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Supersymmetric Quantum Mechanics

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Abstract

This bachelor thesis contains an introduction into supersymmetric quantum mechanics(SUSYQM). SUSYQM provides a different way of solving quantum mechanical problems. The thesis explains the basic concepts of SUSYQM, such as the factorization of a Hamiltonian, the definition of its superpotential and the shape invariant potentials. The SUSYQM framework was applied to some problems such as the infinite square well potential, the harmonic oscillator, the radial solution to the Hydrogen atom and isospectral deformation of potentials.

Populärvetenskapligt sammanfattning

Man skulle förvänta sig att de mikroskopiska fysikaliska lagarna har samma karaktär som de man upplever i den vardagliga tillvaron, men så är det inte. Fysikaliska system som är mycket små uppvisar egenskaper som inte har någon motsvarighet i större fysikaliska system.

Exempelvis, så går det inte att bestämma en partikels läge och hastighet noggrant samtidigt. Om man försöker beräkna hastigheten exakt tappar man noggrannhet i positionen och omvänt. Kvantfysiken skapar en bild av hur världen fungerar på den mikroskopiska skalan som skiljer sig avsevärt från den klassiska fysikens värld. Den förklarar de funktioner som är bakomliggande för hur atomer fungerar och interagerar.

Själva benämningen kvantfysik kommer från ordet kvanta vilket beskriver de minsta paket av energi som kan överföras. Partiklar så som elektroner eller fotoner visar i vissa situationer partikelegenskaper och i andra situationer vågegenskaper. I kvantmekaniken använder man en vågfunktionen som innehåller all information om de kvantmekaniska systemet.

För att beskriva system som atomer, metaller, molekyler och subatomära system är kvantmekaniken nödvändig. Där har man oftast en modell där partikeln är fångad i en potential. Där det är möjligt att beräkna vilka energier som kvantmekaniska systemet kan anta, där de möjliga energierna kallas energitillstånd.

I kvantmekaniken finns det en del udda oförklarade beteende så som faktumet att orelaterade potentialer har samma energitillstånd. I supersymmetrisk kvantmekanik så utnyttjar man en underliggande symmetri till potentialen som förklarar dessa beteenden.

Supersymmetrisk kvantmekanik är en metod för att finna energitillstånd för en potential och hitta potentialer med identiskt energitillstånd samt liknande energitillstånd. Genom att använda supersymmetrisk kvantmekanik så kan man identifiera analytiska lösningar till potentialer som tidigare ej haft en analytisk lösning.

Man kan använda supersymmetrisk kvantmekanik till att lösa ett antal olika problem så som att finna energitillstånden för väteatomen och den klassiska partikel i låda problemet, där man placerar en partikel i en oändligt djup låda och kan hitta de möjliga energitillstånden.

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1 Introduction

In classical quantum mechanics with the Schrödinger equation and its relativistic counterpart, the Dirac equation there are some odd properties, such as the fact that very different and seemingly unrelated potentials give rise to the same spectrum of allowed energies, as well as the fact that certain Hamiltonians seem like they can be split into two factors but with an unexplained leftover.

Early on in the development of quantum mechanics Dirac found a method to factorize the harmonic oscillator and Schrödinger noted symmetries in the solutions to his equation. Later on Schrödinger also found a way to factorize the Hamiltonian for the hydrogen atom. This indicates an underlying symmetry.

In an attempt to unify the forces in the standard model, a mathematical concept called supersymmetry was created that led to an extension of the standard model. Supersymmetry relates fermions (half integer spin) to bosons (integer spin) and vice versa [7] in order to obtain a unified description of all basic interactions of nature. However supersymmetry is in itself not observed, so in order for it to exist the symmetry need to be spontaneously broken.

To study supersymmetry breaking in a simple setting, supersymmetric quantum mechanics was introduced. This led to the concept of a superpotential and partner potentials, where the partner potentials are derivable from the superpotential. Supersymmetric quantum mechanics was originally considered a toy model, more of a thought experiment. But it got real consideration when it started to explain some of the odd behaviours of quantum mechanics as well as providing solutions to previously unsolved potentials.

Supersymmetric quantum mechanics provide a new view of quantum mechanics. Instead of asking “For a given potential, what are the allowed energies it produces?” one could ask “For a given energy, what are the potentials that could have produced it?” [1]. It provides a deeper understanding of why certain potentials are analytically solvable.

This thesis will go through the background of supersymmetric quantum mechanics, the derivation of the superpotential and the corresponding partner potentials that can be created using it. As well as shape invariant potentials and how they can be used to find the energy spectrum and how to perform an isospectral deformation. These concepts will then be applied to the infinite square well to find its partner potentials. The concept of shape invariant potentials will be used to find the energy spectrum for the harmonic oscillator. The supersymmetric framework will also be applied to both the relativistic and non-relativistic hydrogen atom. In the end the an isospectral deformation will be performed to the finite square well as well as on a custom wavefunction.

2 Background

2.1 Hamiltonian Factorization and the Superpotential

Consider a one dimensional quantum system described by a Hamiltonian $H^{(1)}$ with a time independent potential $V^{(1)}(x)$ for unbroken cases of SUSYQM.

$$H^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x). \quad (2.1)$$

In order to use SUSYQM the ground state energy of the Hamiltonian needs to be zero, $E_0^{(1)} = 0$. This can be achieved by shifting the potential with the the ground state energy, this will result in a shifted energy spectrum, but it will have no other effect on quantum system.

So it is possible to study how the Hamiltonian acts on the ground state wave function

without loss of generality, by using the Schrödinger equation

$$H^{(1)}\psi_0(x) = -\frac{\hbar^2}{2m}\psi_0''(x) + V^{(1)}(x)\psi_0(x) = 0, \quad (2.2)$$

where ψ_0 refers to the nodeless ground state wave function belonging to the Hamiltonian $H^{(1)}$. It is possible to rewrite eq. (2.2) to obtain the potential

$$V^{(1)}(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0(x)}. \quad (2.3)$$

Since the ground state is node less, the potential $V^{(1)}$ will be well defined. So it will be possible to construct the Hamiltonian either by knowing the potential $V^{(1)}$ or by knowing the ground state wave function.

It is possible to factorize the Hamiltonian as

$$H^{(1)} = A^\dagger A \quad (2.4)$$

with the operators

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \quad (2.5)$$

$W(x)$ is a potential known as the super potential. So by using the operators A and A^\dagger it will be possible to write $V^{(1)}(x)$ in terms of the superpotential($W(x)$)

$$\begin{aligned} H^{(1)} &= A^\dagger A = \left(-\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \left(\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) \end{aligned} \quad (2.6)$$

$$\implies V^{(1)}(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x). \quad (2.7)$$

The obtained expression eq. (2.7) is called the Riccati equation. Using the Riccati equation, eq. (2.7) together with the fact that the Hamiltonian can be factorized, eq. (2.4), as well as the ground state energy condition gives

$$-\frac{\hbar^2}{2m}\psi_0''(x) + \left(W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) \right) \psi_0(x) = 0. \quad (2.8)$$

Rearranging the expression and using the chain rule for differentiation gives

$$\frac{\psi_0''(x)}{\psi_0(x)} = \left(\frac{\sqrt{2m}}{\hbar} W(x) \right)^2 - \frac{\sqrt{2m}}{\hbar} W'(x) \quad (2.9)$$

$$\implies \left(\frac{\psi'_0(x)}{\psi_0(x)} \right)^2 + \left(\frac{\psi'_0(x)}{\psi_0(x)} \right)' = \left(\frac{\sqrt{2m}}{\hbar} W(x) \right)^2 - \frac{\sqrt{2m}}{\hbar} W'(x). \quad (2.10)$$

One solution to this equation is

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi'_0(x)}{\psi_0(x)} = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln(\psi_0(x)). \quad (2.11)$$

Note that this implies

$$A\psi_0^{(1)}(x) = 0 \implies -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \psi_0^{(1)}(x) + W(x)\psi_0^{(1)}(x) = 0 \quad (2.12)$$

$$\implies \psi_0^{(1)}(x) = N \exp \left(-\frac{\sqrt{2m}}{\hbar} \int_{-\infty}^x W(y) dy \right) \quad (2.13)$$

where N is a normalization constant.

2.2 Partner Potentials

In the last section the Hamiltonian was factorized as $H^{(1)} = A^\dagger A$, by instead considering the factorization

$$H^{(2)} = AA^\dagger = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(2)}(x). \quad (2.14)$$

This can be rewritten to form a new potential

$$V^{(2)}(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x), \quad (2.15)$$

the new potential $V^{(2)}$ and the first potential $V^{(1)}$ are related to the same super potential $W(x)$,

$$V^{(2)}(x) - V^{(1)}(x) = H^{(2)} - H^{(1)} = [A, A^\dagger] = 2\frac{\hbar}{\sqrt{2m}} W'(x). \quad (2.16)$$

The two potentials are called partner potentials.

2.3 Energy Level Relations

Now consider the energy eigenvalues $E_n^{(1,2)}$ of the two Hamiltonians, the energy levels of the first Hamiltonian can be obtained from the Schrödinger equation

$$H^{(1)}\psi_n^{(1)}(x) = A^\dagger A\psi_n^{(1)}(x) = E_n^{(1)}\psi_n^{(1)}(x). \quad (2.17)$$

For the second Hamiltonian, $H^{(2)} = AA^\dagger$ the ansatz $\psi_m^{(2)}(x) = A\psi_n^{(1)}(x)$ is used, where $\psi_n^{(1)}(x)$ is the wavefunction corresponding to the first Hamiltonian and $\psi_n^{(2)}(x)$ is the wavefunction corresponding to the second Hamiltonian. Making it possible to find the energy levels for the second Hamiltonian, $H^{(2)}$

$$H^{(2)}\psi_m^{(2)}(x) = H^{(2)}(A\psi_n^{(1)}(x)) = AA^\dagger A\psi_n^{(1)}(x) = E_n^{(1)}(A\psi_n^{(1)}(x)). \quad (2.18)$$

This means that the eigenfunction of $H^{(2)}$ is $A\psi_n^{(1)}(x)$ with the eigenvalues $E_n^{(1)}$. In the same way the Schrödinger equation for the second Hamiltonian gives

$$H^{(2)}\psi_m^{(2)}(x) = AA^\dagger\psi_m^{(2)}(x) = E_m^{(2)}\psi_m^{(2)}(x), \quad (2.19)$$

which gives a similar relation for the first Hamiltonian.

$$H^{(1)}(A^\dagger\psi_m^{(2)}(x)) = A^\dagger AA^\dagger\psi_m^{(2)}(x) = A^\dagger(H^{(2)}\psi_m^{(2)}(x)) = E_m^{(2)}(A^\dagger\psi_m^{(2)}(x)), \quad (2.20)$$

This shows that the eigenfunctions of $H^{(1)}$ are $A^\dagger\psi_m^{(2)}(x)$ with the corresponding eigenvalues $E_m^{(2)}$. So the spectra of $H^{(1)}$ and $H^{(2)}$ are equal except for the ground state, that only exist in the first quantum system $H^{(1)}$.

Studying eq. (2.17) and (2.20) and normalizing the ground state gives $m = n + 1$ and the relations

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad (2.21)$$

$$\psi_n^{(2)}(x) = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A\psi_{n+1}^{(1)}(x), \quad \psi_{n+1}^{(1)}(x) = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger\psi_n^{(2)}(x), \quad (2.22)$$

$$n = 0, 1, 2, \dots \quad (2.23)$$

The operators A and A^\dagger relate the energy states for the two Hamiltonians, and their energy spectra are degenerate except for the ground state for the first Hamiltonian. So if we have a quantum system that is exactly solvable with at least one known bound state, it is possible to construct a partner potential through SUSYQM with the same energy spectrum except for the ground state energy. This can be used to find more analytically solvable potentials and is one of the main achievements of SUSYQM.

2.4 Chain of Hamiltonians

The two partner Hamiltonians have the same energy spectrum except for the one level as seen in the last subsection. This was seen by shifting the first Hamiltonian by the ground state energy to get a zero energy ground state, in order to factorize it. By instead

shifting the second Hamiltonian to have a zero energy ground state and ignoring the first Hamiltonian. It is possible to construct a third partner potential.

Lets start with the original system

$$H^{(1)} = A_1^\dagger A_1 - E_0^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x) \quad (2.24)$$

with the potential

$$V^{(1)}(x) = W^1(x)^2 - \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W^{(1)}(x) - E_0^{(1)} \quad (2.25)$$

where

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(1)}(x), A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(1)}(x) \quad (2.26)$$

and $E_0^{(1)}$ is the ground state energy of $H^{(1)}$. From this it is possible to get the partner Hamiltonian and the corresponding potential, using eq. (2.14) and (2.15) and shifting it by the original energy ground state energy.

$$H^{(2)} = A_1 A_1^\dagger - E_0^{(1)} = -\frac{\hbar}{2m} \frac{d^2}{dx^2} + V^{(2)}(x) \quad (2.27)$$

$$\begin{aligned} V^{(2)} &= W^{(1)}(x)^2 + \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W^{(1)}(x) - E_0^{(1)} \\ &= V^{(1)}(x) + \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W^{(1)}(x) = V^{(1)}(x) - \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \ln \left(\psi_0^{(1)}(x) \right). \end{aligned} \quad (2.28)$$

It is possible to construct a third Hamiltonian with corresponding partner potential by setting the ground state energy for the second Hamiltonian to zero. Using $E_0^{(2)} = E_1^{(1)}$ gives

$$H^{(2)} = A_1 A_1^\dagger - E_0^{(1)} = A_2^\dagger A_2 - E_0^{(2)} = A_2^\dagger A_2 - E_1^{(1)} \quad (2.29)$$

with the factorization

$$A_2 = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(2)}(x), A_2^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(2)}(x) \quad (2.30)$$

where

$$W^{(2)}(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\frac{d}{dx} \psi_0^{(2)}(x)}{\psi_0^{(2)}(x)} = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left(\psi_0^{(2)}(x) \right). \quad (2.31)$$

As previously the partner potential is constructed by

$$H^{(3)} = A_2 A_2^\dagger + E_1^{(1)} = -\frac{\hbar}{2m} \frac{d^2}{dx^2} + V^{(3)}(x), \quad (2.32)$$

and the corresponding potential with

$$\begin{aligned} V^{(3)}(x) &= W^{(2)}(x)^2 + \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W^{(2)}(x) + E_1^{(1)} = V^{(2)}(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left(\psi_0^{(2)}(x) \right) \\ &= V^{(1)}(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left(\psi_0^{(1)}(x) \psi_0^{(2)}(x) \right). \end{aligned} \quad (2.33)$$

From this it is possible to relate the energy levels and wave functions for the third partner Hamiltonian to the first Hamiltonian.

$$E_n^{(3)} = E_{n+1}^{(2)} = E_{n+2}^{(1)} \quad (2.34)$$

$$\begin{aligned} \psi_n^{(3)}(x) &= \frac{1}{\sqrt{E_{n+1}^{(2)} - E_0^{(2)}}} A_2 \psi_{n+1}^{(1)}(x) \\ &= \frac{1}{\sqrt{E_{n+2}^{(1)} - E_1^{(1)}}} \frac{1}{\sqrt{E_{n+2}^{(1)} - E^{(1)}}} A_2 A_1 \psi_{n+1}^{(1)}(x) \end{aligned} \quad (2.35)$$

In the same way it is possible to expand this to the k 'th Hamiltonian and corresponding partner potential where the number of possible Hamiltonians and corresponding partner potentials are limited by the number of bound states(m) in the initial Hamiltonian i.e. $k = m - 1$. So knowing that $H^{(1)}$ has m bound states it is possible to write

$$H^{(k)} = A_k^\dagger A_k - E_{k-1}^{(1)} = -\frac{d^2}{dx^2} + V^{(k)}(x) \quad (2.36)$$

where

$$A_k = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(k)}(x), \quad A_k^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(k)}(x) \quad (2.37)$$

$$W^{(k)}(x) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left(\psi_0^{(k)} \right), \quad k = 2, 3, \dots, m \quad (2.38)$$

with the relations

$$E_n^{(k)} = E_{n+1}^{(k-1)} = E_{n+k-1}^{(1)} \quad (2.39)$$

$$\psi_n^{(k)} = \frac{1}{\sqrt{E_{n+k-1}^{(1)} - E_{k-2}^{(1)}}} \cdots \frac{1}{\sqrt{E_{n+k-1}^{(1)} - E_0^{(1)}}} A_{k-1} \cdots A_1 \psi_{n+k-1}^{(1)}(x) \quad (2.40)$$

$$V^{(k)} = V^{(1)}(x) - 2 \left(\frac{\hbar}{\sqrt{2m}} \right)^2 \frac{d^2}{dx^2} \ln \left(\psi_0^{(1)} \cdots \psi_0^{(k-1)} \right). \quad (2.41)$$

If the ground state energy level was shifted in order to set it to zero, the entire energy spectra needs to be shifted back with the same constant in order to obtain the actual spectra.

2.5 Shape Invariant Potential

For a certain class of super potentials $W(x, a)$, where the partner potentials $V^{(1,2)}(x, a)$ obey

$$V^{(2)}(x, a_1) = V^{(1)}(x, a_2) + R(a_1). \quad (2.42)$$

a describes the strength of the interaction and could be multiple parameters, $R(a_1)$ is an additive constant independent of x , and the parameters a_1 and a_2 represent characteristics of the potential, and is related through some function $a_2 = f(a_1)$. This condition is called shape invariance and when it is fulfilled one does not need to know the energy spectrum of one partner Hamiltonian to know the other.

2.5.1 Energy spectrum

Consider two partner Hamiltonians related through unbroken SUSYQM and fulfill the shape invariant potentials condition, eq. (2.42). The wave function is given by eq. (2.22), this gives

$$E_0^{(1)}(a_1) = 0, \quad (2.43)$$

$$\psi_0^{(1)}(x, a_1) = N \exp\left(-\int_{-\infty}^x W(y, a_1) dy\right). \quad (2.44)$$

Constructing a chain of Hamiltonians that obeys eq. (2.42), starting with the first Hamiltonian

$$H^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_1) \quad (2.45)$$

and its partner potential

$$H^{(2)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(2)}(x, a_1) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_2) + R(a_1) \quad (2.46)$$

By shifting the second Hamiltonian so it can be factorized using SUSYQM and then doing the same steps again to obtain a third Hamiltonian. The shifted second Hamiltonian is

$$\tilde{H}^{(2)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_2) \quad (2.47)$$

and the third Hamiltonian is

$$\tilde{H}^{(3)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(2)}(x, a_2) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_3) + R(a_2). \quad (2.48)$$

Shifting the Hamiltonians back in order to obtain the real spectra gives

$$H^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_1), \quad (2.49)$$

$$H^{(2)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_2) + R(a_1), \quad (2.50)$$

$$H^{(3)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_3) + R(a_2) + R(a_1). \quad (2.51)$$

Expressing the parameters a_2 and a_3 terms of a_1

$$a_2 = f(a_1), \quad a_3 = f(a_2) = f(f(a_1)). \quad (2.52)$$

It is possible to use the same technique to obtain the k 'th Hamiltonian

$$H^{(k)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_k) + \sum_{i=1}^{k-1} R(a_i). \quad (2.53)$$

$$\begin{aligned} H^{(2)}\psi_0^{(1)}(x, a_2) &= \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_2) \right] \psi_0^{(1)}(x, a_2) + R(a_1)\psi_0^{(1)}(x, a_2) \\ &= R(a_1)\psi_0^{(1)}(x, a_2). \end{aligned} \quad (2.54)$$

So $\psi_0^{(1)}(x, a_2)$ is an eigenfunction for the second Hamiltonian but with the parameter $a_2 = f(a_1)$ and the remainder $R(a_1)$ as the ground state energy.

$$E_0^{(2)} = R(a_1), \quad (2.55)$$

$$\implies E_{1(a_i)}^{(1)} = E_0^{(2)}(a_2) \quad (2.56)$$

Considering k 'th partner potential eq. (2.53)

$$\begin{aligned} H^{(k)}\psi_0^{(1)}(x, a_k) &= \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(1)}(x, a_k) \right] \psi_0^{(1)}(x, a_{k-1}) + \sum_{i=1}^{k-1} R(a_i)\psi_0^{(1)}(x, a_k) \\ &= \sum_{i=0}^{k-1} R(a_i)\psi_0^{(1)}(x, a_k) \end{aligned} \quad (2.57)$$

gives the ground state energy for the k 'th partner potential

$$E_0^{(k)} = \sum_{i=1}^{k-1} R(a_i). \quad (2.58)$$

Since the ground state energy of the k 'th Hamiltonian corresponds to the $k - 1$ energy level for the first Hamiltonian, it is possible to express the entire spectrum for the first Hamiltonian by

$$E_n^{(1)}(a_1) = \sum_{i=1}^n R(a_i), \quad E_0^{(1)} = 0. \quad (2.59)$$

If the initial problem does not have $E_0^{(1)} = 0$, the spectrum needs to be shifted back to the initial problem after using the SIP method in order to obtain the real spectrum.

2.5.2 Eigenfunctions for Shape Invariant Potentials

Using eq. (2.22) and the fact that eq. (2.57) shows that the Hamiltonian $H^{(k)}$ has the ground state $\psi_0^{(1)}(x, a_k)$, it is possible to write

$$\psi_1^{(1)}(x, a_{(k-1)}) \propto A^\dagger(x, a_{k-1})\psi_0^{(1)}(x, a_k). \quad (2.60)$$

It can be normalized as

$$\psi_1^{(1)}(x, a_{(k-1)}) = \frac{1}{\sqrt{E_{k-1}^{(1)}}} A^\dagger(x, a_{k-1})\psi_0^{(1)}(x, a_k). \quad (2.61)$$

Which is the first excited state for the Hamiltonian $H^{(k-1)}$, repeating this step several times gives

$$\psi_k^{(1)}(x, a_1) \propto A^\dagger(x, a_1)A^\dagger(x, a_2) \cdots A^\dagger(x, a_k)\psi_0^{(1)}(x, a_{k+1}) \quad (2.62)$$

which is an unnormalized eigenfunction for the k 'th excited state. Using eq. (2.22) it is possible to conclude

$$\psi_k^{(1)}(x, a_1) = \frac{1}{\sqrt{E_{k-1}^{(1)}}} A^\dagger(x, a_1)\psi_{k-1}^{(1)}(x, a_2) \quad (2.63)$$

Thus it is possible to find the eigenfunctions and eigenstates for the first Hamiltonian if first ground state and the function $f(a)$ is known assuming the SIP requirement is fulfilled.

2.6 Supersymmetric Hamiltonian

In SUSYQM the Hamiltonians are given by $H^{(1)} = A^\dagger A$ and $H^{(2)} = AA^\dagger$ where A and A^\dagger are operators. These operators ensures that their eigenvalues are never negative,

$$\langle \phi | H^{(1)} | \phi \rangle = \langle \phi | A^\dagger A | \phi \rangle = |A|\phi\rangle|^2 \geq 0, \quad \langle \phi | H^{(2)} | \phi \rangle = \langle \phi | AA^\dagger | \phi \rangle = |A^\dagger|\phi\rangle|^2 \geq 0. \quad (2.64)$$

If we consider the conditions for unbroken and broken supersymmetry. By introducing the operators Q and Q^\dagger one can define the Hamiltonian, H in matrix form

$$H = \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(2)} \end{pmatrix} \quad (2.65)$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \quad (2.66)$$

The Hamiltonian is given by

$$\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = H, \quad (2.67)$$

the operators Q and Q^\dagger commute with the Hamiltonian H ,

$$[Q, H] = [Q^\dagger, H] = 0. \quad (2.68)$$

This indicates an underlying symmetry and

$$QH|\psi\rangle - HQ|\psi\rangle = 0 \quad (2.69)$$

where $|\psi\rangle$ is an eigenstate of the Hamiltonian. This means that $H(Q|\psi\rangle) = E(Q|\psi\rangle)$ i.e. the operators Q and Q^\dagger do not change the energy of the state. Thus this is the supersymmetry in SUSYQM.

The operators Q and Q^\dagger have an other property

$$(Q)^2 = (Q^\dagger)^2 = 0, \quad (2.70)$$

i.e. if the operator is applied twice on any state it will not create a new state. This is very similar to the Pauli exclusion principle where it is not possible to create two fermions in one state. So due to the this behaviour one could call the operators Q and Q^\dagger fermionic.

If we consider the eigenfunctions $\psi^{(1)}(x)$ and $\psi^{(2)}(x)$ of the Hamiltonians $H^{(1)}$ and $H^{(2)}$ and place them in column vectors

$$\begin{pmatrix} \psi_n^{(1)}(x) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \psi_n^{(2)}(x) \end{pmatrix}. \quad (2.71)$$

Applying the operators Q and Q^\dagger to these vectors gives

$$Q \begin{pmatrix} \psi_n^{(1)}(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi_n^{(1)}(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi_n^{(1)}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_{n-1}^{(2)}(x) \end{pmatrix} \quad (2.72)$$

and

$$Q^\dagger \begin{pmatrix} 0 \\ \psi_n^{(2)}(x) \end{pmatrix} = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_n^{(2)}(x) \end{pmatrix} = \begin{pmatrix} A^\dagger \psi_n^{(2)}(x) \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_{n+1}^{(1)}(x) \\ 0 \end{pmatrix}. \quad (2.73)$$

In this case the operators Q and Q^\dagger relates two states with the same eigenvalues of the Hamiltonian H . The application of the operators can be considered as changing the character of the state from bosonic to fermionic, and vice versa. So this is the same supersymmetry as in theoretical particle physics.

Now consider the symmetry

$$HQ - QH = 0, \quad HQ^\dagger - Q^\dagger H = 0, \quad (2.74)$$

it is not fulfilled the super symmetry is considered explicitly broken.

The ground state system, $|0\rangle$ must be unchanged when acted up on by an operator in order to have symmetry. So It must fulfilled

$$Q|0\rangle = Q^\dagger|0\rangle = |0\rangle \quad (2.75)$$

But if it instead has

$$Q|0\rangle \neq |0\rangle, \quad Q^\dagger|0\rangle \neq |0\rangle, \quad (2.76)$$

the supersymmetry will be spontaneously broken. So the supersymmetry is only unbroken when the energy of the ground state is zero. So for unbroken symmetry we have

$$\begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \begin{pmatrix} \psi_0^{(1)}(x) \\ \psi_0^{(2)}(x) \end{pmatrix} = 0 \quad (2.77)$$

$$\implies A\psi_0^{(1)}(x) = 0, \quad A^\dagger\psi_0^{(2)}(x) = 0, \quad (2.78)$$

and is this not fulfilled the symmetry will be spontaneously broken.

2.7 Spontaneous Breaking of Supersymmetry

If the conditions previously derived for the lowest energy solution are not fulfilled, the supersymmetry is broken and the SUSYQM framework is no longer applicable. I.e. the ground state wave function has to be normalizable, so the superpotential $W(r)$ can be created using eq. (2.11).

Assume that for a given $W(r)$ it is possible to construct the ground state wave function and the partner wave function from eq. (2.13), by convention $H^{(1)}$ has the lowest ground state energy. Thus it should be chosen so that the ground state wave function is normalizable. Thus

$$\psi_0^{(1)}(x) = N \exp \left(-\frac{\sqrt{2m}}{\hbar} \int_{-\infty}^x W(y) dy \right). \quad (2.79)$$

were N is a normalization constant, $\psi_0^{(1)}(x)$ needs to vanish at $\pm\infty$ i.e. the exponent needs to converge to $-\infty$ as $x \rightarrow \pm\infty$. This will be fulfilled if $W(y)$ is an odd function. But if $W(y)$ is even or have no point where it is zero, it will lead to spontaneous broken symmetry see fig. 1.

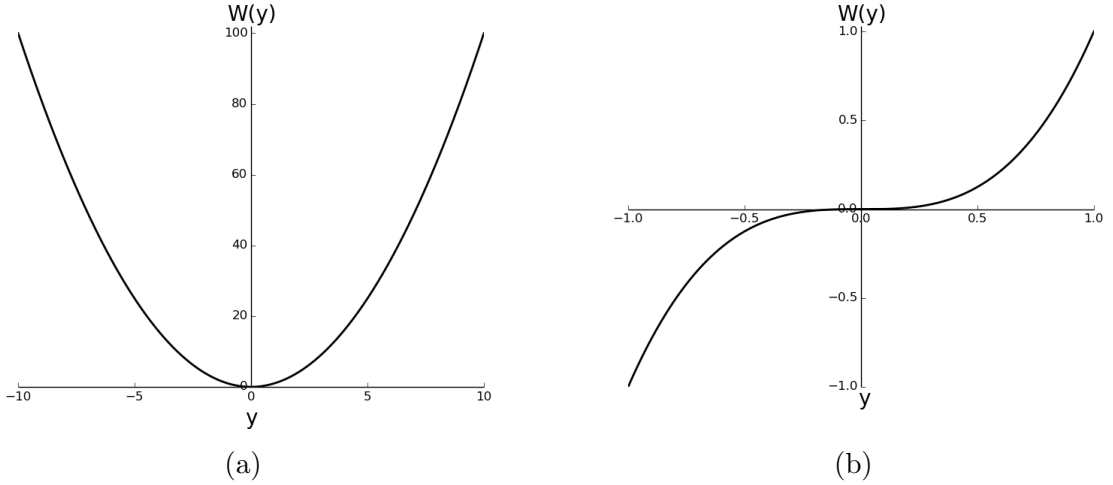


Figure 1: (a): $W(y)$ is even and gives spontaneous broken symmetry. (arb. units) (b): $W(y)$ is odd and gives unbroken symmetry. (arb. units)

2.8 Isospectral Deformation

It is possible to use SUSYQM to build a family of new potentials with the same energy spectrum and the same reflection and transmission coefficients as the original potential. In this section we will assume $\frac{\hbar}{2m}=1$. As seen in section 2.3

$$E_0^{(1)} = 0, \quad E_{n+1}^{(1)} = E_n^{(2)} \quad (2.80)$$

$$\psi_n^{(2)}(x) = \frac{\frac{d}{dx} + W(x)}{\sqrt{E_n^{(2)}}} \psi_{n+1}^{(1)}(x), \quad \psi_{n+1}^{(1)}(x) = -\frac{\frac{d}{dx} + W(x)}{\sqrt{E_n^{(2)}}} \psi_{n+1}^{(2)}(x) \quad (2.81)$$

were

$$W(x) = -\frac{\frac{d}{dx}\psi_0^{(1)}(x)}{\psi_0^{(1)}(x)} = -\frac{d}{dx} \ln \left(\psi_0^{(1)}(x) \right) \quad (2.82)$$

The reflection and transmission amplitudes for the partner potentials $V^{(1,2)}$ are related by [1]

$$r^{(1)}(k) = \frac{W^{(1)} + ik}{W^{(1)} - ik} r^{(2)}(k), \quad t^{(1)}(k) = \frac{W^{(2)} + ik'}{W^{(1)} - ik'} t^{(2)}(k) \quad (2.83)$$

where $k = \sqrt{E - W^{(1)2}}$, $k' = \sqrt{E - W^{(2)2}}$ with $W^{(1,2)} = \lim_{x \rightarrow \pm\infty} W(x)$. To study isospectrality between the partner potentials created from a super potential $W(x)$ one can assume that there exist other superpotentials $\tilde{W}(x)$ that gives the same potential $V^{(2)}$ and has the same partner system.

$$V^{(2)}(x) = \tilde{W}^2(x) + \tilde{W}'(x) \Leftrightarrow W^2(x) + W'(x) = \tilde{W}^2(x) + \tilde{W}'(x) \quad (2.84)$$

It is possible to solve eq. (2.84) with

$$\tilde{W}(x) = W(x) + f(x) \quad (2.85)$$

putting it to eq. (2.84) and making the substitution $f(x) = 1/y(x)$ one obtains a differential equation

$$y'(x) = 1 + 2W(x)y(x) \quad (2.86)$$

with the solution

$$y(x) = \left[\int_{-\infty}^x \exp \left(-2 \int_{-\infty}^t W(u) du \right) dt + \lambda \right] \exp \left(\int_{-\infty}^x 2W(t) dt \right) \quad (2.87)$$

where λ is an integration constant. Using eq. (2.82) to rewrite eq. (2.87) gives

$$y(x) = \frac{\int_{-\infty}^x [\psi_0^{(1)}(t)]^2 dt + \lambda}{[\psi_0^{(1)}(t)]^2}, \quad (2.88)$$

$$f(x) = \frac{1}{y(x)} = \frac{[\psi_0^{(1)}(x)]^2}{\int_{-\infty}^x [\psi_0^{(1)}(t)]^2 dt + \lambda} = \frac{d}{dx} \ln [I(x) + \lambda], \quad (2.89)$$

where $I(x) = \int_{-\infty}^x [\psi_0^{(1)}(t)]^2 dt$. So using eq. (2.85) one gets the general superpotential

$$\tilde{W}(x) = W(x) + \frac{d}{dx} \ln [I(x) + \lambda]. \quad (2.90)$$

So using $\tilde{W}(x)$ to find the potentials $\tilde{V}^{(1)}(x)$

$$\tilde{V}^{(1)}(x) = \tilde{W}(x)^2 - \tilde{W}(x)' = V^{(1)}(x) - 2\frac{d^2}{dx^2} \ln[I(x) + \lambda], \quad (2.91)$$

which has the same energy spectrum as $V^{(2)}$. Note that when $\lambda \rightarrow \pm\infty$, $\tilde{V}^{(1)}(\lambda, x) \rightarrow V^{(1)}$.

In order to study where eq. (2.91) is solvable, the parameter λ needs to be studied. By considering the ground state wave function, $\tilde{\psi}_0^{(1)}(x)$, corresponding to $\tilde{V}^{(1)}$, with the requirement that it is square integrable, eq. (2.90) and (2.82), with some calculations one can obtain a formula for the ground state wavefunction for the isospectral deformed potentials[1]m

$$\tilde{\psi}_0^{(1)}(x) = N \cdot \exp\left(-\int_{-\infty}^x \tilde{W}(t)dt\right) = \frac{\sqrt{\lambda(\lambda+1)}}{I(x)+\lambda} \psi_0^{(1)}(x). \quad (2.92)$$

$I(x)$ is constrained to the interval $(0, 1)$, so one have to study three cases were λ on the intervals (1) $\lambda < -1$ or $\lambda > 0$, (2) $\lambda = -1$ or $\lambda = 0$ and (3) $-1 < \lambda < 0$. For case (1) the wave function is nodeless and thus square integrable, so $\tilde{V}(x)$ has the same energy spectrum as $V(x)$. So if $V(x)$ is exactly solvable $\tilde{V}(x)$ will be exactly solvable. For case (2) SUSYQM will be broken due to the fact that the wavefunction will be zero as can be seen in eq. (2.92) and the fact that $I(x)$ vanishes as $x = \pm\infty$ so the wave function will no longer be nodeless and $\tilde{V}(x)$ will miss an energy level. Case (3) gives rise to singularities in $\tilde{V}^{(1)}(\lambda, x)$, that will make the physical interpretation uncertain.

2.8.1 Uniqueness of the Isospectral Deformation

One has to consider if the family of isospectral deformations is unique or if it is possible to find other families.

If one uses eq. (2.91) to create a isospectral defomed potential and does once more an isospectral deformation gives

$$\tilde{\tilde{V}}^{(1)} = \tilde{V}^{(1)}(x) - 2\frac{d^2}{dx^2} \ln[\tilde{I} + \tilde{\lambda}] \quad (2.93)$$

were the dependence on λ and x is dropped for simplicity. Calculating $\tilde{I}(x)$ with $I(x) = \int_{-\infty}^x [\psi_0^{(1)}(t)]^2 dt$ gives

$$\begin{aligned} \tilde{I}(x) &= \int_{-\infty}^x [\tilde{\psi}_0^{(1)}(t)]^2 dt = \lambda(\lambda+1) \int_{-\infty}^x \frac{[\psi_0^{(1)}(t)]^2}{[I(t)+\lambda]^2} dt \\ \lambda(\lambda+1) \int_{-\infty}^x \frac{\frac{d}{dt} I(t)}{[I(t)+\lambda]^2} dt &= \lambda(\lambda+1) \left(\frac{1}{\lambda} - \frac{1}{I(x)+\lambda} \right). \end{aligned} \quad (2.94)$$

Then we used it in eq. (2.93)

$$\begin{aligned}\tilde{V}^{(1)} &= V^{(1)} - 2\frac{d^2}{dx^2} \ln \left[(I(x) + \lambda) \left(\lambda(\lambda + 1) \left(\frac{1}{\tilde{\lambda}} - \frac{1}{I(x) + \lambda} \right) \right) \right] \\ &= V^{(1)} - 2\frac{d^2}{dx^2} \ln [I(x) + \mu], \quad \mu = \frac{\tilde{\lambda}\lambda}{\tilde{\lambda} + \lambda + 1}.\end{aligned}\tag{2.95}$$

So this deformation is not new but simply yields deformation parameter μ instead of λ . It is possible to prove that as long as λ and $\tilde{\lambda}$ are in the same range outside $[-1, 0]$, this gives the deformation parameter μ to be in the same region. So the isospectral deformation is unique.

3 Applications

Supersymmetric quantum mechanics can be applied to many different problems, in this thesis SUSYQM is applied to the infinite square well, harmonic oscillator, the hydrogen atom and to the isospectral deformation of the finite square well.

3.1 One Dimensional Infinite Square Well

The infinite square well is one of the first examples of a potential in books on classical quantum mechanics. It describes a particle free to move around in a small region surrounded by impenetrable barriers.

The infinite square well in one dimension classically has the potential

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq L \\ \infty, & \text{otherwise} \end{cases}.\tag{3.1}$$

With the Hamiltonian

$$H\psi(x) = \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi(x)\tag{3.2}$$

But in this example the potential will be shifted so it is symmetric around zero i.e.

$$V(x) = \begin{cases} 0, & \text{if } -L/2 \leq x \leq L/2 \\ \infty, & \text{otherwise} \end{cases}.\tag{3.3}$$

The infinite square well is a well known problem and the solution for the ground state wave function and the ground state energy can be found in several introductory books on quantum mechanics [4]. So using the $y = x - \frac{L}{2}$ variable substitution and trigonometry on a standard solution in order to get the correct wave function for the shifted potential. The wave functions and the spectra are given by

$$\begin{cases} \psi_n^{(1)}(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n}{L}x\right), n = 1, 3, 5\dots \\ \psi_n^{(1)}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L}x\right), n = 2, 4, 6\dots \end{cases} \quad (3.4)$$

$$E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2} \quad (3.5)$$

In order to solve it using SUSYQM the energy ground state needs to be zero in order to factorise the Hamiltonian. $H^{(1)}(x) = H(x) - E_0$.

$$E_n^{(1)} = \frac{n^2 \hbar^2 \pi^2}{2mL^2} - \frac{\hbar^2 \pi^2}{2mL^2} = (n^2 - 1) \frac{\hbar^2 \pi^2}{2mL^2}, n = 1, 2\dots \quad (3.6)$$

Using the ground state wave function and its derivative it is possible to calculate the infinite square well's superpotential $W(x)$ using eq. (2.11)

$$W(x) = -\frac{\pi \hbar}{\sqrt{2mL}} \tan\left(\frac{\pi x}{L}\right). \quad (3.7)$$

From the super potential it's possible to obtain the infinite square well's SUSYQM partner potential using eq. (2.15)

$$V^{(2)}(x) = \frac{\pi^2 \hbar^2}{2mL^2} \frac{\sin^2\left(\frac{\pi x}{L}\right) + 1}{\cos^2\left(\frac{\pi x}{L}\right)} = \frac{\pi^2 \hbar^2}{2mL^2} \left(2 \sec^2\left(\frac{\pi x}{L}\right) - 1\right). \quad (3.8)$$

The energy spectrum for the shifted Hamiltonian eq. (3.6) has a ground state energy equal to zero so using SUSYQM the energy spectra for the partner potential can be calculated using $E_n^{(2)} = E_{n+1}^{(1)}$ then shifting the energy levels back to the original problem gives

$$E_n^{(1)} = \frac{(n+1)^2 \pi^2 \hbar^2}{2mL^2}, n = 0, 1, 2\dots,$$

$$E_n^{(2)} = \frac{(n+2)^2 \pi^2 \hbar^2}{2mL^2}, n = 0, 1, 2\dots,$$

$$\psi_n^{(2)}(x) = \frac{1}{\sqrt{\left(E_n^{(1)}\right)}} A \psi_n^{(1)}(x). \quad (3.9)$$

Where A is given by eq. (2.5) and $E_1^{(1)}$ by eq. (3.5). Using eq. (3.9) gives the wave functions for the partner potential.

$$\psi_1^{(2)}(x) = -\sqrt{\frac{8}{3L}} \cos^2\left(\frac{\pi}{L}x\right), \quad (3.10)$$

$$\psi_2^{(2)}(x) = -\frac{4}{\sqrt{L}} \cos^2\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{L}x\right). \quad (3.11)$$

Note that $\psi_1^{(2)}(x)$ is the ground state wave function, the indexing is due to the definition of the wave functions for the infinite square well eq. (3.4).

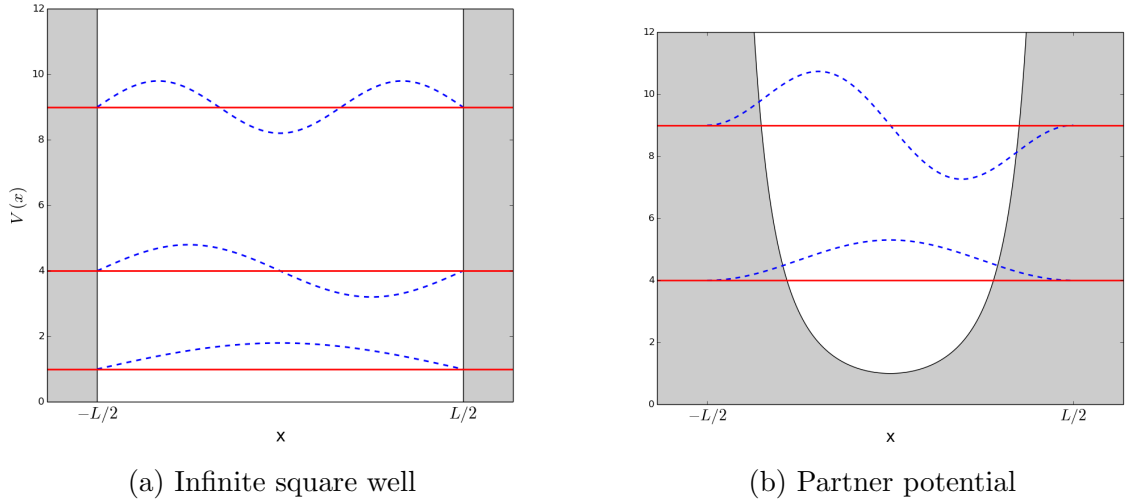


Figure 2: Infinite square well and the partner potential with the first energy levels and corresponding eigenfunctions.(Here $\hbar = 2m = 1$ in order to simplify plotting.)

As can be seen in fig.2 the operators A destroy and A^\dagger create nodes on the wave functions that they act upon. It also shows that the two potentials have the same energy spectra, except H_1 has one more bound state[6]. Why several potentials has the same energy spectrum is not explained in standard quantum mechanics which does not include the super potential but it is explained by SUSYQM.

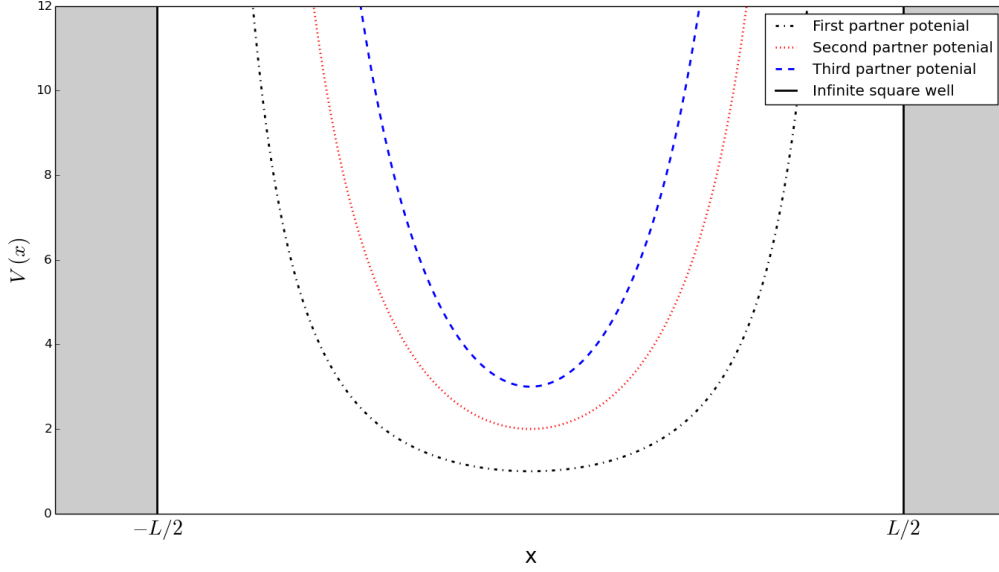


Figure 3: Infinite square well and the first three partner potentials in the chain of Hamiltonians

Using the methodology outlined in section 2.4 the first three partner potentials were found and plotted in fig.3.

3.2 Harmonic Oscillator

The quantum harmonic oscillator is a well know problem in quantum mechanics and one of the few with an analytical solution. It can be used to approximate any attractive potential near a minimum, it has implications in a range of different problems such as the simple diatomic molecule, vibration modes in molecules and the motion of atoms in a solid lattice.

Using supersymmetric quantum mechanics it is possible to find the energy spectra and eigenfunctions without solving the Schrödinger equation. The potential for the quantum harmonic oscillator in one dimension is

$$V(x) = \frac{1}{2}m\omega^2x^2. \quad (3.12)$$

It has the superpotential

$$W(x) = \sqrt{\frac{m}{2}}\omega x. \quad (3.13)$$

From the super potential it's possible to obtain the harmonic oscillator and its partner potential, using eq. (2.15) gives

$$V^{(1)}(x) = \frac{1}{2}m\omega^2x^2 - \frac{\hbar\omega}{2} \quad (3.14)$$

$$V^{(2)}(x) = \frac{1}{2}m\omega^2x^2 + \frac{\hbar\omega}{2} \quad (3.15)$$

Due to the linearity of the super-potential the two partner potential are just shifted by a constant $V^{(2)}(x) - V^{(1)}(x) = \hbar\omega$. Since the different Hamiltonians describes the same system but are only shifted from each other by a constant thus $E_1^{(1)} = E_0^{(2)} = \hbar\omega$, this gives the energies for the ground state and the first energy.

Taking the same steps with second Hamiltonian and creating a third Hamiltonian as a partner potential shifted by $\Delta E = \hbar\omega$ from the second partner potential. The ground state for $H^{(3)}$ can be used in the same way to find the second excited state for the first Hamiltonian, due to fact that the different Hamiltonians are related trough supersymmetry, which gives $E_2^{(1)} = E_1^{(2)} = E_0^{(3)} = 2\hbar\omega$

This can be repeated for an infinite amount of potentials, and thus gives a formula for the energy spectrum of the harmonic oscillator

$$E_n^1 = \sum_{i=1}^n \Delta E = \left(n + \frac{1}{2}\right) \hbar\omega, \quad (3.16)$$

here shifted back to the original problem .

In the same way the eigenfunctions can be found, using the fact that the partner potential are just the shifted and that the Hamiltonians have the same ground state wave function in the partner series.

$$\psi_n(x) = (A^\dagger)^n \psi_0(x). \quad (3.17)$$

The derivation of the ground state wave function can be found in most introductory textbooks to quantum mechanics e.g. [5]

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right). \quad (3.18)$$

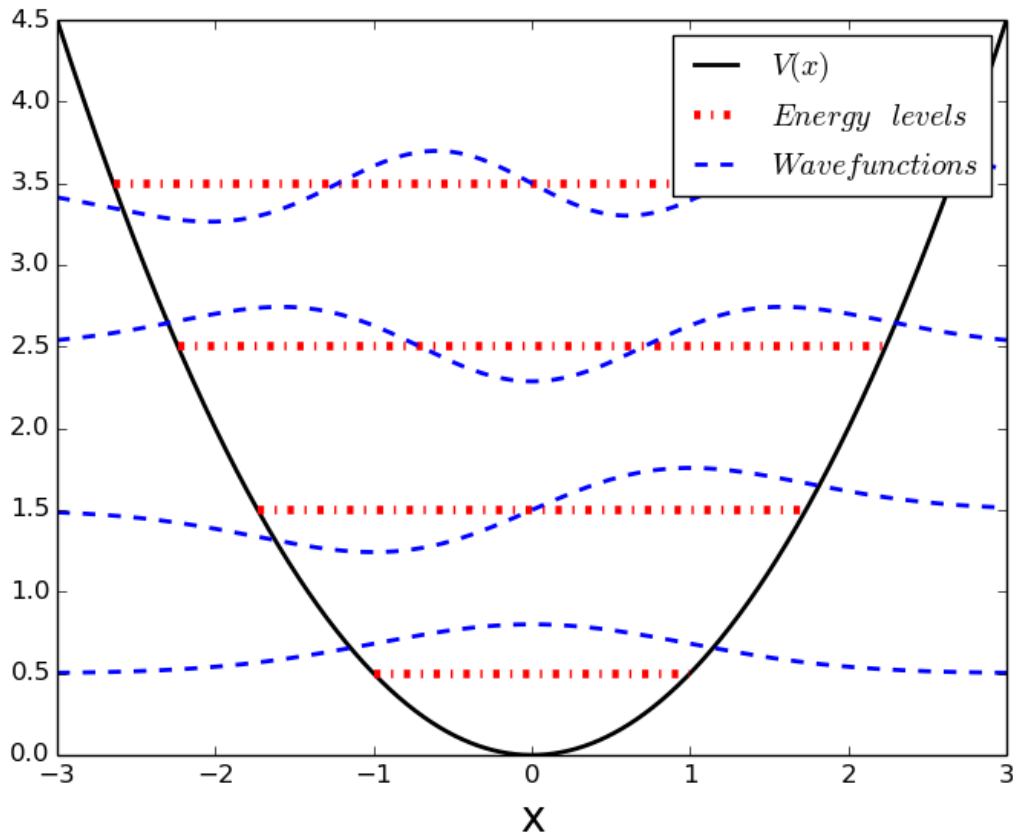


Figure 4: Harmonic oscillator with first four energy levels and corresponding eigenfunctions. (Plotted with $\hbar = \omega = m = 1$ for convince and the eigenfunctions are scaled to create a better image.)

So using the fact that the harmonic oscillator is a shape invariant potential it is possible to find the energy spectrum by only knowing the ground state energy and the superpotential of the harmonic oscillator. Without even solving the Schrödinger equation and making the calculation faster and easier than "traditional" quantum mechanics.

3.3 The Hydrogen Atom

The hydrogen atom is a common quantum mechanics problem.

In this problem only the radial equation will be treated, as it for fills the requirements for a shape invariant potential. This makes it possible to obtain the energy spectra using supersymmetric quantum mechanics.

The electron is in a potential affected by the Coulomb force

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} \quad (3.19)$$

and the radial Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u(r) = E u(r). \quad (3.20)$$

The effective potential can be seen in fig.5

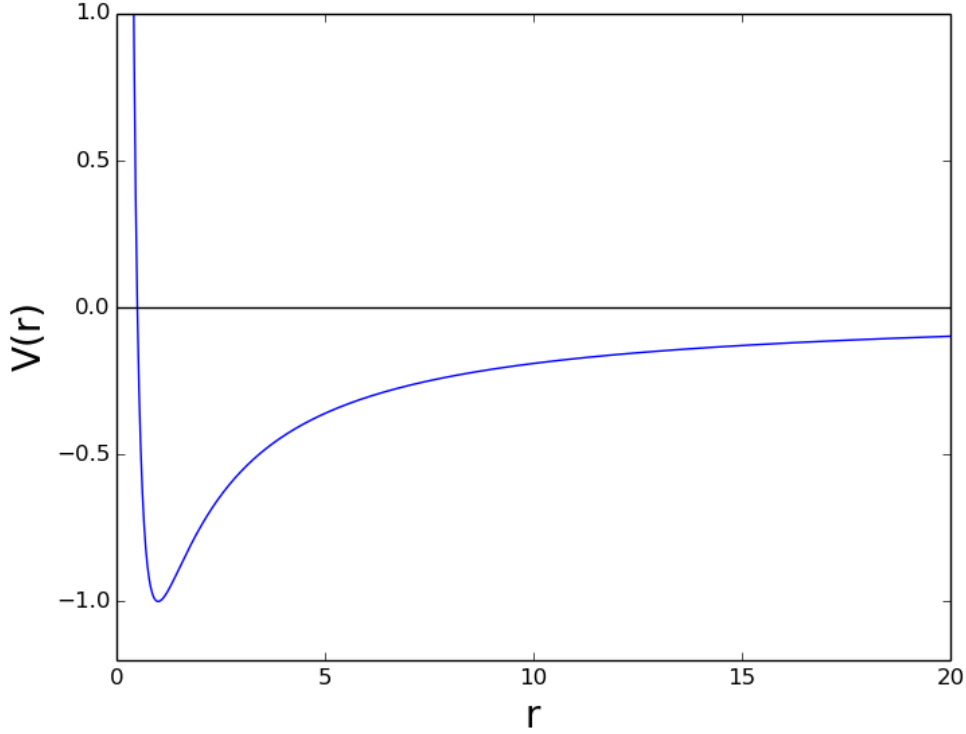


Figure 5: Radial coulomb potential ($l = 1$, $\frac{e^2}{4\pi\epsilon_0} = 2$, $\frac{\hbar^2}{2m} = 1$)

In order to use supersymmetry, the ground state energy needs to be zero. So the shifting potential gives

$$V_{eff}(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} + \left[\frac{\hbar^2 l(l+1)}{2m} \right] \frac{1}{r^2} - E_0. \quad (3.21)$$

The superpotential can be found by solving the differential equation given by $V_{eff}(r) = W(r)^2 - \frac{\hbar}{\sqrt{2m}} W'(r)$. Making the ansatz that the super potential will have the form

$$W(r) = B - \frac{C}{r} \quad (3.22)$$

and inserting it into the differential equation gives

$$V_{eff}(r) = B^2 - \frac{2BC}{r} + \left[C^2 - \frac{\hbar}{\sqrt{2m}} C \right] \frac{1}{r^2}. \quad (3.23)$$

By comparing the $V_{eff}(r)$ in eq. (3.21) and $V_{eff}(r)$ in eq. (3.23) given by the differential equation, makes it is possible to identify $E_0 = -B^2$ given the fact that B^2 is the only term independent of r . The other terms are identified as

$$2BC = \frac{e^2}{4\pi\epsilon_0} \quad (3.24)$$

$$C^2 - \frac{\hbar}{\sqrt{2m}}C = \frac{\hbar^2 l(l+1)}{2m} \quad (3.25)$$

Solving these two equation, keeping in mind that we are looking for a positive l gives

$$C = \frac{\hbar}{\sqrt{2m}}(l+1) \quad (3.26)$$

$$B = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{8\pi\epsilon_0(l+1)} \quad (3.27)$$

Using the obtained constants B and C in the anzats gives the superpotential

$$W(r) = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{8\pi\epsilon_0(l+1)} - \frac{\hbar}{\sqrt{2m}}(l+1)\frac{1}{r} \quad (3.28)$$

Using eq. (2.15) gives the partner potential

$$V^{(2)}(r) = \left(-\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r} + \left(\frac{\hbar^2(l+1)(l+2)}{2m}\right)\frac{1}{r^2} + \left(\frac{e^4 m}{32\pi^2 \hbar^2 \epsilon_0^2 (l+1)^2}\right), \quad (3.29)$$

it is shown in fig.6

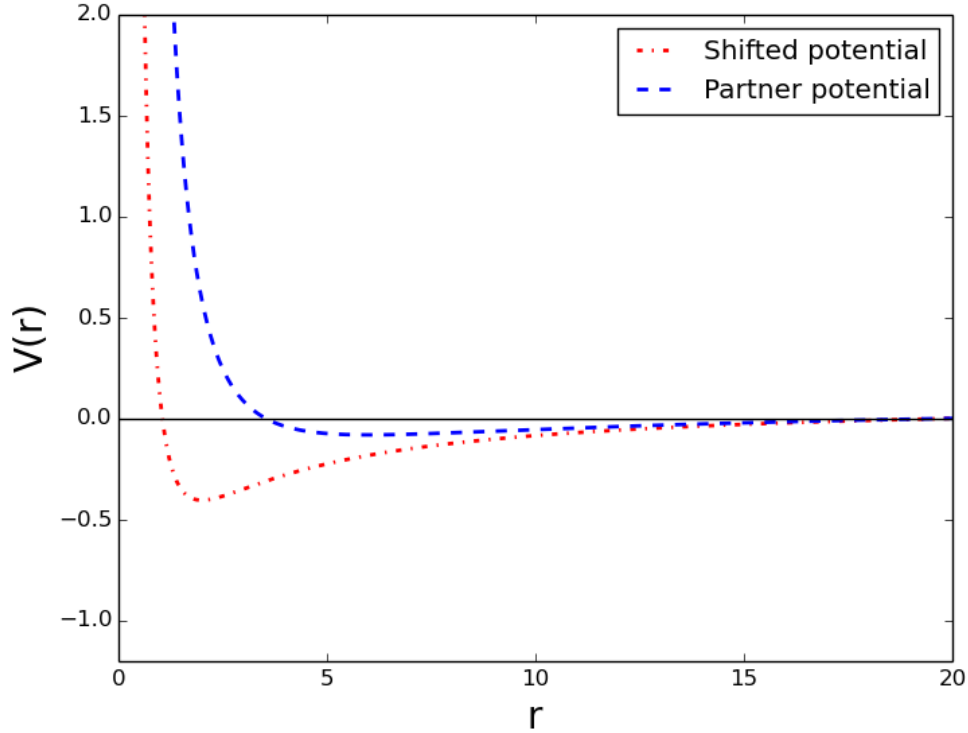


Figure 6: Shifted potential and partner potential ($l = 1$, plotted with $\frac{e^2}{4\pi\epsilon_0} = 2$, $\frac{\hbar^2}{2m} = 1$)

From the obtained value for B and the relation $E_0 = -B^2$ makes it possible to calculate the ground state energy for hydrogen were $l = 0$,

$$E_0 = -B^2 = - \left[\frac{\sqrt{2m}}{\hbar} \frac{e^2}{8\pi\epsilon_0} \right]^2 \approx -13.6eV. \quad (3.30)$$

Looking at the shape invariant potential condition $a_2 = f(a_1)$, by comparing the two partner potentials $V_{eff}(r)$ and $V^{(2)}(r)$ it is possible to identify the relationship between the parameters

$$f(l) = l + 1 \quad (3.31)$$

From this it is possible to find the remainder, $R(l)$, that can be used to find the energy difference between the ground states for the two potentials.

$$\begin{aligned} \left(\frac{e^4 m}{32\pi^2 \hbar^2 \epsilon_0^2 (l+1)^2} \right) &= \left(\frac{e^4 m}{32\pi^2 \hbar^2 \epsilon_0^2 (l+2)^2} \right) + R(l) \\ \Rightarrow R(l) &= \left(\frac{e^4 m (2l+3)}{32\pi^2 \hbar^2 \epsilon_0^2 (l+1)^2 (l+2)^2} \right). \end{aligned} \quad (3.32)$$

To find the first excited state for the original problem, the energies need to be shifted by the ground state

$$E_1 = -\frac{e^2 m}{32\pi\hbar\epsilon_0(l+1)^2} + \frac{e^4 m(2l+3)}{32\pi^2\hbar^2\epsilon_0^2(l+1)^2(l+2)^2}, \quad (3.33)$$

where n is the principal quantum number. Using the remainder eq. (3.32) and the relation for the parameter eq. (3.31) it is possible to obtain a formula the whole energy spectrum.

$$E_n = E_0 + \sum_{i=1}^n \frac{e^4(2(l+i-1)+3)}{32\pi^2\hbar^2\epsilon_0^2(l+i)^2(l+i+1)^2} \quad (3.34)$$

For the case $l = 0$ it is possible to rewrite the sum to

$$E_n = \frac{e^4 m}{32\pi^2\hbar^2\epsilon_0^2(n+1)^2}, \quad (3.35)$$

which is the known formula for the energy levels of the hydrogen atom that can be seen in fig.7.



Figure 7: Energy spectrum for the hydrogen atom ($l = 0$)

SUSYQM once again proved useful in providing an easier and faster solution to the problem, and showing how useful it can be applied to more complex problem.

3.4 Relativistic Hydrogen Atom

The Schrödinger equation does not take in to account special relativity, in order to do so one has to use the Dirac equation. Using the Dirac equation to obtain the energy spectrum for hydrogen, one gets a slightly different spectrum with splitting of energy levels. This is due to the fact that the Dirac equation takes into account relativistic and spin effects which leads to breaking of degeneracy for the energy levels.

For a central field potential it is possible to write the Dirac equation($\hbar = c = 1$) [3] as

$$H\psi = (\vec{\alpha} \cdot \vec{p} + \beta m + V)\psi \quad (3.36)$$

where

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = -\frac{e^2}{r} \quad (3.37)$$

σ^i are the Pauli spin matrices. In the case of the central field, the Dirac equation can be separated into spherical coordinates. For the hydrogen atom is only required to use the radial part to calculate the energy spectra.

In order obtain the radial functions G and F for hydrogen the Dirac equation has the be rewritten as a pair of first order equations [2].

$$\left(-\frac{d}{d\rho} + \frac{\tau}{\rho}\right) F = \left(-\nu + \frac{\gamma}{\rho}\right) G \quad (3.38)$$

$$\left(\frac{d}{d\rho} + \frac{\tau}{\rho}\right) G = \left(\frac{1}{\nu} + \frac{\gamma}{\rho}\right) F \quad (3.39)$$

$$\gamma = Ze^2, \quad k = \sqrt{m^2 - E^2}, \quad \rho = kr,$$

$$\nu = \sqrt{\frac{m-E}{m+E}}, \quad \tau = \left[j + \frac{1}{2}\right] \tilde{\omega} \quad (3.40)$$

where $j = l \pm \frac{1}{2}, \tilde{\omega} = \pm 1$ for states of parity $(-1)^{j+1/2}$. It is possible to rewrite eq. (3.38) and (3.39) in a more compact form

$$\begin{pmatrix} dG/dr \\ dF/dr \end{pmatrix} + \frac{1}{r} \begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} (m+E)G \\ (m-E)F \end{pmatrix} \quad (3.41)$$

The matrix that is multiplied by $\frac{1}{r}$ mixes G and F . Needs to be diagonalized in order to be rewritten in SUSYQM form. This is done by a linear transformation of the functions G, F into \tilde{G}, \tilde{F}

$$D = \begin{pmatrix} k+s & -\gamma \\ -\gamma & k+s \end{pmatrix}, \quad s = \sqrt{k^2 - \gamma^2} \quad (3.42)$$

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = D \begin{pmatrix} G \\ F \end{pmatrix} \quad (3.43)$$

and the supersymmetric form for the coupled equations

$$A\tilde{F} = \left(\frac{m}{E} - \frac{k}{s} \right) \tilde{G}, \quad A^\dagger \tilde{G} = - \left(\frac{m}{E} + \frac{k}{s} \right) \tilde{F} \quad (3.44)$$

where

$$A = \frac{d}{d\mu} - \frac{s}{\mu} + \frac{\gamma}{s}, \quad A^\dagger = -\frac{d}{d\mu} - \frac{s}{\mu} + \frac{\gamma}{s} \quad (3.45)$$

with the introduction of the variable $\mu = Er$.

Decoupling the equations for \tilde{F} and \tilde{G} gives the eigenvalue equations for \tilde{F} and \tilde{G}

$$H^{(1)}\tilde{F} = A^\dagger A\tilde{F} = \left(\frac{k^2}{s^2} - \frac{m^2}{E^2} \right) \tilde{F}, \quad (3.46)$$

$$H^{(2)}\tilde{G} = AA^\dagger\tilde{G} = \left(\frac{k^2}{s^2} - \frac{m^2}{E^2} \right) \tilde{G}. \quad (3.47)$$

Note that the two Hamiltonians eq. (3.46) and (3.47) are SUSYQM partner systems. The two Hamiltonians are also shape invariant super symmetric partners due to fact that they can be written on the form of a shape invariant potential, eq. (2.42)

$$H^{(2)}(\rho, s, \gamma) = H^{(1)}(\rho, s+1, \gamma) + \frac{\gamma^2}{s^2} - \frac{\gamma^2}{(s+1)^2}. \quad (3.48)$$

Looking at the Shape Invariance condition gives

$$a_2 = s+1, \quad a_1 = s, \quad R(a_1) = \frac{\gamma^2}{a_1^2} - \frac{\gamma^2}{(a_1+1)^2} \quad (3.49)$$

so the energy spectra for $H^{(1)}$ are given by

$$E_n^{(1)} = E_{n-1}^{(2)} = \left(\frac{k}{s} \right)^2 - \left(\frac{m^2}{E_n^2} \right) = \sum_{k=1}^n R(a_k) = \gamma^2 \left(\frac{1}{s^2} - \frac{1}{(s+n)^2} \right). \quad (3.50)$$

Note that the $H^{(1,2)}$ is not the Dirac Hamiltonian with the sought after energy spectra. It is possible to get the energy spectrum(E_n) for the hydrogen by relating it to $E_n^{(1)}$ through eq. (3.46), eq. (3.47) and eq. (3.50).

$$E_n = \frac{m}{\sqrt{1 + \frac{\gamma^2}{(s+n)^2}}}, \quad n = 0, 1, 2, \dots \quad (3.51)$$

Reintroducing \hbar and c and setting $Z = 1$ in order to do calculations for hydrogen gives $\gamma = \alpha = \frac{e^2}{\hbar c}$, using this in eq. (3.51)

$$E_{nj} = m \left(1 + \left(\frac{\alpha}{n - (j + \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - \alpha^2})} \right)^2 \right)^{-\frac{1}{2}}, \quad n = 0, 1, 2, \dots \quad (3.52)$$

In fig.8 one can see the energy spectra and the non degeneracy caused by the quantum number l and in fig.9 is a comparison between the energy spectra for the relativistic and non relativistic hydrogen atom.

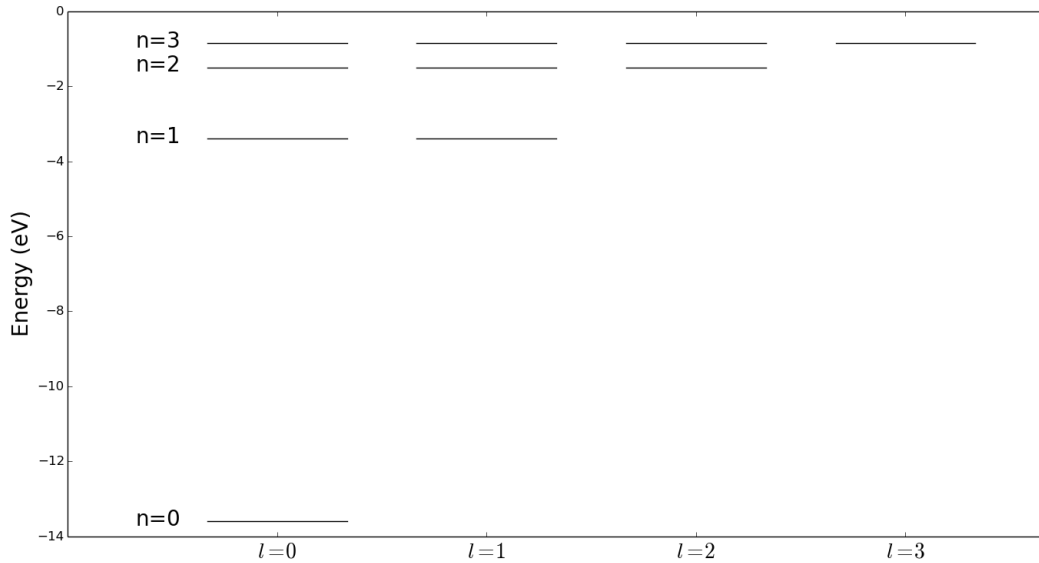


Figure 8: Energy spectra for relativistic hydrogen atom. The splitting of energy levels is so small that it can't be seen in the plot.



Figure 9: Comparison between the energy spectra for the relativistic and non relativistic hydrogen atom. (Spectrum is scaled and shifted to create a better fig.)

Thus we proved that even the Dirac equation can be solved using the SUSYQM framework.

3.5 Isospectral Deformation of the Finite Square Well

Application of isospectral deformation to the finite square well, we are assuming that $\frac{\hbar}{\sqrt{2m}} = 1$.

The finite square well has the potential.

$$V_0(x) = \begin{cases} 0, & -L/2 \leq x \leq L/2 \\ V, & \text{otherwise} \end{cases} . \quad (3.53)$$

The finite square well is a well known problem and the solution for the ground state wave function and the ground state energy can be found several introductory book to quantum mechanics [4].

The groundstate wavefunction is given by for each region of the potential

$$\psi_0(x) = \begin{cases} Ae^{\alpha x}, & x < L/2 \\ B \cos(kx), & -L/2 \leq x \leq L/2 \\ Ce^{-\alpha x}, & x > L/2 \end{cases} . \quad (3.54)$$

where the groundstate wavefunction is continuous. Using eq. (2.91) where $I(x) = \int_{-\infty}^x [\psi_0^{(1)}(t)]^2 dt$ to find the isospectral deformation of the potential, so by studying $I(x)$ in region I,

$x \in (-\infty, -L/2)$ gives

$$I_I(x) = \int_{-\infty}^x [\psi_0(t)]^2 dt = \int_{-\infty}^x [Ae^{\alpha x}]^2 dt = \frac{A^2}{2\alpha} e^{2\alpha x} \quad (3.55)$$

and studying the isospectral deformation of the potential in the same region gives

$$\tilde{V}_I(x) = V_I - 2 \frac{d^2}{dx^2} \ln[I_I(x) + \lambda] = V - \frac{16\alpha^3 2A^2 e^{2\alpha x} \lambda}{(A^2 e^{2\alpha x} + 2\alpha\lambda)^2} \quad (3.56)$$

In region II, $x \in [-L/2, L/2]$ gives

$$\begin{aligned} I_{II}(x) &= \int_{-\infty}^x [\psi_0(t)]^2 dt = \int_{-L/2}^x [B \cos(kt)]^2 dt + I_I(-L/2) \\ &= \frac{B^2(\sin(2kx) + 2kx + \sin(kL) + kL)}{4k} + I_I(-L/2) \end{aligned} \quad (3.57)$$

with the isospectral deformation of the potential

$$\tilde{V}_{II}(x) = V_{II} - 2 \frac{d^2}{dx^2} \ln[I_{II}(x) + \lambda] \quad (3.58)$$

$$= \frac{8B^2 k^2 (2B^2 \cos(2kx) + (2B^2 kx + 4k\lambda + B^2(\sin(kL) + kL) + 4I_I(-L/2)k) \sin(2kx) + 2B^2)}{(B^2 \sin(2kx) + 2B^2 kx + 4k\lambda + B^2(\sin(kL) + kL) + 4I_I(-L/2)k)^2} \quad (3.59)$$

In region III, $x \in (L/2, \infty)$

$$\begin{aligned} I(x)_{III} &= \int_{-\infty}^x [\psi_0(t)]^2 dt = \int_{L/2}^x [Ce^{-\alpha t}]^2 dt + I_{II}(L/2) \\ &= \frac{1}{2\alpha} C^2 (e^{2\alpha x} - e^{\alpha L}) e^{-2\alpha x - \alpha L} + I_{II}(L/2) \end{aligned} \quad (3.60)$$

and studying in isospectral deformation of the potential in the same region gives

$$\begin{aligned} \tilde{V}_{III}(x) &= V_{III} - 2 \frac{d^2}{dx^2} \ln[I_{III}(x) + \lambda] \\ &= V + \frac{8\alpha^2 C^2 e^{2\alpha x + \alpha L} (2\alpha e^{\alpha L} \lambda + 2\alpha e^{\alpha L} I_{II}(L/2) + C^2)}{((e^{2\alpha x} (2\alpha e^{\alpha L} \lambda + 2\alpha e^{\alpha L} I_{II}(L/2) + C^2) + C^2) - C^2 e^{\alpha L})^2} \end{aligned} \quad (3.61)$$

The isospectral deformation of the finite square well for a few lambdas is plotted in fig.10.

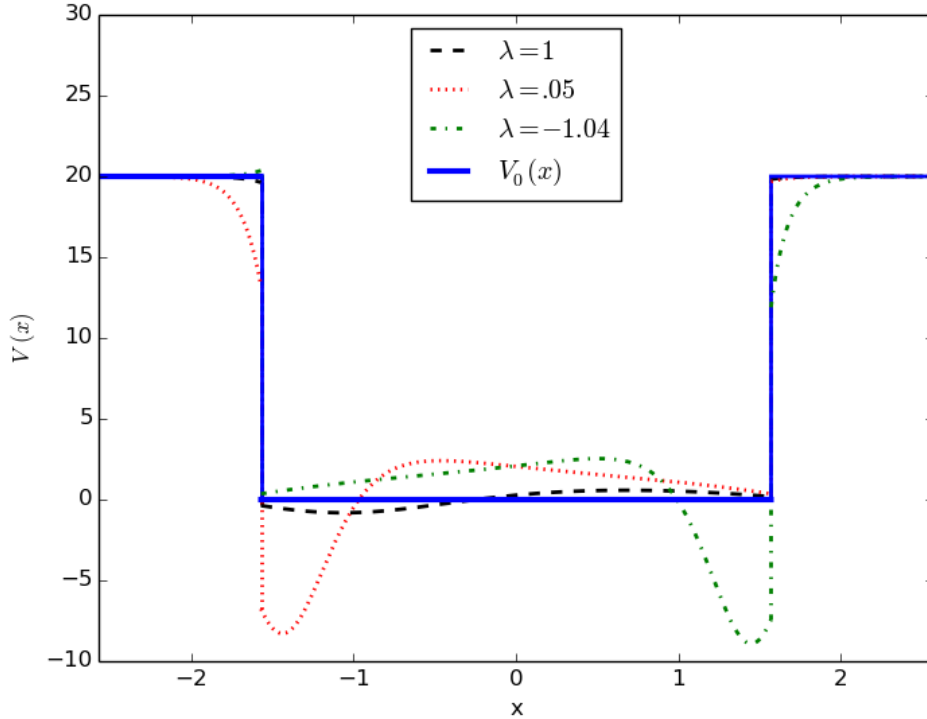


Figure 10: Isospectral deformation of finite square well, plotted with $\frac{\hbar^2}{2m} = 1$, $V = 20$ and $L = \pi$.

3.6 Isospectral Deformation from a Wave Function

It is possible to construct a potential from just knowing the ground state wave function, assuming it fills the SUSYQM requirements i.e. it needs to be nodeless. Using eq. (2.3) and the assumption $\frac{\hbar^2}{2m} = 1$.

$$V^{(1)}(x) = \frac{\psi_0''(x)}{\psi_0(x)}. \quad (3.62)$$

So doing this with the wave function

$$\psi_0(x) = A(1 + \beta x^2)e^{-\alpha x^2}, \quad (3.63)$$

where A is the normalization constant, note that it is nodeless for $\beta \geq 0$.

$$V(x) = \frac{\psi_0''(x)}{\psi_0(x)} = \frac{(4\alpha^2\beta x^4 + 2\alpha(2\alpha - 5\beta)x^2 - 2(\alpha - \beta))}{(1 + \beta x^2)} \quad (3.64)$$

So from this it is possible to construct an isospectral deformation of the potential using eq. (2.91) where $I(x) = \int_{-\infty}^x [\psi_0(t)]^2 dt$, so by studying $I(x)$ and using the gamma function, $\Gamma(s)$ and the incomplete gamma function, $\gamma(s, t)$ were

$$\gamma(s, t) = \int_0^x t^{s-1} e^{-t} dt, \quad \Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt \quad (3.65)$$

to rewrite it, gives

$$\begin{aligned} I(x) &= \int_{-\infty}^x [A(1 + \beta t^2) e^{-\alpha t^2}]^2 dt \\ &= A^2 \int_{-\infty}^x e^{-2\alpha t^2} dt + A^2 \int_{-\infty}^x 2\beta t^2 e^{-2\alpha t^2} dt + A^2 \int_{-\infty}^x \beta^2 t^4 e^{-2\alpha t^2} dt \\ &= \frac{A^2}{\sqrt{8\alpha}} \int_{-\infty}^{2\alpha x^2} \frac{1}{\sqrt{z}} e^{-z} dz + \frac{2A^2\beta}{\sqrt{8\alpha^3}} \int_{-\infty}^{2\alpha x^2} \sqrt{z} e^{-z} dz + \frac{A^2\beta^2}{\sqrt{32\alpha^{5/2}}} \int_{-\infty}^{2\alpha x^2} z^{3/2} e^{-z} dz \end{aligned}$$

At this point it is easy to normalize the wave function simply by introducing the normalization condition by changing the integration region for $I(x)$ i.e.

$$\begin{aligned} &\int_{-\infty}^{\infty} [A(1 + \beta t^2) e^{-\alpha t^2}]^2 dt = 1 \\ &= A^2 \int_{-\infty}^{\infty} e^{-2\alpha t^2} dt + A^2 \int_{-\infty}^{\infty} 2\beta t^2 e^{-2\alpha t^2} dt + A^2 \int_{-\infty}^{\infty} \beta^2 t^4 e^{-2\alpha t^2} dt = 1 \\ &= \frac{2A^2}{\sqrt{8\alpha}} \Gamma\left(\frac{1}{2}\right) + \frac{2A^2\beta}{\sqrt{8\alpha^3}} \Gamma\left(\frac{3}{2}\right) + \frac{A^2\beta^2}{\sqrt{32\alpha^{5/2}}} \Gamma\left(\frac{5}{2}\right) = 1 \\ &\implies A = \left(\frac{2}{\sqrt{8\alpha}} \sqrt{\pi} + \frac{\beta}{\sqrt{8\alpha^3}} \sqrt{\pi} + \frac{3}{4} \frac{\beta^2}{\sqrt{32\alpha^{5/2}}} \sqrt{\pi} \right)^{-\frac{1}{2}} \quad (3.66) \end{aligned}$$

Getting back to $I(x)$

$$\begin{aligned} I(x) &= \frac{A^2}{\sqrt{8\alpha}} \left(\sqrt{\pi} \pm \gamma\left(\frac{1}{2}, 2\alpha x^2\right) \right) + \frac{A^2\beta}{\sqrt{8\alpha^3}} \left(\frac{1}{2} \sqrt{\pi} \pm \gamma\left(\frac{3}{2}, 2\alpha x^2\right) \right) \\ &\quad + \frac{\sqrt{2}A^2\beta^2}{16\alpha^{5/2}} \left(\frac{3}{4} \sqrt{\pi} \pm \gamma\left(\frac{5}{2}, 2\alpha x^2\right) \right), \quad \begin{cases} +, x > 0 \\ -, x < 0 \end{cases} \quad (3.67) \end{aligned}$$

and the isospectral deformation of the potential is given by

$$\tilde{V}(x) = V(x) - 2\frac{d^2}{dx^2} \ln[I(x) + \lambda] = V(x) - 2\frac{(I(x) + \lambda)I''(x) - I'(x)I'(x)}{(I(x) + \lambda)^2} \quad (3.68)$$

where

$$I'(x) = [\psi_0(x)]^2 = \left[A(1 + \beta x^2)e^{-\alpha x^2} \right]^2, \quad (3.69)$$

$$I''(x) = (4\beta^2 A^2 x^3 - 4\alpha\beta^2 A^2 x^5)e^{-2\alpha x^2} \quad (3.70)$$

In fig.11 the potential is plotted for some different lambdas and in fig. 12 the corresponding wave functions are plotted using eq. (2.92), with $\alpha = 1/2$ and $\beta = 2$.

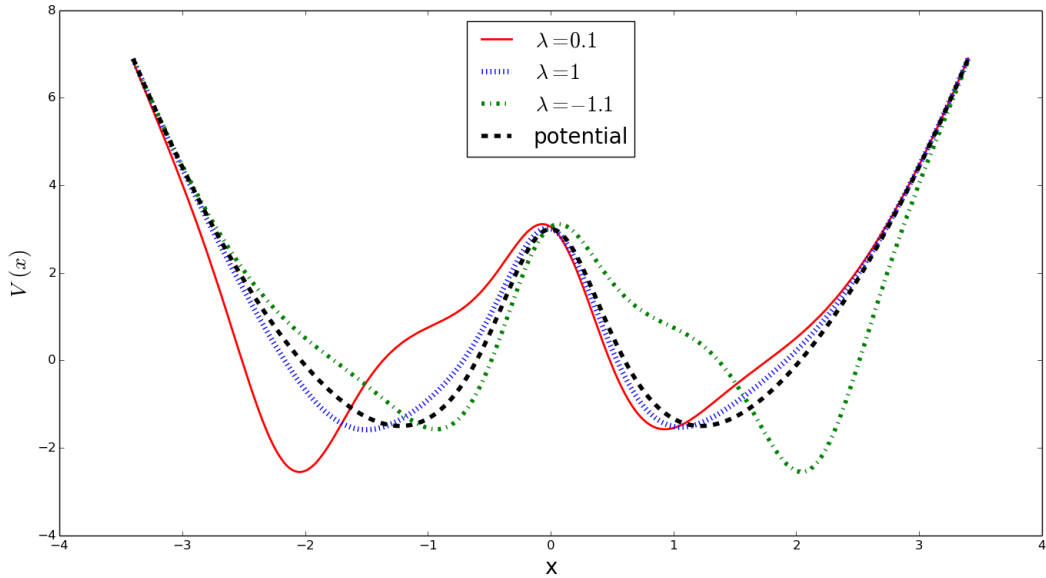


Figure 11: Isospectral deformation of the potential created with the wave function eq. (3.63), plotted with $\frac{\hbar^2}{2m} = 1$, $\alpha = 1/2$ and $\beta = 2$ (arb. units).

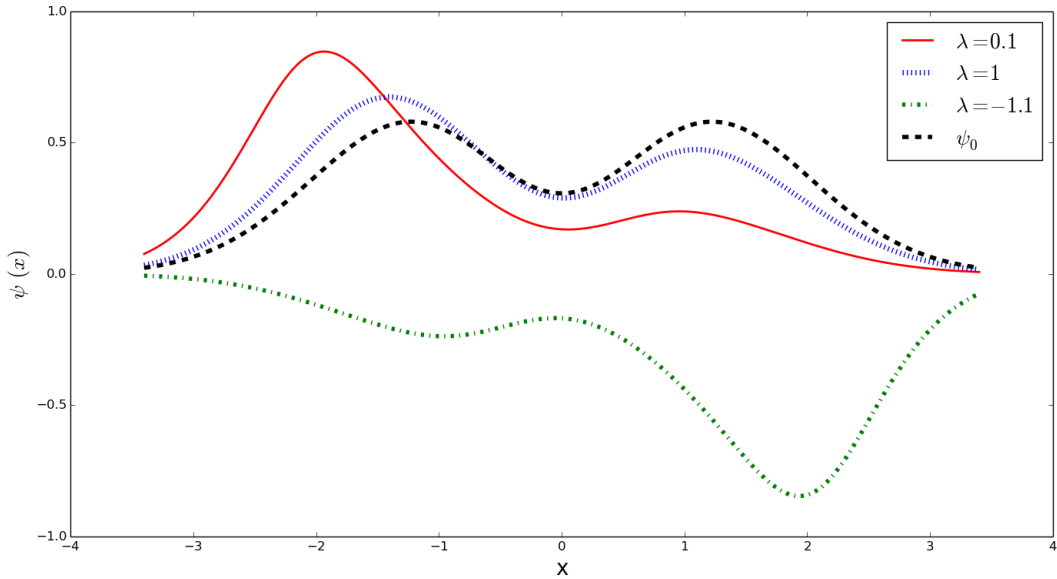


Figure 12: Wave functions corresponding to the isospectral deformation of the potential created with the wave function eq. (3.63), plotted with $\frac{\hbar^2}{2m} = 1$, $\alpha = 1/2$ and $\beta = 2$ (arb. units).

4 Conclusions

Supersymmetric quantum mechanics provides a different approach to solving a number of quantum mechanical problems. Using SUSYQM and the shape invariant condition it is possible to calculate the energy spectra without solving the Schrödinger equation or the Dirac equation, by finding the superpotential and using it to generate partner potentials. Were the partner potentials has the same energy spectra except for the ground state.

Supersymmetric quantum mechanics provides a new aspect to quantum mechanics. Shows the difficulty of the inverse problem, instead of asking “For a given potential, what are the allowed energies it produces?” one could ask “For a given set of energy levels energy, what are the potentials that could have produced it?” [1].

The fact that seemingly unrelated potentials have almost identical energy spectra is caused by the fact that they have the same underlying superpotential or are a part of the same isospectral deformation family. This is not explained in traditional quantum mechanics and are one of the oddities in traditional quantum mechanics. Thus it provides a deeper understanding of a quantum mechanics.

Supersymmetric quantum mechanics can be applied to a wide range of quantum mechanics problems, such as obtaining the energy spectra for a hydrogen atom as well as obtaining the fine structure for the energy spectrum of the hydrogen atom from the Dirac equation without solving it explicitly.

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