators can be used to create two supersymmetric Hamiltonians which can be applied to  $F$  and  $G$ . Their eigenvalues can be determined by using the equations (3.47)

$$H^{(1)}F = A_0^\dagger A_0 F = \tilde{E}F = \left(\frac{\gamma^2}{s^2} + 1 - \frac{m^2}{E^2}\right) F \quad (3.49)$$

$$H^{(2)}G = A_0 A_0^\dagger G = \tilde{E}G = \left(\frac{\gamma^2}{s^2} + 1 - \frac{m^2}{E^2}\right) G \quad (3.50)$$

with  $\tilde{E}$  as new energy eigenvalue for the supersymmetric Hamiltonians. It is used in both equations which means that every energy eigenvalue of  $A_0^\dagger A_0$  is also an eigenvalue of  $A_0 A_0^\dagger$

except for the case  $A_0 F = 0$ . This case can be used to determine the ground state wave function to

$$F_0(\rho) = \rho^s e^{-\frac{\gamma \rho}{s}}. \quad (3.51)$$

The ground state energy eigenvalue  $\tilde{E}_0$  is supposed to be zero, which leads to the condition

$$\begin{aligned} H^{(1)} F_0 = \tilde{E}_0 F_0 &= \left( \frac{\gamma^2}{s^2} + 1 - \frac{m^2}{E_0^2} \right) F_0 = 0 \\ E_0 &= \frac{m}{\sqrt{1 + \frac{\gamma^2}{s^2}}}. \end{aligned} \quad (3.52)$$

for the energy  $E_0$ . At this energy the equation (3.50) leads to an unnormalizable eigenstate for  $G$ . To sum up, all eigenstates except for the ground state of  $F$  are paired on the same energy levels. They are linked with the relations  $F \propto A_0^\dagger G$  and  $G \propto A_0 F$ , so the main relations of supersymmetric quantum mechanics are obtained.

According to equation (2.4) the superpotential can be gained from the operators  $A_0$  and  $A_0^\dagger$ , it is determined to

$$W(\rho) = -\frac{s}{\rho} + \frac{\gamma}{s} \quad (3.53)$$

when the substitution  $x = \frac{\hbar}{\sqrt{2m}} \rho$  is made. This superpotential can be used to calculate the first two partner potentials

$$V^{(1)} = \frac{s(s-1)}{\rho^2} + \frac{\gamma^2}{s^2} - \frac{2\gamma}{\rho} \quad (3.54)$$

$$V^{(2)} = \frac{(s+1)s}{\rho^2} + \frac{\gamma^2}{s^2} - \frac{2\gamma}{\rho} \quad (3.55)$$

with the help of (2.6) and (2.10). The ground state wave function  $F_0(\rho)$  is normalizable, the energy eigenvalue  $\tilde{E}_0$  is zero and the comparison between the two potentials shows a possible shape invariance. In order to prove this, the shape invariance condition (2.40) is tested

$$\frac{(s_1+1)s_1}{\rho^2} + \frac{\gamma^2}{s_1^2} - \frac{2\gamma}{\rho} = \frac{s_2(s_2-1)}{\rho^2} + \frac{\gamma^2}{s_2^2} - \frac{2\gamma}{\rho} + R(s_1). \quad (3.56)$$

Comparing the  $\rho^{-2}$ -dependent terms leads to the conclusion

$$s_2 = s_1 + 1 \Rightarrow f^{(n)}(s) = s + n \quad (3.57)$$

which can be put into (3.56) to determine the  $\rho$ -independent remainder to

$$R(s_i) = \frac{\gamma^2}{s_i^2} - \frac{\gamma^2}{(s_i+1)^2}. \quad (3.58)$$



The energy levels  $E_n$  can now be calculated with (2.45) and the supersymmetric energy eigenvalue  $\tilde{E}_n$  is determined to

$$\begin{aligned}\tilde{E}_n &= \sum_{i=1}^n \left( \frac{\gamma^2}{s_i^2} - \frac{\gamma^2}{(s_i + 1)^2} \right) = \sum_{i=1}^n \left( \frac{\gamma^2}{(s + i - 1)^2} - \frac{\gamma^2}{(s + i)^2} \right) \\ &= \frac{\gamma^2}{s^2} - \frac{\gamma}{(s + n)^2} \stackrel{(3.49)}{=} \left( \frac{\gamma^2}{s^2} + 1 - \frac{m^2}{E_n^2} \right).\end{aligned}\quad (3.59)$$

Solving (3.59) for the energy levels  $E_n$  gives

$$E_n = \frac{m}{\sqrt{1 + \frac{\gamma^2}{(s+n)^2}}}.\quad (3.60)$$

The shape invariance makes it also possible to determine all eigenfunctions  $F_n(\rho)$  by the repeated application of  $A_n^\dagger$  according to (2.49):

$$F_n(\rho) \propto (A_0^\dagger A_1^\dagger \dots A_{n-1}^\dagger) \rho^{s+n} e^{-\frac{\gamma\rho}{(s+n)}}\quad (3.61)$$

with

$$A_n = \left( \frac{d}{d\rho} - \frac{s+n}{\rho} + \frac{\gamma}{s+n} \right); \quad A_n^\dagger = \left( -\frac{d}{d\rho} - \frac{s+n}{\rho} + \frac{\gamma}{s+n} \right).\quad (3.62)$$

Like written above the calculation of  $F_n$  determines also  $G_n$  according to (3.47). For the case  $n = 0$ ,  $G_0$  has no normalizable eigenstate at the energy  $E_0$ . When the two cases of positive and negative values for  $k$  are distinguished, inserting in the equations (3.47) shows that normalizable solutions for  $F_0$  and  $G_0$  can be found for negative values of  $k$  with  $G_0 = 0$  [3]. This is not possible for positive values of  $k$ . The energy (3.60) only depends on  $k^2$  and is therefore degenerate, each energy level is a doublet for every level with  $n > 0$ . For  $n = 0$  only the negative value of  $k$  allows a normalization of  $F_0$  and  $G_0$ , which leads to a singlet state.

In the calculations above a fixed  $k$  is used to determine the fixed parameter  $s = \sqrt{k^2 - \gamma^2}$ . In order to determine all energies and eigenfunctions for the values of  $J = |k| - \frac{1}{2}$  the relation for  $s$  can be put into the formulas above. The energy spectrum can then be written as

$$E_n = \frac{m}{\sqrt{1 + \frac{\gamma^2}{(n + \sqrt{k^2 - \gamma^2})^2}}}\quad (3.63)$$

which is the same result as obtained in textbooks, for example [7] and [12]. The principal quantum number  $N$  can be obtained from  $k$  and  $n$  by  $N = n + |k|$  which leads to

$$E_{NJ} = \frac{m}{\sqrt{1 + \frac{\gamma^2}{(N - J - \frac{1}{2} + \sqrt{(J + \frac{1}{2})^2 - \gamma^2})^2}}}\quad (3.64)$$

The energy depends on the principal quantum number  $N$  and the total spin  $J$  which means that the degeneracy of the nonrelativistic calculation is partly removed. (3.64) shows the fine structure in the central Coulomb potential. Together with the doublet structure and the singlet in the ground state the  $J$  dependence creates the eigenvalue spectrum of the Dirac equation.

## 4 Two-Dimensional SUSY QM

The former treatment of supersymmetric quantum mechanics was only done in one dimension. An extension to more space dimensions allows the observation of new phenomena. The description of the formalism is mostly based on [9], [13], [14] and [15].

Before the formalism for a higher dimensional treatment is derived, the main relations of chapter 2 are presented. From now on for simplicity the factor  $\frac{\hbar}{\sqrt{2m}}$  is set to one. With the abbreviation  $\partial = \frac{d}{dx}$  the two partner Hamiltonians (2.1) and (2.9) can be written as

$$H^{(1)} = -\partial^2 + V^{(1)}(x) = A^\dagger A \quad \text{and} \quad H^{(2)} = -\partial^2 + V^{(2)}(x) = AA^\dagger. \quad (4.1)$$

According to (2.16) and (2.17) the wave functions of these Hamiltonians are connected via

$$\psi_n^{(2)} \propto A\psi_{n+1}^{(1)} \quad \text{and} \quad \psi_{n+1}^{(1)} \propto A^\dagger\psi_n^{(2)}. \quad (4.2)$$

The reason for this connection is that the Hamiltonians are intertwined. The intertwining relations (2.19) play an important role in the description of higher dimensional supersymmetric quantum mechanics.

The described extension to a higher dimensional treatment is mainly based on the use of vector intertwining operators  $A_l(\vec{x}); l = 1, 2, \dots, d$  with

$$A_l(\vec{x}) = \partial_l + (\partial_l Z)(\vec{x}), \quad A_l^\dagger(\vec{x}) = -\partial_l + (\partial_l Z)(\vec{x}) \quad (4.3)$$

with  $\vec{x} = (x_1, x_2, \dots, x_d), \quad \partial_l = \frac{\partial}{\partial x_l}$

where  $Z(\vec{x})$  is related to the superpotential by  $W_l = \partial_l Z(\vec{x})$ . The initial Hamiltonian  $H^{(1)}$  can then be quasifactorized in terms of these operators

$$\begin{aligned} H^{(1)}(\vec{x}) &= A_l^\dagger A_l = -\partial_l \partial_l + (\partial_l Z)^2(\vec{x}) - (\partial_l^2 Z)(\vec{x}) \\ &= -\partial_l \partial_l + V^{(1)}(\vec{x}) \end{aligned} \quad (4.4)$$

with the implied sum over all values of  $l$ . The first step is the extension to  $d = 2$  dimensions as simplest case. Similar to the one-dimensional case (2.10), a partner Hamiltonian with another scalar potential can be constructed

$$\begin{aligned} H^{(3)}(\vec{x}) &= A_l A_l^\dagger = -\partial_l \partial_l + (\partial_l Z)^2(\vec{x}) + (\partial_l^2 Z)(\vec{x}) \\ &= -\partial_l \partial_l + V^{(3)}(\vec{x}). \end{aligned} \quad (4.5)$$

In contrast to the one-dimensional case, these two Hamiltonians do not intertwine, but both of them intertwine with a third Hamiltonian  $H_{ik}^{(2)}$  that depends on a  $2 \times 2$  matrix potential  $V_{ik}^{(2)}$ ;

$$\begin{aligned}
H_{ik}^{(2)}(\vec{x}) &= \delta_{ik}H^{(1)}(\vec{x}) + [A_i, A_k^\dagger] \\
&= -\delta_{ik}\partial_l^2 + \delta_{ik}((\partial_l Z)^2(\vec{x}) - (\partial_l^2 Z)(\vec{x})) + 2(\partial_i \partial_k Z)(\vec{x}) \\
&= -\delta_{ik}\partial_l^2 + \delta_{ik}V^{(1)}(\vec{x}) + 2(\partial_i \partial_k Z)(\vec{x}) \\
&= -\delta_{ik}\partial_l^2 + V_{ik}^{(2)}(\vec{x})
\end{aligned} \tag{4.6}$$

with  $\delta_{ik}$  the Kronecker delta. The intertwining operators are mutually orthogonal, they obey

$$[A_l^\dagger, A_k^\dagger] = 0 \quad [A_l, A_k] = 0 \quad \text{for } l \neq k. \tag{4.7}$$

This orthogonality relation allows to construct the intertwining relations for the three Hamiltonians  $H^{(1)}$ ,  $H_{ik}^{(2)}$  and  $H^{(3)}$ :

$$\begin{aligned}
H^{(1)}A_i^\dagger &= A_k^\dagger H_{ki}^{(2)}; & H_{ik}^{(2)}A_k &= A_i H^{(1)}, \\
H_{ik}^{(2)}\epsilon_{kl}A_l^\dagger &= \epsilon_{il}A_l^\dagger H^{(3)}; & \epsilon_{kl}A_l H_{ki}^{(2)} &= H^{(3)}\epsilon_{il}A_l \quad \text{with } i, k, l = 1, 2
\end{aligned} \tag{4.8}$$

Analogous to the one-dimensional case the intertwining relations determine the similarity of the energy spectra of the participating Hamiltonians.  $H_{ik}^{(2)}$  intertwines with both other Hamiltonians, while  $H^{(1)}$  and  $H^{(3)}$  do not intertwine directly. Therefore the two scalar potentials have the same energy levels as the vector potential up to zero modes of the operators  $A_l, A_l^\dagger$ . This means that the SUSY formalism allows to reduce a problem related to a vector potential to the solutions of two scalar potential problems. In contrast to the one-dimensional case the superhamiltonian is not defined as the anticommutator of the supercharges. This does not influence the connection between the spectra because the intertwining relations still hold. Like in (4.2) the intertwining relations cause a connection between the scalar wave functions of  $H^{(1)}$  and  $H^{(3)}$  and the vector wave functions of the Hamiltonian  $H_{ik}^{(2)}$ .

$$\begin{aligned}
\psi_i^{(2)}(\vec{x}) &\propto A_i \psi^{(1)}(\vec{x}); & \psi^{(1)}(\vec{x}) &\propto A_i^\dagger \psi_i^{(2)}(\vec{x}) \\
\psi_i^{(2)}(\vec{x}) &\propto \epsilon_{ik} A_k^\dagger \psi^{(3)}(\vec{x}); & \psi^{(3)}(\vec{x}) &\propto \epsilon_{ik} A_k \psi_i^{(2)}(\vec{x}) \quad \text{with } i, k = 1, 2
\end{aligned} \tag{4.9}$$

The relations between the Hamiltonians are as in the one-dimensional case based on supersymmetry. So it is again possible to express the Hamiltonians and factorization operators via a superhamiltonian  $\mathbf{H}$  and supercharges  $Q, Q^\dagger$ .

$$\mathbf{H} = \begin{pmatrix} H^{(1)} & 0 & 0 & 0 \\ 0 & H_{11}^{(2)} & H_{12}^{(2)} & 0 \\ 0 & H_{21}^{(2)} & H_{22}^{(2)} & 0 \\ 0 & 0 & 0 & H^{(3)} \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & 0 \end{pmatrix} = (Q^\dagger)^\dagger \tag{4.10}$$

Rearranging the terms can emphasize the similarity to the one-dimensional case and the SUSY algebra (2.21)

$$\tilde{\mathbf{H}} = \begin{pmatrix} \tilde{H}^{(1)} & 0 \\ 0 & \tilde{H}^{(2)} \end{pmatrix}; \quad \tilde{H}^{(1)} = \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(3)} \end{pmatrix}; \quad \tilde{H}^{(2)} = \left( H_{ik}^{(2)} \right) \quad (4.11)$$

$$\tilde{Q} = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} = \left( \tilde{Q}^\dagger \right)^\dagger; \quad q = \begin{pmatrix} A_1 & A_2^\dagger \\ A_2 & -A_1^\dagger \end{pmatrix}; \quad q^\dagger = \begin{pmatrix} A_1^\dagger & A_2^\dagger \\ A_2 & -A_1 \end{pmatrix} \quad (4.12)$$

## 4.1 The Pauli Hamiltonian in Two Dimensions

A nonrelativistic fermion in a given electromagnetic potential is described by the Pauli Hamiltonian [9], [16], [17]

$$H_P = (i\partial_i + eA_i)^2 - \mu\sigma_i B_i + U \quad \text{with } i = 1, 2, 3 \quad (4.13)$$

where  $e$  is the charge,  $\mu$  the magnetic momentum,  $\vec{B}(\vec{x}) = \text{rot}\vec{A}(\vec{x})$  is the magnetic field which is determined from the vector potential  $\vec{A}(\vec{x})$  and  $U(\vec{x})$  is the scalar electric potential.  $\sigma_i$  are the Pauli matrices.  $H_P$  is a  $2 \times 2$  matrix which acts on a two-component spinor wave function. If the potentials are restricted to potentials that do not depend on one of the coordinates ( $x_2$  is chosen) and  $B_2 = 0$  it can be shown that  $H_P$  can be identified with  $H_{ik}^{(2)}$  from section 4.

The fermion can move freely in the direction of  $x_2$ , its wave function can then be written

$$\psi(\vec{x}) = e^{ikx_2}\psi(x_1, x_3) \quad (4.14)$$

and this form can be used to express the Hamiltonian as

$$H_P = -(\partial_i + eA_i)^2 + U + (k + eA_2)^2 - \mu\sigma_i B_i \quad \text{with } i = 1, 3 \quad (4.15)$$

The last part of this Hamiltonian in the matrix form is written

$$-\mu\sigma_i B_i = -\mu \begin{pmatrix} B_3 & B_1 \\ B_1 & -B_3 \end{pmatrix} \quad \text{for } i = 1, 3 \quad (4.16)$$

and the comparison with  $H_{ik}^{(2)}$  in (4.6) shows that the components of the magnetic field can be identified with

$$\begin{aligned} -\mu B_1 &= 2(\partial_1 \partial_3 Z)(\vec{x}), \\ -\mu B_3 &= (\partial_1^2 - \partial_3^2)Z(\vec{x}). \end{aligned} \quad (4.17)$$

This allows also the identification of the rest of the Pauli Hamiltonian with a correspondent part of the supersymmetric partner Hamiltonian

$$-(\partial_i + eA_i)^2 + U + (k + eA_2)^2 = -\partial_i^2 + (\partial_i Z)^2(\vec{x}) \quad \text{with } i = 1, 3 \quad (4.18)$$

If the field sources are assumed to be absent, the equations

$$\text{rot}\vec{B} = \text{div}\vec{B} = 0 \quad (4.19)$$

lead to the conditions

$$\begin{aligned} \partial_1 B_1 + \partial_3 B_3 &= 0 \Leftrightarrow \partial_3(3\partial_1^2 - \partial_3^2)Z(\vec{x}) = 0, \\ \partial_3 B_1 - \partial_1 B_3 &= 0 \Leftrightarrow \partial_1(3\partial_3^2 - \partial_1^2)Z(\vec{x}) = 0. \end{aligned} \quad (4.20)$$

The highest order solution for  $Z(\vec{x})$  is according to [17] in fourth order in the variables  $x_1$  and  $x_3$  when the terms with negative power are dropped to get a regular solution without poles,

$$Z(\vec{x}) = \frac{1}{4}a(x_1^2 + x_3)^2 + (bx_1 + cx_3)(x_1^2 + x_3^2) + dx_1^2 + fx_3^2 + 2gx_1x_3 + hx_1 + tx_3 \quad (4.21)$$

depends on eight arbitrary parameters  $a, b, c, \dots$ . The parameter  $a$  is restricted to  $a > 0$  if  $\psi_0 = e^{-Z}$  is a normalizable ground state of  $H^{(1)}$ . Equations (4.17) can be used to determine the components  $B_1$  and  $B_3$  of the magnetic field to

$$\begin{aligned} -\mu B_1 &= 4(ax_1x_3 + cx_1 + bx_3 + g), \\ -\mu B_3 &= 4\left(\frac{a}{2}x_1^2 - \frac{a}{2}x_3^2 + bx_1 - cx_3 + \frac{d}{2} - \frac{f}{2}\right). \end{aligned} \quad (4.22)$$

The vector potential generates the magnetic field

$$\vec{B} = \text{rot}\vec{A} = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix}. \quad (4.23)$$

The component  $B_2$  is zero, so  $\partial_3 A_1 = \partial_1 A_3$ . For the trivial solution  $A_1 = A_3 = 0$ , the component  $A_2$  can be calculated straightforwardly from  $B_1$  and  $B_3$ ,

$$A_2 = \frac{1}{2\mu} \left( -\frac{1}{3}ax_1^3 + ax_1x_3^2 - b(x_1^2 - x_3^2) + 2cx_1x_3 + 2gx_3 + (f - d)x_1 + \gamma \right). \quad (4.24)$$

With  $A_1 = A_3 = 0$ , equation (4.18) can be transformed in order to determine the scalar potential  $U(\vec{x})$

$$\begin{aligned} -\partial_i^2 + U(\vec{x}) + (k + eA_2)^2 &= -\partial_i^2 + (\partial_i Z)^2(\vec{x}) \\ \rightarrow U(\vec{x}) &= (\partial_i Z)^2(\vec{x}) - (k + eA_2)^2 = (\partial_i Z)^2(\vec{x}) - e^2 \tilde{A}_2^2 \end{aligned} \quad (4.25)$$

with  $\tilde{A}_2 = \frac{k}{e} + A_2$ . This scalar potential has terms up to sixth order in the coordinates.

To sum up for the given constraints it is possible to identify the Pauli Hamiltonian  $H_P$  as the Hamiltonian  $H_{ik}^{(2)}$  which is part of the superhamiltonian  $\mathbf{H}$ . Instead of solving an eigenvalue problem for the  $2 \times 2$  matrix Pauli Hamiltonian it is now possible to use the scalar Hamiltonians  $H^{(1)}$  and  $H^{(3)}$ .

## 4.2 Exact Solution of a Two-Dimensional Model with Real Spectrum

The two-dimensional treatment of supersymmetric quantum mechanics allows to handle different types of problems. A second approach is to deal just with the two scalar Hamiltonians and to solve a potential which is not amenable to separation of variables via the solution of a partner potential that is.

### 4.2.1 The Two-Dimensional Real Singular Morse Potential

The way to keep the focus only on the scalar potentials is to use supercharges of second order in derivatives [18]. These supercharges exist in two different variations. In the reducible form they create two partner Hamiltonians that differ only by a constant which means that if one Hamiltonian is amenable to separation of variables, the other one is it, too. In order to create the case described in the introduction of this section, it is necessary to use irreducible second order components of supercharges. These are written [18],[19]

$$Q^\dagger = g_{ik}(\vec{x})\partial_i\partial_k + C_i(\vec{x})\partial_i + B(\vec{x}), \quad Q = (Q^\dagger)^\dagger \quad (4.26)$$

while the Hamiltonians

$$H^{(i)} = -\partial_l\partial_l + V^{(i)}(\vec{x}); i = 1,2 \quad (4.27)$$

still satisfy the intertwining relations

$$H^{(1)}Q^\dagger = Q^\dagger H^{(2)}; \quad QH^{(1)} = H^{(2)}Q. \quad (4.28)$$

It is important to note that the relation  $\{Q, Q^\dagger\} = \mathbf{H}$  is now dropped. Nevertheless the Hamiltonians and supercharges are chosen in a way that the intertwining relations still hold. In this particular example the metric  $g_{ik}(\vec{x})$  is chosen to  $g_{ik}(\vec{x}) = \text{diag}(1, -1)$  leading to the supercharges

$$Q^\dagger = (\partial_1^2 - \partial_2^2) + C_i\partial_i + B = 4\partial_+\partial_- + C_+\partial_- + C_-\partial_+ + B \quad (4.29)$$

$$Q = (\partial_1^2 - \partial_2^2) - C_i\partial_i + B = 4\partial_+\partial_- - C_+\partial_- - C_-\partial_+ + B; \quad i = 1,2 \quad (4.30)$$

where  $x_\pm = x_1 \pm x_2$ ,  $\partial_\pm = \frac{d}{dx_\pm}$  and  $C_\pm$  depend on  $x_\pm$ :

$$C_+ = C_1 - C_2 = C_+(x_+); \quad C_- = C_1 + C_2 = C_-(x_-). \quad (4.31)$$

The potentials and the function  $B(\vec{x})$  can be expressed in terms of  $C_\pm$  and the functions  $F_1(2x_1)$ ,  $F_2(2x_2)$  which satisfy

$$\partial_-(C_-F) = -\partial_+(C_+F). \quad (4.32)$$

With

$$F = F_1(2x_1) + F_2(2x_2) = F_1(x_+ + x_-) + F_2(x_+ - x_-), \quad (4.33)$$

the general expressions are

$$\begin{aligned}
B &= \frac{1}{4}(C_+C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)), \\
V^{(1)} &= \frac{1}{2}(C'_+ + C'_-) + \frac{1}{8}(C_+^2 + C_-^2) + \frac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)), \\
V^{(2)} &= -\frac{1}{2}(C'_+ + C'_-) + \frac{1}{8}(C_+^2 + C_-^2) + \frac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)). \tag{4.34}
\end{aligned}$$

This particular choice of supercharges and potentials satisfies (4.28).

As described at the beginning of this section the formalism allows to solve a problem for one Hamiltonian via finding the solution for the partner Hamiltonian. The case studied here is the two-dimensional generalization of the one-dimensional Morse potential. The definitions of  $C_{\pm}$  and  $F_{1/2}$  define the potentials  $V^{(1)}$  and  $V^{(2)}$ :

$$C_+ = 4a\alpha; \quad C_- = 4a\alpha \cdot \coth\left(\frac{\alpha x_-}{2}\right) \tag{4.35}$$

$$-V_M(x_1) = \frac{1}{4}F_1(2x_1) = -A(e^{-2\alpha x_1} - 2e^{-\alpha x_1}),$$

$$V_M(x_2) = \frac{1}{4}F_2(2x_2) = A(e^{-2\alpha x_2} - 2e^{-\alpha x_2}),$$

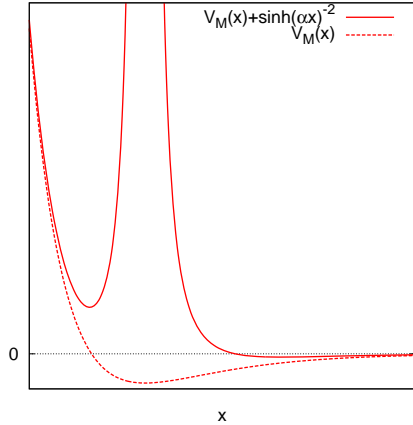
$$V^{(1)}(\vec{x}) = \alpha^2 a(2a - 1) \sinh^{-2}\left(\frac{\alpha x_-}{2}\right) + 4a^2 \alpha^2 + A(e^{-2\alpha x_1} - 2e^{-\alpha x_1} + e^{-2\alpha x_2} - 2e^{-\alpha x_2}) \tag{4.36}$$

$$V^{(2)}(\vec{x}) = \alpha^2 a(2a + 1) \sinh^{-2}\left(\frac{\alpha x_-}{2}\right) + 4a^2 \alpha^2 + A(e^{-2\alpha x_1} - 2e^{-\alpha x_1} + e^{-2\alpha x_2} - 2e^{-\alpha x_2}) \tag{4.37}$$

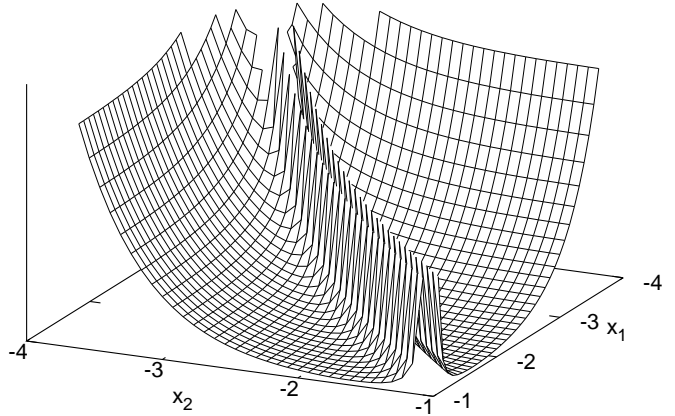
with the real numbers  $A > 0, \alpha > 0$  and  $a$ . Both potentials consist of an  $x$ -independent term, two Morse potentials  $V_M(x) = A(e^{-2\alpha x} - 2e^{-\alpha x})$  which depend on only one coordinate and a  $\sinh^{-2}$ -term that depends on  $x_- = x_1 - x_2$ . The mixing of coordinates in the  $\sinh^{-2}$ -term is responsible for the fact that the potentials cannot be solved with separation of variables. If the parameters of the model are chosen in a way that separation of variables is amenable to one of the Hamiltonians belonging to the potentials, the intertwining relations allow it to gain all informations about the other Hamiltonian. The choice of  $a = -\frac{1}{2}$  lets the  $\sinh^{-2}$ -term in  $V^{(2)}$  vanish and makes  $H^{(2)}$  amenable to separation of variables, while  $V^{(1)}$  still contains a mixing term.

$$\begin{aligned}
V^{(1)}(\vec{x}) &= \alpha^2 \left(1 + \sinh^{-2}\left(\frac{\alpha x_-}{2}\right)\right) + V_M(x_1) + V_M(x_2), \\
V^{(2)}(\vec{x}) &= \alpha^2 + V_M(x_1) + V_M(x_2) \tag{4.38}
\end{aligned}$$

Figure 6 shows the one-dimensional Morse potential and the two-dimensional potential  $V^{(1)}(\vec{x})$ . The  $\sinh^{-2}$ -term creates a singularity at  $x_1 = x_2$  and as it is shown in figure 6(a) this singularity deforms the shape of the simple Morse potential.  $V^{(2)}(\vec{x})$  on the other hand is simply the two-dimensional form of  $V_M(x)$  and matches with  $V^{(1)}(\vec{x})$  except for



(a) One-dimensional Morse potential  $V_M(x)$  and Morse potential with an added  $\sinh^{-2}$ -term.



(b) Potential  $V^{(1)}(x_1, x_2)$  for  $a = -\frac{1}{2}$  and  $A = \alpha = 1$ .

Figure 6: One-dimensional Morse potential and the two-dimensional model of a Morse potential.

the singularity.

The Hamiltonian  $H^{(2)}$  can be determined to

$$H^{(2)} = h_1(x_1) + h_2(x_2) + \alpha^2; \quad h_i(x_i) = -\partial_i^2 + V_M(x_i); \quad i = 1, 2 \quad (4.39)$$

and it leads to the energy eigenvalues

$$E_{m,n}^{(2)} = \epsilon_n + \epsilon_m + \alpha^2 \quad (4.40)$$

of the symmetric or antisymmetric (for  $n \neq m$ ) functions

$$\psi_{n,m}^{(2),S/A} = \phi_n(x_1)\phi_m(x_2) \pm \phi_m(x_1)\phi_n(x_2) \quad (4.41)$$

where  $\epsilon_k$  and  $\phi_k(x)$  solve the one-dimensional Morse potential problem [10],[20]:

$$(-\partial^2 + V_M(x)) \phi_k(x) = \epsilon_k \phi_k(x); \quad \epsilon_k = -\alpha^2 s_k^2 \quad (4.42)$$

$$\phi_k = e^{-\frac{\xi}{2}} \cdot \xi^{s_k} \cdot \Phi(-k, 2s_k + 1; \xi); \quad \xi \equiv \frac{2\sqrt{A}}{\alpha} e^{-\alpha x} \quad (4.43)$$

$$\Phi(a, c; x) = 1 + \frac{a x}{c 1!} + \frac{a(a+1) x^2}{c(c+1) 2!} + \dots \quad s_k = \frac{\sqrt{A}}{\alpha} - k - \frac{1}{2} > 0, \quad k = 0, 1, 2, \dots \quad (4.44)$$

The function  $\Phi(a, c; x)$  is the confluent hypergeometric function. The equation for the energy levels of the two-dimensional Hamiltonian  $H^{(2)}$  (4.40) acting on the symmetric and antisymmetric wave functions shows that the energy levels are 2-fold degenerate for all levels with  $n \neq m$ .

The intertwining relations can be used to obtain the majority of the levels of  $H^{(1)}$ , but not all of them. In general it is possible to think of three different cases which have to be considered in order to determine the spectrum and wave functions completely.



1. The first attempt to determine the wave functions of  $H^{(1)}$  is to use the intertwining relations (4.28). The supercharges allow to calculate these wave functions from the wave functions of  $H^{(2)}$ . The energy levels for the wave functions of  $H^{(1)}$  and  $H^{(2)}$  coincide and are given by (4.40).
2. As seen in the one-dimensional case, possible zero modes of  $Q$  might occur for  $H^{(1)}$ . If  $Q$  acts on such zero modes no corresponding bound state wave function for  $H^{(2)}$  exist. This means that the intertwining relations cannot be used to obtain the wave functions of  $H^{(1)}$  from  $H^{(2)}$  for this case because no corresponding states in  $H^{(2)}$  exist.
3. If the action of  $Q$  on a bound state wave function of  $H^{(1)}$  results in a nonnormalizable wave function for  $H^{(2)}$  the same problem as in the former case occurs. A state in  $H^{(1)}$  without a correspondent state in  $H^{(2)}$  cannot be calculated via the intertwining relations from the solved system  $H^{(2)}$ .

In general the intertwining relations (4.28) lead to the connection  $\psi_{n,m}^{(1)} = Q^\dagger \psi_{n,m}^{(2)}$ . The explicit relations can be expanded by using the definition of  $Q^\dagger$  in (4.29), the explicit form of the wave function (4.43) and the Schrödinger equation (4.42) of the one-dimensional Morse potential. The derivation is done in appendix A.1 and leads to

$$\psi_{n,m}^{(1),S} = Q^\dagger \psi_{n,m}^{(2),A} = (\epsilon_m - \epsilon_n) \psi_{n,m}^{(2),S} + D \psi_{n,m}^{(2),A} \quad (4.45)$$

$$\psi_{n,m}^{(1),A} = Q^\dagger \psi_{n,m}^{(2),S} = (\epsilon_m - \epsilon_n) \psi_{n,m}^{(2),A} + D \psi_{n,m}^{(2),S} \quad (4.46)$$

with the operator

$$D = \frac{\alpha^2}{\xi_2 - \xi_1} [\xi_1 + \xi_2 + 2\xi_1 \xi_2 (\partial_{\xi_1} + \partial_{\xi_2})] \quad (4.47)$$

for  $a = -\frac{1}{2}$ . The operator  $Q^\dagger$  is antisymmetric with respect to the exchange of  $x_1$  and  $x_2$  which explains the calculation of the symmetric functions from the antisymmetric and vice versa.  $D$  has a singularity at the line  $\xi_1 = \xi_2$  or  $x_1 = x_2$  respectively. The functions  $\psi_{n,m}^{(1),S/A}$  are therefore only normalizable if the functions the operator acts on vanish for  $\xi_1 = \xi_2$ . For this line the symmetrical functions are written

$$\begin{aligned} \psi_{n,m}^{(2),S}(\xi_1, \xi_2 = \xi_1) &\propto \phi_n(\xi_1) \phi_m(\xi_1) \\ &= e^{-\xi_1} \xi_1^{2\frac{\sqrt{A}}{\alpha} - 1 - n - m} \Phi(-n, 2s_n + 1; \xi_1) \Phi(-m, 2s_n + 1; \xi_1). \end{aligned} \quad (4.48)$$

with the definitions of  $\phi_k$  and  $\Phi(a, c; x)$  in (4.43) and (4.44).  $\Phi$  is just a polynomial and so  $\psi_{n,m}^{(2),S}(\xi_1, \xi_2 = \xi_1)$  is nonzero for all  $\xi_1$  except for  $\xi_1 = 0$ . Thus the singularity in  $D$  can not be compensated generally and no normalizable antisymmetric functions  $\psi_{n,m}^{(1),A}$  exist.

The existence of symmetric functions depends on the behavior of  $\psi_{n,m}^{(2),A}(\xi_1, \xi_2 = \xi_1)$ . The functions are written

$$\begin{aligned} & \lim_{\xi_2 \rightarrow \xi_1} \psi_{n,m}^{(2),A}(\xi_1, \xi_2) = \\ & = e^{-\xi_1} \Phi(-n, 2s_n + 1; \xi_1) \Phi(-m, 2s_m + 1; \xi_1) \lim_{\xi_2 \rightarrow \xi_1} (\xi_1^{s_n} \xi_2^{s_m} - \xi_1^{s_m} \xi_2^{s_n}) \\ & = e^{-\xi_1} \xi_1^{\frac{2\sqrt{A}}{\alpha} - 1} \Phi(-n, 2s_n + 1; \xi_1) \Phi(-m, 2s_m + 1; \xi_1) \lim_{\xi_2 \rightarrow \xi_1} (\xi_1^{-n} \xi_2^{-m} - \xi_1^{-m} \xi_2^{-n}). \end{aligned} \quad (4.49)$$

The limit of the last factor should be evaluated with the multiplied singularity from the operator  $D$ :

$$\begin{aligned} \lim_{\xi_2 \rightarrow \xi_1} \frac{\xi_1^{-n} \xi_2^{-m} - \xi_1^{-m} \xi_2^{-n}}{\xi_2 - \xi_1} & = \xi_1^{-2m} \lim_{\xi_2 \rightarrow \xi_1} \frac{\xi_1^{-n+m} - \xi_2^{-n+m}}{\xi_2 - \xi_1} \\ & = \xi_1^{-2m} \lim_{\xi_2 \rightarrow \xi_1} \frac{(n-m)\xi_2^{-n+m-1}}{1} = (n-m)\xi_1^{-n-m+1}, \end{aligned} \quad (4.50)$$

where l'Hospital's rule is used. So the action of  $D$  leads to normalizable results for  $\psi_{n,m}^{(1),S}$ . The behavior of these functions in dependency on  $m$  and  $n$  can be further investigated. In order to calculate the norm of the wave functions  $\psi_{n,m}^{(1),S}$  it is helpful to introduce the operator  $T = QQ^\dagger$ . It is shown in appendix A.2 that for  $a = -\frac{1}{2}$  this operator can be written as

$$T = (h_1 - h_2)^2 + 2\alpha^2(h_1 + h_2) + \alpha^4, \quad (4.51)$$

with the definition of  $h_i$  in (4.39). The action on the antisymmetric functions  $\psi_{n,m}^{(2),A}$  shows that they are eigenfunctions of  $T$ , the eigenvalues  $t_{n,m}$  can be calculated to (cf. A.2)

$$T\psi_{n,m}^{(2),A} = [(\epsilon_n - \epsilon_m)^2 + 2\alpha^2(\epsilon_n + \epsilon_m) + \alpha^4]\psi_{n,m}^{(2),A} \quad (4.52)$$

$$= \alpha^4[(n-m)^2 - 1][(s_n + s_m)^2 - 1]\psi_{n,m}^{(2),A} \equiv t_{n,m}\psi_{n,m}^{(2),A}. \quad (4.53)$$

Relation (4.53) can be used to determine the norm of the symmetric wave functions

$$\|\psi_{n,m}^{(1),S}\|^2 = \langle \psi_{n,m}^{(2),A} | QQ^\dagger | \psi_{n,m}^{(2),A} \rangle = t_{n,m} \|\psi_{n,m}^{(2),A}\|^2. \quad (4.54)$$

Because the antisymmetric function  $\psi_{n,m}^{(2),A}$  vanishes for  $m = n$ ,  $\psi_{n,m}^{(1),S}$  is also zero. The representation of  $t_{n,m}$  in (4.53) shows additionally that  $t_{n,n\pm 1}$  vanishes and with it also  $\|\psi_{n,n\pm 1}^{(1),S}\|$ . The norms of  $\psi_{n,m}^{(1),S}$  with  $|n-m| > 1$  are finite and positive. The values  $n, m$  have an upper limit because of the condition  $s_k > 0$  from the solution of the one-dimensional Morse potential. With the relation (4.44), the range of the parameters is determined to  $0 \leq k < (\frac{\sqrt{A}}{\alpha} - \frac{1}{2})$  with  $k = n, m$ .

In addition to these calculations the possible cases for energy levels and wave functions of  $H^{(1)}$  without a normalizable counterpart in the spectrum  $H^{(2)}$  have to be examined.

The second case of the three cases described above concerns the states of  $H^{(1)}$  for which  $Q\psi_{n,m}^{(1)} = 0$ . In order to determine these zero modes a property of the system is used. The comparison of the formulas for  $Q$  and  $Q^\dagger$  shows that generally an exchange  $a \leftrightarrow -a$  leads to the exchanges  $Q \leftrightarrow Q^\dagger$  and  $H^{(1)} \leftrightarrow H^{(2)}$ . It is shown in [11] that the range for  $a$  that provides the condition of normalizability of zero modes of  $Q^\dagger$  and the absence of a fall to the center is

$$a \in \left( -\infty, -\frac{1}{4} - \frac{1}{4\sqrt{2}} \right). \quad (4.55)$$

Because of the reflection symmetry of  $Q$  and  $Q^\dagger$ , the condition for normalizable zero modes of  $Q$  is  $a > \left( \frac{1}{4} + \frac{1}{4\sqrt{2}} \right)$ . Because the value  $a = -\frac{1}{2}$  in the investigated model is not in this range, no normalizable zero modes of  $Q$  exist. Additionally the restriction for  $a$  destroys the reflection symmetry because in contrast to  $Q$ ,  $Q^\dagger$  has zero modes.

In the last possible case the action of  $Q$  on a normalizable eigenfunction of  $H^{(1)}$  leads to a nonnormalizable eigenfunction of  $H^{(2)}$ . In order to find out if this case exists, it has to be examined if  $Q$  can affect the normalizability of a state. The Hamiltonian  $H^{(1)}$  acts near  $x_- = 0$  effectively as

$$\begin{aligned} H^{(1)} &\sim -\partial_1^2 - \partial_2^2 + \alpha^2 \sinh^{-2} \left( \frac{\alpha x_-}{2} \right) \\ &\sim \frac{1}{2}(-\partial_+^2 - \partial_-^2) + \frac{4}{x_-^2}. \end{aligned} \quad (4.56)$$

This means that only two different behaviors of the eigenfunctions at the line  $x_- = 0$  are possible,

$$\psi \sim x_-^2 \quad \text{or} \quad \psi \sim \frac{1}{x_-} \quad (4.57)$$

where only the first one is normalizable at  $x_- = 0$ . The operator  $Q$  on the other hand acts effectively as

$$\begin{aligned} Q &\sim 4\partial_+\partial_- + 2\alpha\partial_- + 2\alpha^2 \coth \left( \frac{\alpha x_-}{2} \right) + 2\alpha \coth \left( \frac{\alpha x_-}{2} \right) \partial_+ \\ &\sim 4\partial_+\partial_- + 2\alpha\partial_- + \frac{4\alpha}{x_-} + \frac{4}{x_-}\partial_+. \end{aligned} \quad (4.58)$$

If the normalizability of a function is unaffected by the action of  $H^{(1)}$ , it is not transformed to a nonnormalizable function by the action of  $Q$ . So this third case does also not lead to new eigenfunctions of  $H^{(1)}$  and all eigenfunctions are determined above.

The energy levels of  $V^{(1)}$  are the same as the ones of  $V^{(2)}$  except for the zero modes of  $Q^\dagger$  calculated above. This can be shown with the intertwining relations

$$\begin{aligned} Q^\dagger H^{(2)} \psi_{n,m}^{(2)} &= E_{n,m}^{(2)} Q^\dagger \psi_{n,m}^{(2)} = E_{n,m}^{(2)} \psi_{n,m}^{(1)} \\ &= H^{(1)} Q^\dagger \psi_{n,m}^{(2)} = H^{(1)} \psi_{n,m}^{(1)} = E_{n,m}^{(1)} \psi_{n,m}^{(1)}. \end{aligned} \quad (4.59)$$

The analysis performed above shows that the antisymmetric wave functions  $\psi_{n,m}^{(1),A}$  vanish. Hence the energy levels of  $H^{(1)}$  are nondegenerate.

To summarize the calculations the wave functions of  $H^{(1)}$  are the normalizable symmetric wave functions  $\psi_{n,m}^{(1),S} = Q^\dagger \psi_{n,m}^{(2),A}$  for  $|n-m| > 1$  and  $0 \leq n, m < (\frac{\sqrt{A}}{\alpha} - \frac{1}{2})$  at the nondegenerate energy levels (4.40). The two investigated cases of normalizable wave functions of  $H^{(1)}$  without a partner in  $H^{(2)}$  do not appear, they do not add any levels to the spectrum. The energy eigenvalues and wave functions of  $H^{(1)}$  can be determined completely from the solutions of the problem for  $V^{(2)}$  which is amenable for separation of variables.

#### 4.2.2 Shape Invariance of the Two-Dimensional Morse Model

The one-dimensional Morse potential belongs to the known shape invariant potentials, its properties are listed in Table I of [4]. The SIP-condition (2.40) can also be applied to the two-dimensional potentials  $V^{(1)}$  and  $V^{(2)}$ . The parameter for shape invariance is supposed to be  $a$ , so it is not set to  $-\frac{1}{2}$  anymore, the shape invariance condition reads then

$$V^{(1)}(\vec{x}, a_{k-1}) = V^{(2)}(\vec{x}, a_k) + R(a_{k-1}). \quad (4.60)$$

In this section it is shown that this condition can be used to create a chain of two-dimensional potentials with known solutions basing on the former calculations. In order to simplify the calculations, a shift in the model by  $4\alpha^2 a^2$  is done. The potentials can then be expressed by

$$\begin{aligned} V^{(1)}(\vec{x}, a_k) &= \alpha^2 a_k (2a_k - 1) \sinh^{-2} \left( \frac{\alpha x_-}{2} \right) + V_M(x_1) + V_M(x_2), \\ V^{(2)}(\vec{x}, a_k) &= \alpha^2 a_k (2a_k + 1) \sinh^{-2} \left( \frac{\alpha x_-}{2} \right) + V_M(x_1) + V_M(x_2) \end{aligned} \quad (4.61)$$

and the shift changes physically only the energy of the bound states. The intertwining relations and all relations deduced by the supercharges  $Q$  and  $Q^\dagger$  do not change, because the supercharges are not changed by the shift and commute with every constant factor in the Hamiltonians. The choice of  $a$  as parameter for the shape invariance excludes the original Morse potential parts  $V_M(x_1)$  and  $V_M(x_2)$  from the analysis because they are independent of  $a$ . The  $a_k$ -dependent part of (4.60) is

$$\alpha^2 a_{k-1} (2a_{k-1} - 1) \sinh^{-2} \left( \frac{\alpha x_-}{2} \right) = \alpha^2 a_k (2a_k + 1) \sinh^{-2} \left( \frac{\alpha x_-}{2} \right) + R(a_{k-1}). \quad (4.62)$$

The  $\sinh^{-2}$ -dependent terms can be compared in order to determine the connection between  $a_{k-1}$  and  $a_k$  because  $R$  is independent of  $x_1$  and  $x_2$ . Solving this quadratic equation

$$a_{k-1} (2a_{k-1} - 1) = a_k (2a_k + 1) \quad (4.63)$$

leads to different possible values for  $a_k$ . The first one is  $a_k = -a_{k-1}$  which does not lead to a hierarchy of potentials, this choice would just cause an exchange between the two known

potentials. So the choice for the connection of the parameters is the second solution

$$a_k = a_{k-1} - \frac{1}{2} = a_0 - \frac{k}{2}. \quad (4.64)$$

The remainder  $R(a_k)$  as  $x_-$ -independent term is not needed, it can be set to zero for all  $a_k$ .

So the potentials  $V^{(1)}(\vec{x}, a_{k-1})$  and  $V^{(2)}(\vec{x}, a_k)$  are shape invariant and at the same time the potentials with the same parameter  $a_k$  are supersymmetric partner potentials. This allows to construct a chain of potentials which are alternately connected via the intertwining relations and shape invariance. Using the sign  $\leftrightarrow$  for a connection via the intertwining relations, this chain can be expressed as follows

$$\begin{aligned} V^{(2)}(\vec{x}, a_0) &\leftrightarrow V^{(1)}(\vec{x}, a_0) = V^{(2)}(\vec{x}, a_1) \leftrightarrow V^{(1)}(\vec{x}, a_1) = V^{(2)}(\vec{x}, a_2) \leftrightarrow \dots \\ &\dots = V^{(2)}(\vec{x}, a_{k-1}) \leftrightarrow V^{(1)}(\vec{x}, a_{k-1}) = V^{(2)}(\vec{x}, a_k) \leftrightarrow V^{(1)}(\vec{x}, a_k). \end{aligned} \quad (4.65)$$

The model is already examined for the case  $a = -\frac{1}{2}$  and the condition for  $a$  in order to provide the condition of normalizability of zero modes is  $a < (-\frac{1}{4} - \frac{1}{4\sqrt{2}})$  according to (4.55). Increasing  $k$  by one leads to a decrease of  $a_k$  by  $\frac{1}{2}$  as it can be seen in equation (4.64); it changes in half-integer steps. It is convenient to set  $a_0 = -\frac{1}{2}$ , which is the biggest possible half-integer value, additionally the model with this value for  $a$  is already completely solved.

The wave function of  $H^{(2)}(\vec{x}, a_1)$  can simply be derived from the wave function of  $H^{(1)}(\vec{x}, a_0)$  by a change of parameters

$$\psi_{n,m}^{(2),S}(\vec{x}, a_1) = \psi_{n,m}^{(1),S}(\vec{x}, a_0) = Q^\dagger(a_0)\psi_{n,m}^{(2),A}(\vec{x}, a_0) \quad (4.66)$$

for  $|n - m| > 1$ . At the same time it is clear that no antisymmetric functions  $\psi_{n,m}^{(2),A}(\vec{x}, a_1)$  exist, because they do not exist for  $H^{(1)}(a_0)$ . The energy levels can be determined with the formula (4.40) and the SIP-condition (4.60), the shift of the potentials leads to a shift of  $\alpha^2$  compared to the energies determined for  $a_0$  in the former section

$$\begin{aligned} E^{(2)}(a_1) \psi_{n,m}^{(2),S}(\vec{x}, a_1) &= H^{(2)}(\vec{x}, a_1) \psi_{n,m}^{(2),S}(\vec{x}, a_1) \\ &= H^{(1)}(\vec{x}, a_0) - R(a_0) \psi_{n,m}^{(1),S}(\vec{x}, a_0) \\ &= [\epsilon_n + \epsilon_m] \psi_{n,m}^{(1),S}(\vec{x}, a_0) \\ &= [\epsilon_n + \epsilon_m] \psi_{n,m}^{(2),S}(\vec{x}, a_1) \end{aligned} \quad (4.67)$$

with the limitation  $|n - m| > 1$  like above.

The intertwining relations allow it to do the next step in order to calculate the wave functions of  $H^{(1)}(\vec{x}, a_1)$ . In general they can be determined via

$$\psi_{n,m}^{(1),A}(\vec{x}, a_1) = Q^\dagger(a_1)\psi_{n,m}^{(2),S}(\vec{x}, a_1) = Q^\dagger(a_1)Q^\dagger(a_0)\psi_{n,m}^{(2),A}(\vec{x}, a_0) \quad (4.68)$$

and no symmetric wave functions exist. Like in the first step the factor  $Q^\dagger(a_0)\psi_{n,m}^{(2),A}(\vec{x}, a_0)$  could contain zero modes of  $Q^\dagger(a_1)$  leading to vanishing wave functions which would lead

to a change of the range for  $|n - m|$ . The energy levels of  $H^{(1)}(\vec{x}, a_1)$  are the same as the ones determined in (4.67). Instead of calculating the zero modes for this particular  $Q^\dagger(a_1)$ , the analysis of these zero modes is performed for the generalized case for all  $a_k$ .

The steps performed above can be used repeatedly in order to determine the wave functions of all members of the chain (4.65). The wave functions

$$\psi_{m,n}^{(1)}(\vec{x}, a_k) = Q^\dagger(a_k)Q^\dagger(a_{k-1})\dots Q^\dagger(a_0)\psi_{n,m}^{(2),A}(\vec{x}, a_0). \quad (4.69)$$

change between symmetric and antisymmetric behavior, depending on the number of actions of  $Q^\dagger(a)$ . The energy levels are

$$E^{(1)}(a_k) = E^{(2)}(a_0) - \sum_{i=0}^{k-1} R(a_i) = E^{(2)}(a_0) = \epsilon_n + \epsilon_m \quad (4.70)$$

because  $R(a_k) = 0$  and have no degeneracy. The last step to the complete solution for all Hamiltonians  $H^{(1)}(a_k)$  is the determination of the zero modes of  $Q^\dagger(a_k)$  in order to find out which levels belong to nonvanishing wave functions. The determination is done in A.3. It requires some algebraic steps and it turns out that the wave functions are only nonvanishing for  $|n - m| > (k + 1)$ .

As in the analysis for  $a_0 = -\frac{1}{2}$  the other possibilities for wave functions in the spectrum of  $H^{(1)}(\vec{x}, a_k)$  that cannot be calculated from the spectrum of  $H^{(2)}(\vec{x}, a_0)$  have to be examined. Zero modes of  $Q(a_k)$  would again require a positive value of  $a_k$ , this is not possible due to the restricted range for  $a$ . For the third case the behavior of the Hamiltonian and the supercharge near the singularity are studied. Near  $x_- = 0$ , the Hamiltonian acts as

$$H^{(1)}(a_k) \sim \frac{1}{2}(-\partial_+^2 - \partial_-^2) + \frac{2(k+1)(k+2)}{x_-^2} \quad (4.71)$$

and the supercharge as

$$Q(a_k) \sim 4\partial_+\partial_- + 2\alpha\partial_- + \frac{4(k+1)\alpha}{x_-} + \frac{4(k+1)^2}{x_-}\partial_+ \quad (4.72)$$

and again  $Q(a_k)$  is not able to affect the normalizability of a wave function. Thus the property of shape invariance and the intertwining relations can be used not only to determine the wave functions and energy levels of the potential  $V^{(2)}(\vec{x}, a_0)$ , which is amenable to separation of variables, but also to solve the problem for a whole chain of potentials  $V^{(1)}(\vec{x}, a_k)$ ,  $V^{(2)}(\vec{x}, a_k)$  via the well-known solution of the one-dimensional Morse potential. It is proven in A.3 by generalizing (4.54) for that the spectra of the related Hamiltonians are nondegenerate because only symmetric or antisymmetric solutions exist. They can be calculated by equation (4.70) where the indices  $n, m$  are restricted by  $|n - m| > k + 1$  and  $n, m < (\frac{\sqrt{A}}{\alpha} - \frac{1}{2})$  which means that the number of potentials in the chain with normalizable wave functions is limited. The wave functions can be determined with the operators  $Q^\dagger(a_k)$  and the original wave functions (4.41) via equation (4.69).

## 5 Conclusions

It is shown in this thesis that the formalism of supersymmetric quantum mechanics has multiple uses. The first use is the possibility of solving well-known problems in a new manner. The factorization method was already used in the early days of quantum mechanics for the solution of the harmonic oscillator but as shown above it can be used with the other tools of supersymmetric quantum mechanics to solve the eigenvalue problems of other potentials. The more important benefit of the method is the possibility to create new potentials and to determine their eigenvalues and bound state wave functions directly from the original potential. So additionally to the plain solution of fairly easy potentials the solution of more complex potentials can be obtained. This can also be seen in the first example of the two-dimensional treatment. The introduced formalism can be used to solve the sketched two-dimensional eigenvalue problem for the Pauli Hamiltonian via scalar Hamiltonians. Of course a two-dimensional treatment is not sufficient for the solution of complicated three-dimensional problems but the example gives an insight in the usefulness of higher-dimensional supersymmetric quantum mechanics. The last example is the most complicated one in this thesis. It is only one example for the solution of a two-dimensional potential that is not amenable to separation of variables, [9] names also generalized Pöschl-Teller and Scarf II potentials as problems that can be solved with the same treatment. This example of the two-dimensional Morse potential shows even more than the others that supersymmetric quantum mechanics can be the key to the solution of complicated potentials. Furthermore the well-known solutions of the one-dimensional problem can be used to determine the bound state wave functions and energy levels of a whole class of two-dimensional Morse potentials with the concept of shape invariance.

The main goals of the thesis were on the one hand to present the basic formalism of supersymmetric quantum mechanics and its application to basic examples and on the other hand to show a more complicated extension of the formalism. The presentation of the content in section 2 on an undergraduate level can already be found in several papers, theses and occasional in quantum mechanics books. The derivation of the formalism is straightforward and does not cause major problems for a student who finished a quantum mechanics course. However the research in the field of supersymmetric quantum mechanics offers many different approaches for the use and the extension of this formalism. Most of these fields of application are presented on a high level and require a lot of effort to reach an understanding for an undergraduate student. So another goal of the second part of this thesis is to explain at least a small part of higher-dimensional supersymmetric quantum mechanics on an educational level. The two presented examples cover only a part of the higher-dimensional treatment and already this topic offers many more examples for the use of supersymmetric quantum mechanics. The summary on an undergraduate level could be done for a wide range of already examined applications. So additionally to the examination of different applications of the formalism, the presentation of already known results gives a perspective for future works and theses on this topic.

## A Derivations

### A.1 Connection between the Wave Functions in the Two-Dimensional Morse Potential

The definition of  $C_+$  and  $C_-$  in (4.35) allows to calculate  $C_1$  and  $C_2$  to

$$\begin{aligned} C_1 &= \frac{1}{2}(C_- + C_+) = 2a\alpha \left[ \coth\left(\frac{\alpha x_-}{2}\right) + 1 \right], \\ C_2 &= \frac{1}{2}(C_- - C_+) = 2a\alpha \left[ \coth\left(\frac{\alpha x_-}{2}\right) - 1 \right] \end{aligned} \quad (\text{A.1})$$

and to insert these expressions in the explicit form of  $Q^\dagger$  in (4.29):

$$\begin{aligned} &Q^\dagger \psi_{n,m}^{(2),A} \\ &= \left\{ (\partial_1^2 - \partial_2^2) - C_i \partial_i + \frac{1}{4} (C_+ C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)) \right\} \psi_{n,m}^{(2),A} \\ &= \left\{ (\partial_1^2 - \partial_2^2) + 2a\alpha \left[ \left( \coth\left(\frac{\alpha x_-}{2}\right) + 1 \right) \partial_1 + \left( \coth\left(\frac{\alpha x_-}{2}\right) - 1 \right) \partial_2 \right] + B \right\} \psi_{n,m}^{(2),A} \quad (\text{A.2}) \\ &\text{with } B = 4a^2\alpha^2 \coth\left(\frac{\alpha x_-}{2}\right) - V_M(x_1) + V_M(x_2). \end{aligned}$$

The action of the single summands on the wave functions can be examined individually. The action of the second derivatives and the one-dimensional Morse potentials can be determined by the Schrödinger equation (4.42)

$$\begin{aligned} &[\partial_1^2 - \partial_2^2 - V_M(x_1) + V_M(x_2)] \psi_{n,m}^{(2),A} \\ &\stackrel{(4.39)}{=} [-h_1 + h_2] [\phi_n(x_1)\phi_m(x_2) - \phi_m(x_1)\phi_n(x_2)] \\ &\stackrel{(4.42)}{=} -\epsilon_n \phi_n(x_1)\phi_m(x_2) + \epsilon_m \phi_m(x_1)\phi_n(x_2) + \epsilon_m \phi_n(x_1)\phi_m(x_2) - \epsilon_n \phi_m(x_1)\phi_n(x_2) \\ &= (\epsilon_m - \epsilon_n) [\phi_n(x_1)\phi_m(x_2) + \phi_m(x_1)\phi_n(x_2)] \\ &= (\epsilon_m - \epsilon_n) \psi_{n,m}^{(2),S}. \end{aligned} \quad (\text{A.3})$$

The second part of (A.2) is

$$\begin{aligned} &2a\alpha \left[ \left( \coth\left(\frac{\alpha x_-}{2}\right) + 1 \right) \partial_1 + \left( \coth\left(\frac{\alpha x_-}{2}\right) - 1 \right) \partial_2 + 2a\alpha \coth\left(\frac{\alpha x_-}{2}\right) \right] \psi_{n,m}^{(2),A} \\ &= 2a\alpha \left[ \frac{2e^{\alpha x_-}}{e^{\alpha x_-} - 1} \partial_1 + \frac{2}{e^{\alpha x_-} - 1} \partial_2 + 2a\alpha \frac{e^{\alpha x_-} + 1}{e^{\alpha x_-} - 1} \right] \psi_{n,m}^{(2),A}. \end{aligned} \quad (\text{A.4})$$



The relation  $\xi = \frac{2\sqrt{A}}{\alpha} e^{-\alpha x}$  given in (4.43) can be used to deduce some helpful equations:

$$\begin{aligned}
\xi_i &= \frac{2\sqrt{A}}{\alpha} e^{-\alpha x_i} \Leftrightarrow x_i = -\frac{1}{\alpha} \ln \left( \frac{\alpha \xi_i}{2\sqrt{A}} \right), \\
\partial_{\xi_i} &= \frac{\partial}{\partial \xi_i} = \frac{\partial}{\partial x_i} \frac{\partial x_i}{\partial \xi_i} = -\frac{1}{\alpha \xi_i} \partial_i \Leftrightarrow \partial_i = -\alpha \xi_i \partial_{\xi_i}, \\
e^{\alpha x_-} &= e^{\alpha(x_1 - x_2)} = e^{\alpha x_1} e^{-\alpha x_2} = \frac{\xi_2}{\xi_1}, \\
e^{\alpha x_-} - 1 &= \frac{\xi_2}{\xi_1} - 1 = \frac{\xi_2 - \xi_1}{\xi_1}, \\
\frac{2e^{(\alpha x_-)}}{e^{\alpha x_-} - 1} &= 2 \frac{\xi_2}{\xi_2 - \xi_1}; \quad \frac{2}{e^{\alpha x_-} - 1} = 2 \frac{\xi_1}{\xi_2 - \xi_1}; \quad \frac{e^{(\alpha x_-)} + 1}{e^{(\alpha x_-)} - 1} = \frac{\xi_2 + \xi_1}{\xi_2 - \xi_1}.
\end{aligned} \tag{A.5}$$

With these equations (A.4) can be rewritten to

$$\begin{aligned}
&2a\alpha \left[ 2 \frac{\xi_2}{\xi_2 - \xi_1} (-\alpha \xi_1 \partial_{\xi_1}) + 2 \frac{\xi_1}{\xi_2 - \xi_1} (-\alpha \xi_2 \partial_{\xi_2}) + 2a\alpha \frac{\xi_2 + \xi_1}{\xi_2 - \xi_1} \right] \psi_{n,m}^{(2),A} \\
&= \frac{4a^2\alpha^2}{\xi_2 - \xi_1} \left[ \xi_1 + \xi_2 - \frac{\xi_1 \xi_2}{a} (\partial_{\xi_1} + \partial_{\xi_2}) \right] \psi_{n,m}^{(2),A} \\
&\equiv D \psi_{n,m}^{(2),A}
\end{aligned} \tag{A.6}$$

which together with the first part leads directly to the given equations.

## A.2 The Operator $T$

The determination of  $T$  requires the multiplication of the two operators  $Q$  (4.30) and  $Q^\dagger$  (4.29), for the examined model they are written

$$\begin{aligned}
Q &= -h_1 + h_2 + 4a^2\alpha^2 \coth \left( \frac{\alpha x_2}{2} \right) - 4a\alpha \partial_- - 4a\alpha \coth \left( \frac{\alpha x_2}{2} \right) \partial_+, \\
Q^\dagger &= -h_1 + h_2 + 4a^2\alpha^2 \coth \left( \frac{\alpha x_2}{2} \right) + 4a\alpha \partial_- + 4a\alpha \coth \left( \frac{\alpha x_2}{2} \right) \partial_+
\end{aligned} \tag{A.7}$$

with the introduced  $h_i = -\partial_i + V_M(x_i)$ . The calculation of  $QQ^\dagger$  contains the application of the chain rule for the partial derivative  $\partial_-$ . The derivation of the Morse potentials  $\partial_\pm(-V_M(x_1) + V_M(x_2))$  is done with the representation  $\partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_2)$  and to simplify the calculation the summands containing a Morse potential can be rearranged separately. The rest of the calculation is done straightforwardly, the factor  $a$  is set to  $-\frac{1}{2}$  and the result is

$$T = (h_1 - h_2)^2 + 2\alpha^2(h_1 + h_2) + \alpha^4 \tag{A.8}$$

The calculations of the eigenvalues  $t_{n,m}$  can be done straightforwardly with the definitions of the wave functions (4.41) and the Schrödinger equation of the one-dimensional

Morse potential (4.42). For clarity the abbreviations  $\eta_a = \phi_n(x_1)\phi_m(x_2)$  and  $\eta_b = \phi_m(x_1)\phi_n(x_2)$  are used.

$$\begin{aligned}
T\psi_{n,m}^{(2),A} &= [h_1^2 - 2h_1h_2 + h_2^2 + 2\alpha^2(h_1 + h_2) + \alpha^4][\eta_a - \eta_b] \\
&= \epsilon_n^2\eta_a - \epsilon_m^2\eta_b - 2\epsilon_n\epsilon_m\eta_a + 2\epsilon_n\epsilon_m\eta_b + \epsilon_m^2\eta_a - \epsilon_n^2\eta_b + \\
&\quad + 2\alpha^2(\epsilon_n\eta_a - \epsilon_m\eta_b + \epsilon_m\eta_a - \epsilon_n\eta_b) + \alpha^4(\eta_a - \eta_b) \\
&= (\epsilon_n^2 - 2\epsilon_n\epsilon_m + \epsilon_m^2)(\eta_a - \eta_b) + 2\alpha^2(\epsilon_n + \epsilon_m)(\eta_a - \eta_b) + \alpha^4(\eta_a - \eta_b) \\
&= [(\epsilon_n - \epsilon_m)^2 + 2\alpha(\epsilon_n + \epsilon_m) + \alpha^4]\psi_{n,m}^{(2),A}. \tag{A.9}
\end{aligned}$$

The equivalence of (4.52) and (4.53) can be shown with the definition of  $s_k$  in (4.44) which leads to the relation  $n - m = s_m - s_n$  and the definition of  $\epsilon_k$  in (4.42).

$$\begin{aligned}
&\alpha^4[(n - m)^2 - 1][(s_n + s_m)^2 - 1] \\
&= \alpha^4[(s_n - s_m)^2 - 1][(s_n + s_m)^2 - 1] \\
&= \alpha^4[(s_n^2 - s_m^2)^2 - (s_n - s_m)^2 - (s_m + s_n)^2 + 1] \\
&= \alpha^4[(s_n^2 - s_m^2)^2 - 2(s_n^2 + s_m^2) + 1] \\
&= (\epsilon_n - \epsilon_m)^2 + 2\alpha^2(\epsilon_n + \epsilon_m) + \alpha^4. \tag{A.10}
\end{aligned}$$

### A.3 Zero Modes of $Q^\dagger(a_k)$

Reference [11] offers a starting point to deduce the zero modes of  $Q^\dagger(a_k)$  by beginning with the calculation of the relation between the operators  $T(a_k)$  and  $T(a_{k-1})$  with  $T(a_k) = Q(a_k)Q^\dagger(a_k)$ , as above. The calculation is similar to the one done in A.2, it additionally requires the use of  $a_k = -\frac{k+1}{2}$  which was determined in (4.64) and leads to the relation

$$Q(a_k)Q^\dagger(a_k) = Q^\dagger(a_{k-1})Q(a_{k-1}) + \alpha^2(2k + 1)[2H^{(1)}(\vec{x}, a_{k-1}) + \alpha^2(2k^2 + 2k + 1)]. \tag{A.11}$$

The operators  $T(a_k)$  can again be used to determine the norm of a wave function  $\psi_{n,m}^{(1)}(\vec{x}, a_k)$  and the form of equation (A.11) can be used to determine a general formula. The norm for  $k = 0$  was computed in (4.54) and contains the factor  $t_{n,m}$ . The norm for  $k = 1$  according to equation (4.69) is

$$\begin{aligned}
\|\psi_{n,m}^{(1)}(\vec{x}, a_1)\|^2 &= \langle \psi_{n,m}^{(2,A)}(\vec{x}, a_0) | Q(a_0)Q(a_1)Q^\dagger(a_1)Q^\dagger(a_0) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle \\
&= \langle \psi_{n,m}^{(2,A)}(\vec{x}, a_0) | Q(a_0) [Q^\dagger(a_0)Q(a_0) + 3\alpha^2[2H^{(1)}(\vec{x}, a_0) + 5\alpha^2]] Q^\dagger(a_0) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle \\
&= (t_{n,m}^2 + t_{n,m} \cdot 3\alpha^2[2(\epsilon_n + \epsilon_m) + 5\alpha^2]) \|\psi_{n,m}^{(2,A)}(\vec{x}, a_0)\|^2 \\
&= t_{n,m} (t_{n,m} + 3\alpha^2[2(\epsilon_n + \epsilon_m) + 5\alpha^2]) \|\psi_{n,m}^{(2,A)}(\vec{x}, a_0)\|^2, \tag{A.12}
\end{aligned}$$

where the intertwining relations are used to determine

$$\begin{aligned}
H^{(1)}(\vec{x}, a_0)Q^\dagger(a_0) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle &= Q^\dagger(a_0)H^{(2)}(\vec{x}, a_0) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle \\
&= (\epsilon_n + \epsilon_m)Q^\dagger(a_0) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle \tag{A.13}
\end{aligned}$$

This simplest case already shows that a recursion occurs. Carrying out the next steps and using the intertwining relations, the SIP-condition and (A.11) lead to the general result

$$\begin{aligned} \|\psi_{n,m}^{(1)}(\vec{x}, a_k)\|^2 &= \langle \psi_{n,m}^{(2,A)}(\vec{x}, a_0) | Q(a_0) \dots Q(a_k) Q^\dagger(a_k) \dots Q^\dagger(a_0) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle \\ &= \langle \psi_{n,m}^{(2,A)}(\vec{x}, a_0) | T(a_0) \cdot (T(a_0) + \Gamma_1) \cdot \dots \cdot (T(a_0) + \Gamma_1 + \dots + \Gamma_k) | \psi_{n,m}^{(2,A)}(\vec{x}, a_0) \rangle \end{aligned} \quad (\text{A.14})$$

$$\text{with } \Gamma_k = \alpha^2(2k+1)[2H^{(1)}(\vec{x}, a_{k-1}) + \alpha^2(2k^2 + 2k + 1)]. \quad (\text{A.15})$$

The goal of the following steps is the factorization of the prefactors in (A.14) in terms of  $(n-m)$ . The sums of the  $\Gamma_k$  terms in the norm can with (A.15) be written as

$$\begin{aligned} \sum_{j=1}^k \gamma_j &= \sum_{j=1}^k [2\alpha^2(2j+1)(\epsilon_n + \epsilon_m) + \alpha^4(2j+1)(2j^2 + 2j + 1)] \\ &= \sum_{j=1}^k [2\alpha^2(2j+1)(\epsilon_n + \epsilon_m) + \alpha^4[(j+1)^4 - j^4]] \\ &= 2\alpha^2 k(k+2)(\epsilon_n + \epsilon_m) + \alpha^4[(k+1)^4 - 1] \\ &= \alpha^4[-2(s_n^2 + s_m^2)k(k+2) + (k+1)^4 - 1] \end{aligned} \quad (\text{A.16})$$

where  $\gamma_k$  denote the eigenvalues of  $\Gamma_k$ . The norm contains the product over the sums plus the eigenvalues  $t_{n,m}$

$$\begin{aligned} &\prod_{j=1}^k \left( t_{n,m} + \sum_{i=1}^j \gamma_i \right) \\ &= \alpha^{4k} \prod_{j=1}^k \{ [(n-m)^2 - 1][(s_n + s_m)^2 - 1] - 2(s_n^2 + s_m^2)j(j+2) + (j+1)^4 - 1 \} \end{aligned} \quad (\text{A.17})$$

which can be factorized with some algebra. The norm can then be determined to

$$\|\psi_{n,m}^{(1)}(\vec{x}, a_k)\|^2 = \alpha^{4k} \|\psi_{n,m}^{(2)}(\vec{x}, a_0)\|^2 t_{n,m} \prod_{j=1}^k [(n-m)^2 - (j+1)^2][(s_n + s_m)^2 - (j+1)^2]. \quad (\text{A.18})$$

It was already shown that the norm of the wave functions  $\psi_{n,m}^{(1)}(\vec{x}, a_k)$  vanishes for  $|n-m| \leq 1$ . The difference in the factor containing  $(n-m)$  in equation(A.18) shows that the lower boundary for nonvanishing wave functions depends on  $k$ . Bound state wave functions only exist for  $|n-m| > (k+1)$ .

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