A theory of baryon resonances at large N_c

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At large number of colors, N_c quarks in baryons are in a mean field of definite space and flavor symmetry. We write down the general Lorentz and flavor structure of the mean field, and derive the Dirac equation for quarks in that field. The resulting baryon resonances exhibit an hierarchy of scales: The crude mass is $\mathcal{O}(N_c)$, the intrinsic quark excitations are $\mathcal{O}(1)$, and each intrinsic quark state entails a finite band of collective excitations that are split as $\mathcal{O}(1/N_c)$. We build a (new) theory of those collective excitations, where full dynamics is represented by only a few constants. In a limiting (but unrealistic) case when the mean field is spherically-and flavor-symmetric, our classification of resonances reduces to the $SU(6)$ classification of the old non-relativistic quark model. Although in the real world N_c is only three, we obtain a good accordance with the observed resonance spectrum up to 2 GeV.

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I. INTRODUCTION

There are currently two main classes of baryon models: three-quark models with certain interaction between quarks, and models where baryons are treated as nonlinear solitons of bosonic fields, such that quarks are implicit. Both points of views have their strong and weak points. The main shortcoming of the 3-quark models is that they ignore the phenomenologically important admixture of quark-antiquark pairs ($Q\bar{Q}$) in a baryon, therefore they are essentially nonrelativistic. A consistent relativistic picture can be only field-theoretic.

This is the advantage of the solitonic approach that stems from the Skyrme model [1, 2] and includes its more recent incarnation in the holographic QCD [3, 4]. Models of this kind effectively take care of $Q\bar{Q}$ pairs in baryons [5]. However, an obvious shortcoming is that they do not possess explicit quarks, therefore it is difficult to address many important issues, for example, what are quark and antiquark (parton) distributions in nucleons.

In this paper, we suggest an approach that is a bridge between the two. On the one hand, we operate in terms of quarks and keep contact with the traditional three-quark models. On the other hand, we do not restrict ourselves to the valence quarks only but allow for an arbitrary amount of additional $Q\bar{Q}$ pairs. Our formalism is relativistically invariant. The key ingredient of our construction is the mean bosonic field for quarks, which, in fact, is nothing but the "soliton" of the second approach.

As in any other solitonic picture of baryons, we formally need to consider the number of colors N_c as a large algebraic parameter. When N_c is large the N_c quarks constituting a baryon can be considered in a mean (non-fluctuating) field that does not change as $N_c \to \infty$ [6]. Quantum fluctuations about a mean field are suppressed as $1/N_c$. While in the real world N_c is only three, we do not expect qualitative difference in the baryon spectrum from its large-N_c limit. The hope is that if one develops a clear picture at large N_c, and controls at least in principle $1/N_c$ corrections, its imprint will be visible at $N_c = 3$.

From the large-N_c viewpoint, baryons have been much studied in the past using general N_c counting rules and group-theoretic arguments, for reviews see [7–9] and references therein. In this framework, many relations for baryon resonances have been derived, with no reference to the underlying dynamics. The key new point of this paper is that we suggest a simple underlying physical picture that results in those relations and disclose the meaning of the otherwise free numerical coefficients therein. We also derive new relations valid in the large-N_c limit.

This work is in the line with our previous chiral quark–soliton model [10] which successfully describes quark and antiquark distributions in nucleons [11], and other properties. In this paper we take off the previous limitation that the mean field is exclusively the pseudoscalar one, and focus on baryons resonances rather than on the ground state.

The advantage of the large-N_c approach is that at large N_c baryon physics simplifies considerably, which enables one to take into full account the important relativistic and field-theoretic effects that are often ignored. Baryons are not just three (or N_c) quarks but contain additional $Q\bar{Q}$ pairs, as it is well known experimentally. The number of antiquarks in baryons is, theoretically, also proportional to N_c [5], which means that antiquarks cannot be obtained from adding one meson to a baryon: one needs $\mathcal{O}(N_c)$ mesons to explain $\mathcal{O}(N_c)$ antiquarks, implying in fact a classical mesonic field.

At the microscopic level quarks experience only color interactions, however gluon field fluctuations are not suppressed if N_c is large; the mean field can be only 'colorless'. An example how originally color interactions are Fierz-transformed into interactions of quarks with mesonic fields is provided by the instanton liquid model [12]. A non-fluctuating confining bag is another example of a 'colorless' mean field. A more modern example of a mean field is given by the 5- or 6-dimensional 'gravitational' and flavor background field in the holographic QCD models.

Since quarks inside baryons are generally relativistic, especially in excited baryons, we shall assume that quarks in the large-N_c baryon obey the Dirac equation in a background mesonic field. In fact, the Dirac equation for quarks may be non-local. All intrinsic quark Dirac levels in the mean field are stable in N_c. All negative-energy levels should be filled in by N_c quarks in the antisymmetric state in color, corresponding to the zero baryon number state. Filling in the lowest positive-energy level by N_c 'valence' quarks makes a baryon. Exciting higher quark levels or making particle-hole excitations produces baryon resonances. The baryon mass is $\mathcal{O}(N_c)$, and the excitation energy is $\mathcal{O}(1)$. When one excites one quark the change of the mean field is $\mathcal{O}(1/N_c)$ that can be neglected to the first approximation.

Moreover, if one replaces one light (u, d or s) quark in light baryons by a heavy (c, b) one, as in charmed or bottom baryons, the change in the mean field is also $\mathcal{O}(1/N_c)$. Therefore, the spectrum of heavy baryons is directly related to that of light baryons. This fact is well known for low-lying multiples, see e.g.[13], and recently has been discussed for more general situation in [14].

Our approach can be illustrated by the chiral quark–soliton model [10, 15] or by the chiral bag model [16] but actually the arguments of this paper are much more general. We argue that the mean field in baryons of whatever nature has a definite symmetry, namely it breaks spontaneously the symmetry under separate $SU(3)_{\rm flavor}$ and $SO(3)_{\rm space}$ rotations but does not change under simultaneous $SU(2)_{\rm iso+space}$ rotations in ordinary space and a compensating rotation in isospace [14, 17].

If the original symmetry (here: flavor and rotational) is spontaneously broken, it means that the ground state is degenerate: all states obtained by a rotation have the same energy. In Quantum Mechanics the rotations must be quantized, which leads to the splitting of the ground state, as well as all one-quark excitations, by the quantized rotational energy. It implies that each intrinsic quark state, be it the ground state or a one-quark excitation in the Dirac spectrum, generates a band of resonances appearing as collective rotational excitations of a given intrinsic state. The quantum numbers of those resonances, their total number and their splittings are unequivocally dictated by the symmetry of the mean field. In this paper we present, for the first time, the theory of the rotational bands stemming from a given intrinsic one-quark excitation. Assuming the $SU(2)_{\rm iso+space}$ symmetry of the mean field, we obtain the resonances observed in Nature. Moreover, certain relations between resonance splittings that are satisfied with high accuracy, are dynamics-independent but follow solely from the particular symmetry of the mean field.

In this paper, we do not consider any specific dynamical model but concentrate mainly on symmetry. A concrete dynamical model would say what is the intrinsic relativistic quark spectrum in baryons. It may get it approximately correct, or altogether wrong. Instead of calculating the intrinsic Dirac spectrum of quarks from a model, we extract it from the experimentally known baryon spectrum by interpreting baryon resonances as collective excitations about the state and about the one-quark transitions. However, we show that the needed intrinsic quark spectrum can be obtained from a natural choice of the mean field satisfying the $SU(2)_{\rm iso+space}$ symmetry.

In summary, we show that it is possible to obtain a realistic spectrum of baryon resonances up to 2 GeV, starting from the large-N_c limit. This means that we are able to find candidates for all rotational $SU(3)$-multiplets generated around intrinsic quark levels and check large N_c relations between their masses. However, this is not the end of the story as $SU(3)$-multiplets in nature are splitted due to non-zero mass of the strange quark. These splittings are not small and not all members of $SU(3)$-mutliplets are known. For this reason even the contents of lowest $SU(3)$ multiplets is under discussion. We will follow analysis of paper[22] (see also [23]).

We will assume that mass of the strange quark is small enough and construct perturbation theory in m_s valid at large N_c. The question of its validity is under discussion as well (see, e.g., [24]) but we consider the success of relations following from this theory (classical Gell-Mann-Okubo or Guadagnini[25] ones) as an argument in favor of this approach. We derive a number of new relations valid at large N_c which are fulfilled with a good accuracy. Let us note our approach give essentially more relations that were derived in the framework of approach [26] (for mass relations and other aspects of broken $SU(3)$ symmetry, see [27]). Also we give the dynamical interpretation of the constants entering mass splitting and provide the formulas which allow one to calculate splittings provided that underlying dynamical model is fixed.

The notion of baryon resonance implies that its width is small. For excited baryons at large N_c this is not granted — the width of the baryon is $O(1)$. The width is small compared to the total mass of baryon ($O(N_c)$) but it appears to be of order the distance between quark levels. This width is due to transitions between different quark levels with emitting of mesons (e.g. one pion decay to the ground level) which is not suppressed by N_c. We will show below that in the leading order the width is universal for all rotational band belonging to the given quark level. Non-zero width of the resonance leads also to some shift in its position. In spite of the fact that this shift is $O(1)$, it is also universal for rotational band and does not ruin rotational spectra.

Mean field approximation cannot be applied directly to unstable quark levels. The correct definition of the baryon resonance comes from the consideration of meson-baryon scattering amplitude. The baryon resonance manifests itself as a pole in the complex plane of the energy with imaginary part being half of the resonance width. Scattering amplitude at large N_c can be found from meson quadratic form which can be obtained by integrating out quark degrees of freedom.

We performed this program for exotic pentaquark states in [5] in the framework of Skyrme model but it looks to be too complicated for the general case of baryons considered in this paper. We will use the fact that only resonances with relatively small width can be observed and neglect the widths of resonance in order to describe their positions. Moreover, as we do not consider any dynamical mechanism, positions of quark levels anyway play a role of phenomenological parameters. At this point our approach is close to the one of the quark model which also neglects influence of the resonance width to its position.

The widths of baryon resonances also have some hierarchy in N_c. Decays with transition from one quark level to another are $O(1)$, decays inside the same rotational band are $O(1/N_c^2)$. In particular, total widths of the baryons belonging to the rotational band of the ground state (like Δ-resonance) are only $O(1/N_c^2)$ while total widths of all excited baryons are all $O(1)$. These widths are the same for all excited baryons belonging to the given band up to corrections of $O(1/N_c)$ which can nevertheless be significant at $N_c = 3$ (to say nothing about corrections in the mass of the strange quark).

One can try two approaches of adjusting large N_c limit results to the real $N_c = 3$ world. The calculation of physical quantity can be divided typically in two stages: translating original quantity to some effective rotational operator and calculation of matrix element of effective operator with wave functions of rotational states representing given baryon. The first stage requires limit of large N_c in order to avoid the mess of strong interactions. The second is, in fact, trivial and leads to some $SU(3)$-Clebsch-Gordan coefficients, calculable at any N_c. The approach pioneered in [26] requires the strict limit $N_c \to \infty$ at both stages. From the other hand in papers [10] and subsequent ones we applied another approach: use the limit $N_c \to \infty$ but substitute Clebsch-Gordan coefficients by its value at $N_c = 3$. The same logic was, in fact, used also in the original paper of [2]. This approach has at least the same accuracy as the first one but allows to avoid large corrections related to the change of Clebsch-Gordan coefficients from $N_c = \infty$ to $N_c = 3$. In this paper we will discuss both approaches but use mainly the second one.

The paper is organized as follows. In Section II we discuss the possible symmetry of mean field and come to the conclusion that it should be a hedgehog one. This is one of the main features distinguishing our model from the quark model (which can be also considered at $N_c \to \infty$). The quark model assumes that mean field inside the baryon has central symmetry (in majority of the versions it is just the confinement potential). We believe that this assumption contradicts to the data and the mean meson field (e.g. mean field of the pion) is at least equally important.

In Section III we derive Dirac equations in the general hedgehog meson field. One has to find intrinsic quark levels in this mean field. To determine the self-consistent meson field, it is necessary to know also the meson part of Lagrangian. This can be done in the concrete dynamical model. We give the classification of quark levels and discuss the possible order of levels in the mean field.

In Section IV we construct the general theory of rotational state around intrinsic quark levels. Previously this theory was discussed for the ground state baryons; we extend it to arbitrary excited baryon states. We derive formulae for baryon masses and obtain the contents of the $SU(3)$ multiplets entering rotational bands. We also discuss their rotational and quark wave functions.

Section V considers the relation of the $SU(3)$-multiplets at $N_c \to \infty$ and $SU(6)$ multiplets of the quark model. We explain that there is one-to-one correspondence between quark model and one-quark excitations in the mean field at $N_c \to \infty$ for negative parity baryons. This is not true for positive parity: here $SU(6)$ multiplets of the quark model correspond mainly to two-quark excitations. Meanwhile, one-particle excitations still exist and have the same structure as in sector with negative parity. We prefer to use excitations of this type in order to describe experimental data as they should have smaller mass and be narrower than two-particle ones. We leave the quark model picture for parity plus sector and arrive at the description unified for both parities: in order to describe experimental baryon spectra we need 6 levels with grand spin $K = 0^\pm, 1^\pm, 2^\pm$. We confront this simple picture to the data in Section VI and see that it can accommodate the experimental baryon spectra up to 2 GeV. At the same time it does not predict extra states which are typical for the quark model. Section VI is devoted to the mass splitting inside $SU(3)$-multiplets. This question can be important for identifying original $SU(3)$ multiplets. We concentrate mainly on general relations which are model independent. We formulate our conclusions in Section VII.

We relegate few important questions to the series of Appendices. In Appendix A the simple exactly solvable model is considered. This model was already investigated in a number of papers; it helps us to illustrate the relations obtained in the main text at $N_c \to \infty$. Appendix B is related to validity of the cranking approximation in the soliton picture; this validity was doubt in some papers. We discuss decays of baryon resonances in Appendix D. The full theory will be published elsewhere. Here we only give N_c counting of the baryon widths due to the different decays and prove the universality of the width in the leading order for the given rotational band. Appendices C and E are devoted to some technical questions. In particular, in appendix E we give the table of $SU(3)$ Clebsch-Gordan coefficients conforming at $N_c = 3$ to the standard conventions.

II. SYMMETRY OF THE MEAN FIELD

In the mean field approximation, justified at large N_c, one looks for the solutions of the Dirac equation for single quark states in the background mean field. In a most general case the background field couples to quarks through all five Fermi variants. If the mean field is stationary in time, it leads to the Dirac eigenvalue equation for the u, d, s quarks in the background field, $H\psi = E\psi$, the Dirac Hamiltonian being schematically

$$H = \gamma^0\Big(-i\partial_i\gamma^i + S(\mathbf{x}) + P(\mathbf{x})i\gamma^5 + V_\mu(\mathbf{x})\gamma^\mu + \\ + A_\mu(\mathbf{x})\gamma^\mu\gamma^5 + T_{\mu\nu}(\mathbf{x})\frac{i}{2}[\gamma^\mu\gamma^\nu]\Big), \quad (1)$$

where S, P, V, A, T are the scalar, pseudoscalar, vector, axial, tensor mean fields, respectively; all are matrices in flavor. In fact, the one-particle Dirac Hamiltonian (1) is generally nonlocal, however that does not destroy symmetries in which we are primarily interested. We include the current and the dynamically-generated quarks masses into the scalar term S.

The key issue is the symmetry of the mean field. We assume the chiral limit for u, d quarks, $m_u = m_d = 0$, which is an excellent approximation. We consider exact $SU(3)$ flavor symmetry as a good starting point. It implies that baryons appear in degenerate $SU(3)$ multiplets $\mathbf{8}$, $\mathbf{10}, \ldots$; the splittings inside $SU(3)$ multiplets can be determined later on as a perturbation in m_s, see *e.g.* Ref. [19] and Section VII.

A natural assumption, then, would be that the mean field is flavor-symmetric, and spherically symmetric. However we know that baryons are strongly coupled to pseudoscalar mesons ($g_{\pi NN} \approx 13$). It means that there is a large pseudoscalar field inside baryons; at large N_c it is a classical mean field. There is no way of writing down the pseudoscalar field (it must change sign under inversion of coordinates) that would be compatible with the $SU(3)_{\rm flav} \times SO(3)_{\rm space}$ symmetry. The minimal extension of spherical symmetry is to write the "hedgehog" *Ansatz* "marrying" the isotopic and space axes [**?**]:

$$\pi^a(\mathbf{x}) = \begin{cases} n^a\,F(r), \;\; n^a = \frac{x^a}{r}, & a = 1,2,3, \\ 0, & a = 4,5,6,7,8. \end{cases} \quad (2)$$

This *Ansatz* breaks the $SU(3)_{\rm flav}$ symmetry. Moreover, it breaks the symmetry under independent space $SO(3)_{\rm space}$ and isospin $SU(2)_{\rm iso}$ rotations, and only a simultaneous rotation in both spaces remains a symmetry, since a rotation in the isospin space labeled by a, can be compensated by the rotation of the space axes. The *Ansatz* (2) implies a spontaneous (as contrasted to explicit) breaking of the original $SU(3)_{\rm flav} \times SO(3)_{\rm space}$ symmetry down to the $SU(2)_{\rm iso+space}$ symmetry. It is analogous to the spontaneous breaking of spherical symmetry by the ellipsoid form of many nuclei; there are many other examples in physics where the original symmetry is spontaneously broken in the ground state.

We list here all possible structures in the S, P, V, A, T fields, compatible with the $SU(2)_{\rm iso+space}$ symmetry and with the C, P, T quantum numbers of the fields [14, 17]. The fields below are generalizations of the 'hedgehog' Ansatz (2) to mesonic fields with other quantum numbers.

Since $SU(3)$ symmetry is broken, all fields can be divided into three categories:

I. Isovector fields acting on u, d quarks

$$\begin{aligned} \text{pseudoscalar: } & P^a(\mathbf{x}) = n^a\,P_0(r), \\ \text{vector, space part: } & V_i^a(\mathbf{x}) = \epsilon_{aik}\,n_k\,P_1(r), \\ \text{axial, space part: } & A_i^a(\mathbf{x}) = \delta_{ai}\,P_2(r) + n_a n_i\,P_3(r), \\ \text{tensor, space part: } & T_{ij}^a(\mathbf{x}) = \epsilon_{aij}\,P_4(r) + \epsilon_{bij}\,n_a n_b\,P_5(r). \end{aligned} \quad (3)$$

II. Isoscalar fields acting on u, d quarks

$$\begin{aligned} \text{scalar: } & S(\mathbf{x}) = Q_0(r), \\ \text{vector, time component: } & V_0(\mathbf{x}) = Q_1(r), \\ \text{tensor, mixed components: } & T_{0i}(\mathbf{x}) = n_i\,Q_2(r). \end{aligned} \quad (4)$$

III. Isoscalar fields acting on s quarks

$$\begin{aligned} \text{scalar: } & S(\mathbf{x}) = R_0(r), \\ \text{vector, time components: } & V_0(\mathbf{x}) = R_1(r), \\ \text{tensor, mixed components: } & T_{0i}(\mathbf{x}) = n_i\,R_2(r). \end{aligned} \quad (5)$$

All the rest fields and components are zero as they do not satisfy the $SU(2)_{\rm iso+space}$ symmetry and/or the needed discrete C, P, T symmetries. The 12 'profile' functions $P_{0,1,2,3,4,5}$, $Q_{0,1,2}$ and $R_{0,1,2}$ should be eventually found self-consistently from the minimization of the mass of the ground-state baryon. We shall call Eqs. (3-5) the hedgehog *Ansatz*. However, even if we do not know those profiles, there are important consequences of this *Ansatz* for the baryon spectrum.

III. u, d, s QUARKS IN THE 'HEDGEHOG' FIELD

Given the $SU(2)_{\rm iso+space}$ symmetry of the mean field, the Dirac Hamiltonian for quarks actually splits into two: one for s quarks and the other for u, d quarks [17]. It should be stressed that the energy levels for u, d quarks on the one hand and for s quarks on the other are completely different, even in the chiral limit $m_s \to 0$.

The energy levels for s quarks are classified by half-integer J^P where P is parity under space inversion, and $\mathbf{J} = \mathbf{L}+\mathbf{S}$ is quark angular momentum; all levels are $(2J+1)$-fold degenerate. The energy levels for u, d quarks are classified by integer K^P where $\mathbf{K} = \mathbf{T} + \mathbf{J}$ is the 'grand spin' (T is isospin), and are $(2K + 1)$-fold degenerate.

All energy levels, both positive and negative, are probably discrete owing to confinement. Indeed, a continuous spectrum would correspond to a situation when quarks are free at large distances from the center, which contradicts confinement. One can model confinement *e.g.* by forcing the effective quark masses to grow linearly at infinity, $S(\mathbf{x}) \to \sigma r$.

The Dirac equation (1) for s quarks in the background field (5) takes the form of a system of two ordinary differential equations for two functions $f(r)$, $g(r)$ depending only on the distance from the center. The system of equations depends on the (half-integer) angular momentum of level under considerations, and on its parity. For s-quark levels with parity $P = (-1)^{J-\frac{1}{2}}$, *e.g.* for the levels $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, ..$, the system takes the form

$$\begin{cases} E\,f \;=\; -g' - \frac{J+\frac{3}{2}}{r}\,g + R_0\,f + R_1\,f + R_2\,g \\ E\,g \;=\; f' + \frac{-J+\frac{1}{2}}{r}\,f - R_0\,g + R_1\,g + R_2\,f. \end{cases} \tag{6}$$

To find an s-quark energy level E with these quantum numbers, one has to solve Eq. (6) with the initial condition $f(r) \sim r^{J-\frac{1}{2}}$, $g(r) \sim r^{J+\frac{1}{2}}$, and both functions decreasing at infinity.

For levels with opposite parity $P = (-1)^{J+\frac{1}{2}}$, *e.g.* $J^P = \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, ..$, one has to solve another system:

$$\begin{cases} E\,f \;=\; -g' - \frac{-J+\frac{1}{2}}{r}\,g + R_0\,f + R_1\,f + R_2\,g \\ E\,g \;=\; f' + \frac{J+\frac{3}{2}}{r}\,f - R_0\,g + R_1\,g + R_2\,f. \end{cases} \tag{7}$$

We note that in the absence of the $R_{1,2}$ fields the energy spectrum is symmetric under simultaneous change of parity and energy signs.

Dirac equation for u, d quarks in the background fields (3,4) is more complicated: one has here a system of four ordinary differential equations. These equations are direct generalizations of the Dirac equations in the 'hedgehog' field [10], and can be derived similarly to how it is done in that reference.

The system of Dirac equations for the radial functions of the states with parity $(-1)^{K+1}$, namely $K^P = 1^+, 2^-, ...$ has the form

$$E\,f = -g' - \frac{1+K}{r}\,g + (Q_0+Q_1+P_2+P_4)f + (Q_2-P_1)g - \frac{P_0-P_1}{2K+1}(g+b_K\,h) + \frac{P_3+P_5}{2K+1}(f+b_K\,j), \tag{8}$$

$$E\,g = f' - \frac{K-1}{r}\,f + (Q_1-Q_0-P_2+P_4)g + (Q_2-P_1)f - \frac{P_0-P_1}{2K+1}(f+b_K\,j) + \frac{P_3-P_5+2P_2-2P_4}{2K+1}(g+b_K\,h), \tag{9}$$

$$E\,h = j' + \frac{2+K}{r}\,j + (Q_1-Q_0-P_2+P_4)h + (Q_2-P_1)j + \frac{P_0-P_1}{2K+1}(j-b_K\,f) - \frac{P_3-P_5+2P_2-2P_4}{2K+1}(h-b_K\,g), \tag{10}$$

$$E\,j = -h' + \frac{K}{r}\,h + (Q_0+Q_1+P_2+P_4)j + (Q_2-P_1)h + \frac{P_0-P_1}{2K+1}(h-b_K\,g) - \frac{P_3+P_5}{2K+1}(j-b_K\,f), \tag{11}$$

where $b_K = 2\sqrt{K(K+1)}$. The radial functions f, g, h, j refer to partial waves with $L = K-1, K, K, K+1$, respectively, and they behave at the origin as r^L. To find the energy levels for a given K^P, one has to solve these equations twice: once with the initial condition $f(r_{\rm min}) \sim r_{\rm min}^{K-1}$, all the rest functions being put to zero at the origin, and another time with the initial condition $h(r_{\rm min}) \sim r_{\rm min}^K$, with all the rest functions zeroes, $r_{\rm min} \to 0$. Evolving the functions according to the equations numerically up to some asymptotically large $r_{\rm max}$ one finds two sets of functions (f_1, g_1, h_1, j_1) and (f_2, g_2, h_2, j_2). The energy levels are found from the zeroes of two (equal) determinants $f_1 h_2 - f_2 h_1 = g_1 j_2 - g_2 j_1$.

For states with parity $(-1)^K$, namely $K^P = 1^-, 2^+, ..$ the system of Dirac equations is:

$$E\,f = -g' - \frac{1+K}{r}\,g + (Q_1-Q_0+P_2-P_4)f - (Q_2+P_1)g + \frac{P_0+P_1}{2K+1}(g+b_K\,h) + \frac{P_3-P_5}{2K+1}(f+b_K\,j), \tag{12}$$

$$E\,g = f' - \frac{K-1}{r}\,f + (Q_0+Q_1-P_2-P_4)g - (Q_2+P_1)f + \frac{P_0+P_1}{2K+1}(f+b_K\,j) + \frac{P_3+P_5+2P_2+2P_4}{2K+1}(g+b_K\,h), \tag{13}$$

$$E\,h = j' + \frac{2+K}{r}\,j + (Q_0+Q_1-P_2-P_4)h - (Q_2+P_1)j - \frac{P_0+P_1}{2K+1}(j-b_K\,f) - \frac{P_3+P_5+2P_2+2P_4}{2K+1}(h-b_K\,g), \tag{14}$$

$$E\,j = -h' + \frac{K}{r}\,h + (Q_1-Q_0+P_2-P_4)j - (Q_2+P_1)h - \frac{P_0+P_1}{2K+1}(h-b_K\,g) - \frac{P_3-P_5}{2K+1}(j-b_K\,f), \tag{15}$$

where again $f \sim r^{K-1}, g \sim r^K, h \sim r^K, j \sim r^{K+1}$, and the levels are found by the same trick. The fields $Q_{1,2}$

and $P_{0,2,3}$ break symmetry with respect to simultaneous change of parity and energy signs.

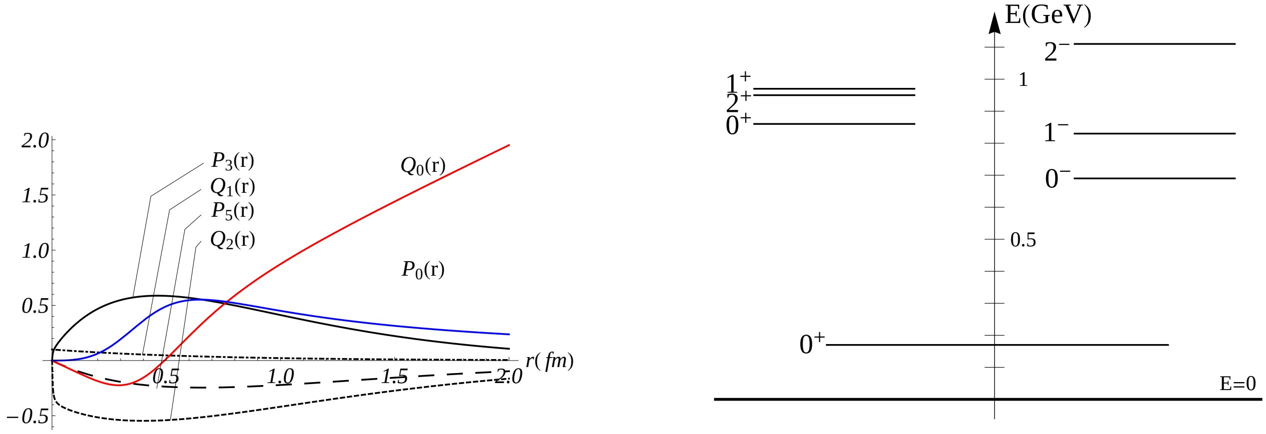

FIG. 1. (Color online) An illustrative example of intrinsic quark levels with quantum numbers K^P (right) generated by the mean fields shown in the left panel.

The case $K = 0$ is special, since the angular momentum is restricted to only one value $J = K + \frac{1}{2} = \frac{1}{2}$. It means that $g = f = 0$, and the system of eqs.(8)-(11) for the $K^P = 0^-$ level reduces to two equations:

$$\begin{aligned} E\,j &= -h' + (Q_0 + Q_1 + P_2 - P_3 + P_4 - P_5)j + (P_0 - 2P_1 + Q_2)h, \\ E\,h &= j' + \frac{2}{r}\,j + (-Q_0 + Q_1 - 3P_2 - P_3 + 3P_4 + P_5)h + (P_0 - 2P_1 + Q_2)j \end{aligned} \tag{16}$$

with $h \sim r^0, j \sim r^1$. Similarly, to find the $K^P = 0^+$ levels one has to solve only two equations:

$$\begin{aligned} E\,j &= -h' + (-Q_0 + Q_1 + P_2 - P_3 - P_4 + P_5)j - (P_0 + 2P_1 + Q_2)h, \\ E\,h &= j' + \frac{2}{r}\,j + (Q_0 + Q_1 - 3P_2 - P_3 - 3P_4 - P_5)h - (P_0 + 2P_1 + Q_2)j. \end{aligned} \tag{17}$$

In Fig. 1 we show an example of quark levels obtained from a 'natural' choice of external fields $Q_{0\text{–}2}, P_{0\text{–}5}$. We take a confining scalar field $S(r) = \sigma r$ with a standard string tension $\sigma = (0.44\,\mathrm{GeV})^2$, and a topological chiral angle field $P(r) = 2\arctan(r_0^2/r^2)$ such that the profile functions introduced in Eqs.(3,4) are $Q_0(r) = S(r)\cos P(r)$, $P_0(r) = S(r)\sin P(r)$; the other profile functions are exponentially decaying at large distances. The external fields are shown in Fig. 1, left, and the resulting quark levels with various K^P are shown in Fig. 1, right. These or similar levels dictate the masses of baryon resonances.

According to the Dirac theory, all *negative*-energy levels, both for s and u, d quarks, have to be fully occupied, corresponding to the vacuum. It means that there must be exactly N_c quarks antisymmetric in color occupying all degenerate levels with J_3 from $-J$ to J, or K_3 from $-K$ to K; they form closed shells. Filling in the lowest level with $E > 0$ by N_c quarks makes a ground state baryon, see Fig. 2. A similar picture arises in the chiral bag model [16]. Excited baryons can be related to different 1,2,3-quark excitations to the other levels. We will try to advocate the point of view that known baryon resonances below 2 GeV are related to one quark excitations only.

The mass of a baryon is the aggregate energy of all filled states, and being a functional of the mesonic field, it is proportional to N_c since all quark levels are degenerate in color. Therefore quantum fluctuations of mesonic field in baryons are suppressed as $1/N_c$ so that the mean field is indeed justified.

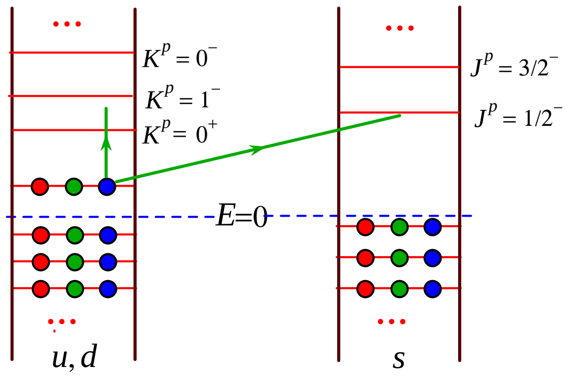

FIG. 2. (Color online) Filling u, d and s shells for the ground-state baryon. Excitations of one valence quarks describe baryon resonances.

IV. ROTATIONAL BANDS ABOUT INTRINSIC QUARK LEVELS

Every intrinsic level is accompanied by the rotational band of the states. It appears as a result of the quantization of the slow rotations both in the flavor and ordinary space. The theory of rotational bands over the ground state was developed years ago[10] but for excited states and for general case of the mean field it has some specifics.

The original symmetry of the theory in the chiral limit is $SU(3)_{\text{flav}} \times SO(3)_{\text{space}}$. Both symmetries are broken by the 'hedgehog' *Ansatz* of the mean field, so the soliton transforms under the space and flavor rotations nontrivially. However, the energy of the rotated soliton is the same as the original one. For this reason constant rotations are zero modes and should be taken into account exactly.

A. Ground states

Rotations slowly depending on time split the energy level into the rotational band. It is convenient to describe this effect with an effective Lagrangian depending on collective coordinates which are rotational matrices.

Let $R(t)$ be an $SU(3)$ matrix describing slow rotation in flavor space, and $\mathcal{S}(t)$ be an $SU(2)$ matrix for slow space (and spin) rotation. They rotate quark wave functions $\phi^{\alpha i}(\boldsymbol{x})$ ($\alpha = 1\ldots 3$ is flavor, i=1…2 – spin indices) in the given mean field as:

$$\tilde{\phi}_n^{\alpha i}(x) = R^{\alpha}_{\alpha'}(t)\mathcal{S}^i_{i'}(t)\phi_n^{\alpha' i'}(O(t)\boldsymbol{x}),$$

$$O_{ik}(t) = \tfrac{1}{2}\mathrm{Tr}\left[\mathcal{S}^+(t)\sigma_k\mathcal{S}(t)\sigma_i\right] \tag{18}$$

Then it is easy to see that simultaneous transformation of the meson fields

$$\tilde{P}^a(\boldsymbol{x}) = O_{ab}[R]P^b(O(\mathcal{S})\boldsymbol{x})$$

$$\tilde{V}^{ai}(\boldsymbol{x}) = O_{ab}[R]O_{ij}[\mathcal{S}]V^{bj}(O(\mathcal{S})\boldsymbol{x}),$$

$$\tilde{A}^{ai}(\boldsymbol{x}) = O_{ab}[R]O_{ij}[\mathcal{S}]A^{bj}(O(\mathcal{S})\boldsymbol{x}), \tag{19}$$

(and so on) leaves invariant Dirac equation in the mean field provided that matrices R and $\mathcal{S}$ are constant in time.

Let us integrate out quarks. Effective action of the theory is a sum of meson Lagrangian and the contribution of constituent quarks which is the determinant of the Dirac equation in the mean field:

$$S_{\text{eff}} = \int dt\, \mathcal{L}(M) - i\sum_c \mathrm{Sp}_{\text{occup}}\mathrm{Log}\left\{i\frac{\partial}{\partial t} - \mathcal{H}[M]\right\} \tag{20}$$

Here sum implies the summation in color indices and $\mathrm{Sp}\{\ldots\}$ is running over all *occupied* states. Since meson field M and the Hamiltonian $\mathcal{H}$ are color blind, the sum in color produces the factor N_c for ground state. For the 1-particle excitations one term in this sum corresponds to some different filling of the levels.

Slow rotations $\mathcal{S}(t), R(t)$ are the particular cases of the quantum fluctuations of the general meson field M. Usually quantum fluctuations are suppressed in the limit of large N_c. Rotations are not suppressed as they are zero modes. Only their frequencies are small in N_c.

Let us parametrize the general meson field as $M = \bar{M} + \delta M$ (where $\bar{M}(\boldsymbol{x})$ is a time independent mean field and $\delta M(\boldsymbol{x}, t)$ are quantum fluctuations) and calculate the effective action eq. (20) on the set of slowly rotated states (18,19) [?]:

$$S_{\text{eff}} = \int dt\, \mathcal{L}_{\text{meson}}(\bar{M} + \delta M, \tilde{\Omega}, \tilde{\omega}) -$$

$$- i\sum_c \mathrm{Sp}_{\text{occup}}\mathrm{Log}\left\{i\frac{\partial}{\partial t} - \mathcal{H}[\bar{M} + \delta M] - \tilde{\Omega}_a t_a - \tilde{\omega}_i j_i\right\} \tag{21}$$

Here $\tilde{\Omega}_a$ and $\tilde{\omega}_i$ are flavor and angular frequencies in the body-fixed frame:

$$\tilde{\Omega}_a = -i\mathrm{Tr}\left[R^+\dot{R}\lambda_a\right], \qquad \tilde{\omega}_i = -i\mathrm{Tr}\left[\mathcal{S}^+\dot{\mathcal{S}}\sigma_i\right] \tag{22}$$

(λ_a are Gell-Mann flavor matrices and σ_i are Pauli spin matrices), t_a and j_i are one-particle operators of flavor and total angular momenta:

$$t_a = \tfrac{1}{2}\lambda_a, \qquad j_i = s_i + l_i = \tfrac{1}{2}\sigma_i + i\varepsilon^{ikl}x_k\frac{\partial}{\partial x_l} \tag{23}$$

Next we expand eq. (21) in small $\delta M, \tilde{\Omega}, \tilde{\omega}$. The linear term should be absent, as mean field $\bar{M}(\boldsymbol{x})$ is a solution of equations of motion. There is a famous exclusion from this rule — Witten-Wess-Zumino term which is linear in Ω_8 and proportional to the baryon charge B of the state:

$$\delta S^{(1)} = -\frac{N_c}{2\sqrt{3}}\int dt\tilde{\Omega}_8 \tag{24}$$

The second order correction is in general:

$$\delta S^{(2)} = \frac{1}{2}\int d^4x\ \delta M W \delta M + \int d^4x \left(\delta M \mathcal{K}^a_\Omega \tilde{\Omega}_a + \delta M \mathcal{K}^i_\omega \tilde{\omega}_i\right) -$$

$$- \frac{1}{2}\int dt \left[I^{(\Omega\Omega)}_{ab}\tilde{\Omega}_a\tilde{\Omega}_b + I^{(\omega\omega)}_{ab}\tilde{\omega}_i\tilde{\omega}_j + I^{(\omega\Omega)}_{ai}\tilde{\Omega}_a\tilde{\omega}_i\right] \quad (25)$$

Here first term is a quadratic form for the quantum fluctuations which are not rotations, second term describes the mixing of rotations and other quantum fluctuations, third term is a generic quadratic form for space and flavor rotations. All coefficients in eq. (25) $(W, \mathcal{K}, I)$ are proportional to N_c. Thus the quantum fluctuations are $\delta M = O(1/\sqrt{N_c})$. As to the frequencies $\tilde{\Omega}, \tilde{\omega}$ we will see that they are $\tilde{\Omega}, \tilde{\omega} = O(1/N_c)$

We are interested in the collective rotational Lagrangian, i.e. the Lagrangian depending only on angular and flavor frequencies. There are two sources for such a Lagrangian. First, the Lagrangian comes from the immediate expansion of the original action eq. (21). Second, in presence of mixing the rotation Lagrangian can arise as a result of integration over non-rotational quantum fluctuations of the meson field δM. Indeed, in the second case the correction to the mean field:

$$\delta M = W^{-1}\left[\mathcal{K}^a_\Omega \tilde{\Omega}_a + \mathcal{K}^i_\omega \tilde{\omega}_i\right] \quad (26)$$

is of the first order in frequencies and should be accounted in the leading order rotational Lagrangian:

$$S^{(2)}_{\rm rot} = -\int dt \left[\tfrac{1}{2}\tilde{\Omega}_a \mathcal{I}^{(\Omega\Omega)}_{ab}\tilde{\Omega}_b + \tfrac{1}{2}\tilde{\omega}_i \mathcal{I}^{(\omega\omega)}_{ij}\tilde{\omega}_j + \tfrac{1}{2}\tilde{\Omega}_a \mathcal{I}^{(\omega\Omega)}_{ai}\tilde{\omega}_i\right]$$

$$\mathcal{I}^{(\Omega\Omega)}_{ab} = I^{(\Omega\Omega)}_{ab} + \mathcal{K}^a_\Omega W^{-1}\mathcal{K}^b_\Omega,$$

$$\mathcal{I}^{(\omega\omega)}_{ij} = I^{(\omega\omega)}_{ij} + \mathcal{K}^i_\omega W^{-1}\mathcal{K}^j_\omega,$$

$$\mathcal{I}^{(\omega\Omega)}_{ai} = I^{(\omega\Omega)}_{ai} + \mathcal{K}^i_\omega W^{-1}\mathcal{K}^a_\Omega + \mathcal{K}^a_\Omega W^{-1}\mathcal{K}^i_\omega \quad (27)$$

i.e. the mixing leads to the renormalization of the moments of inertia. It is essential that the terms arising from mixing are of the same order in N_c (as $\mathcal{K} \sim O(N_c)$ and $W \sim O(N_c)$) and contribute to the collective action.

This phenomenon is well-known from the nuclear physics. The approximation with the mixing is neglected is called the *cranking* one [37]. The importance of the mixing was pointed out by Thouless-Valatin [38]. The mixing of the rotations and quantum fluctuations, however, is absent in many relativistic theories (at least this is true for models based only on pions (see Appendix B). In such theories cranking approximation is exact.

Cranked moments of inertia $I^{(\Omega\Omega)}_{ab}$, $I^{(\omega\omega)}_{ij}$, $I^{(\omega\omega)}_{ai}$ consist of two parts, fermion and meson ones. To obtain the meson part we substitute rotated meson fields eq. (19) in the mean field approximation to the meson Lagrangian. If the last Lagrangian contains some time derivative, we will get some terms quadratic in frequencies $\tilde{\Omega}$, $\tilde{\omega}$ (one should neglect higher terms) which are the contributions to the moments of inertia.

The quark part of moments of inertia can be obtained expanding fermion determinant of eq. (21) in $\tilde{\Omega}$, $\tilde{\omega}$. Corresponding part of the moments of inertia is given by well-known Inglis expression [37]:

$$\left(I^{(\Omega\Omega)}_{ab}\right)_{\rm q} = \frac{N_c}{2}\sum_{n,m}\frac{\langle n|t_a|m\rangle\langle m|t_b|n\rangle + \langle n|t_b|m\rangle\langle m|t_a|n\rangle}{\varepsilon_m - \varepsilon_n} \quad (28)$$

($|n\rangle$ are occupied 1-quark states, $|m\rangle$ are non-occupied states) and analogous expressions for $I^{(\omega\omega)}_{ij}$ and $I^{(\omega\Omega)}_{ij}$ with flavor generator t_a replaced by total quark angular momentum j_i.

Hedgehog symmetry of the mean field leads to the following relations between different moments of inertia:

$$\mathcal{I}^{(\Omega\Omega)}_{ab} = \begin{cases} I_1\delta_{ab}, & a,b = 1\ldots 3 \\ I_2\delta_{ab}, & a,b = 4\ldots 7 \\ 0, & a,b = 8 \end{cases},$$

$$\mathcal{I}^{(\omega\Omega)}_{ai} = -2I_1\delta_{ai} \qquad \mathcal{I}^{(\omega\omega)}_{ij} = I_1\delta_{ij} \quad (29)$$

and hence the quadratic part of the rotational action reduces to:

$$S^{(2)}_{rot} = -\int dt \sum_{i=1}^{3}\frac{I_1}{2}(\tilde{\Omega}_i - \tilde{\omega}_i)^2 + \sum_{a=4}^{7}\frac{I_2}{2}\tilde{\Omega}^2_a \quad (30)$$

This fact does not depend on the origin of the rotational Lagrangian. In particular this result can be checked for the quark part eq. (28) (see, e.g. [10]).

The complete rotational Lagrangian:

$$\mathcal{L}_{rot} = \sum_{i=1}^{3}\frac{I_1}{2}(\tilde{\Omega}_i - \tilde{\omega}_i)^2 + \sum_{a=4}^{7}\frac{I_2}{2}\tilde{\Omega}^2_a + \frac{BN_c}{2\sqrt{3}}\tilde{\Omega}_8 \quad (31)$$

is a Lagrangian for some specific spherical top both in the flavor and usual space. We calculate operators of angular $\tilde{\boldsymbol{J}}$ and flavor momenta $\tilde{\boldsymbol{T}}$:

$$\tilde{\boldsymbol{J}} = -\tfrac{1}{2}{\rm Tr}\left[\mathcal{S}\boldsymbol{\sigma}\frac{\delta}{\delta\mathcal{S}}\right] = \frac{\partial\mathcal{L}_{rot}}{\partial\boldsymbol{\omega}} = I_1(\boldsymbol{\omega} - \boldsymbol{\Omega})$$

$$\tilde{T}_a = -\tfrac{1}{2}{\rm Tr}\left[R\lambda_a\frac{\delta}{\delta R}\right] = \frac{\partial\mathcal{L}_{rot}}{\partial\Omega_a} =$$

$$= \begin{cases} I_1(\Omega_a - \omega_a), & a = 1\ldots 3 \\ I_2\Omega_a, & a = 4\ldots 7 \\ \frac{N_c}{2\sqrt{3}}, & a = 8 \end{cases} \quad (32)$$

We see that the following quantization rules applied to the rotational bands of ground state baryons:

$$\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{T}} = 0, \qquad \tilde{T}_8 = \frac{N_c}{2\sqrt{3}} \quad (33)$$

The second is celebrated Witten quantization rule [2] which claims that hypercharge in the body-fixed frame is $\tilde{Y} = \frac{2}{\sqrt{3}}\tilde{T}_8 = N_c/3$. It is completely the result of the hedgehog symmetry and fact that N_c valence quarks with the hypercharge $\tilde{Y} = 1/3$ are put to some bound state in the sector of u, d-quarks.

The Hamiltonian of rotations determined from eq. (31) should be expressed in terms of momenta $\tilde{T}$, $\tilde{J}$:

$$\mathcal{H}_{rot} = \sum_{a=1}^{3} \frac{\tilde{T}_a^2}{2I_1} + \sum_{a=4}^{7} \frac{\tilde{T}_a^2}{2I_2} =$$

$$= \frac{c_2(r) - \tilde{T}(\tilde{T}+1) - \frac{3}{4}\tilde{Y}}{2I_2} + \frac{\tilde{T}(\tilde{T}+1)}{2I_1} \tag{34}$$

Here $c_2(r) = \sum_a \tilde{T}_a^2$ is Casimir operator in the $SU(3)$ representation r. It is easy to determine also the collective wave function which is an eigenfunction of the Hamiltonian and operators of momenta in the lab fixed frame:

$$T_a = -\frac{1}{2}\mathrm{Tr}\left[\lambda_a R \frac{\delta}{\delta R}\right], \qquad \boldsymbol{J} = -\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\mathcal{S}\frac{\delta}{\delta \mathcal{S}}\right] \tag{35}$$

Wave function is a product of two Wigner $\mathcal{D}$-functions, one for $SU(3)$ and one for $SU(2)$ group:

$$\Psi_{rot}(R, \mathcal{S}) = \sqrt{\dim(r)(2J+1)}\times$$

$$\times \sum_{\tilde{T},\tilde{T}_3,\tilde{J}_3} C^{00}_{\tilde{T}\tilde{T}_3\ J\tilde{J}_3} \mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}(R^+)\mathcal{D}^{J}_{\tilde{J}_3;J_3}(\mathcal{S}^+) =$$

$$= \sqrt{\dim(r)}(-1)^{J+J_3}\mathcal{D}^{(r)}_{\tilde{Y}J,-J_3;YTT_3}(\mathcal{S}R^+) \tag{36}$$

This function is an eigenfunction of spin $\boldsymbol{J}^2 = \tilde{\boldsymbol{J}}^2 = \tilde{\boldsymbol{T}}^2$, J_3, isospin $\boldsymbol{T}^2$ and T_3 and hypercharge Y, index (r) labels the $SU(3)$ representations with dimension $\dim(r)$. According to eq. (33) the hypercharge $\tilde{Y} = N_c/3$. At last Clebsch-Gordan coefficient $C^{00}_{\tilde{T}\tilde{T}_3\ J\tilde{J}_3}$ sums the isospin $\tilde{\boldsymbol{T}}$ and the angular momentum $\tilde{\boldsymbol{J}}$ to zero in order to obey quantization rule eq. (33). In fact, rotational wave function depends only on the combination $R\mathcal{S}^+$. This is natural because owing to the hedgehog symmetry flavor isospin rotation can be compensated by space one.

B. 1-quark excited states

Let us proceed now with 1-quark excitations, i.e. excitations where only one quark is taken from the ground level and is put to some excited level. The effective Lagrangian eq. (21) is only slightly changed: one term in the sum over N_c quarks has a different scheme of occupied levels. The other $N_c - 1$ terms, however, remain the same. This means that in the leading order in N_c the mean field does not change (the correction to the mean field is $O(1/N_c)$). This is also true for moments of inertia I_1 and I_2 — they acquire corrections $O(1)$ as compared to the leading order $O(N_c)$. Therefore effective rotational Lagrangian eq. (31) is valid also in this case. However, additional *linear* terms in frequencies Ω and ω can appear. The reason is that mean field is a solution of equations of motion only for a ground state and not for excited states. Hence, there is no reason why linear terms in a perturbation (which is a rotation in this case) should be absent. The corresponding linear terms are of the form:

$$\delta\mathcal{L}_{\mathrm{rot}} = \langle \mathrm{excited}|(\boldsymbol{\omega}\cdot\boldsymbol{j} + \boldsymbol{\Omega}\cdot\boldsymbol{t}) + \delta M|\mathrm{excited}\rangle \tag{37}$$

where last term accounts for possible change of the contribution of the correction eq. (26) to the mean field as due to rotations. This correction should be also calculated only in the ground state (it is determined mainly by rotation of other $N_c - 1$ quarks) and assumed to be already known. It is also linear in frequencies $\tilde{\omega}, \tilde{\Omega}$.

Excited states are usually degenerate. Indeed, excitations in the s-quark sector have degeneracy $2S+1$ (where $S = \frac{1}{2}, \frac{3}{2}, \ldots$ is a total momentum of the state), excitations in the sector of u, d-quarks are degenerate $2K+1$ fold where K is the grand spin of the state. Any of degenerate states or their mixture can be taken as an excitation. We define:

$$|\mathrm{excited}\rangle = \sum \chi_{K_3}|KK_3JL\rangle \tag{38}$$

(we are dealing now with $K \neq 0$ excitations for definiteness). Here $|KK_3JL\rangle$ is the wave function of some excited state with grand spin K and projection K_3, χ_{K_3} are amplitudes for different values of projection. Energy does not depend on χ_{K_3}. Hence it is a new zero mode and should be considered as a collective coordinate together with S and R. Effective rotational Lagrangian should be written for χ_{K_3} slowly changing with time, evidently the complete Lagrangian is

$$\mathcal{L}_{\mathrm{excited}}[\chi, R, S] = \sum_{K_3} \chi^+_{K_3} i\frac{\partial}{\partial t}\chi_{K_3} + \mathcal{L}_{\mathrm{rot}} + \delta\mathcal{L}_{\mathrm{rot}} \tag{39}$$

where $\mathcal{L}_{\mathrm{rot}}$ is the rotational Lagrangian for the ground state, eq. (31).

Plugging eq. (38) into eq. (37) we obtain:

$$\delta\mathcal{L}_{\mathrm{rot}} = \sum_{K_3K_3'} \chi^+_{K_3'}\chi_{K_3}\Big[\langle KK_3JL|(\boldsymbol{\omega}\cdot\boldsymbol{j} + \boldsymbol{\Omega}\cdot\boldsymbol{t})|KK_3'JL\rangle +$$

$$+(\boldsymbol{\omega} - \boldsymbol{\Omega})\langle KK_3'JL|\frac{\partial \delta M}{\partial \boldsymbol{\omega}}|KK_3JL\rangle\Big] \tag{40}$$

We used here that the fluctuation of the meson field should depend only on the difference of flavor and space frequencies due to the hedgehog symmetry of the ground state: $\delta M \sim \boldsymbol{\omega} - \boldsymbol{\Omega}$.

One-quark flavor momentum $\boldsymbol{t}$ and angular momentum $\boldsymbol{j}$ do not conserve in the hedgehog field. Nevertheless, since they transformed as vectors under simultaneous flavor and spin rotations their matrix elements should be proportional to the matrix elements of the conserved quantity — grand spin $\boldsymbol{K}$:

$$\langle KK_3jl|\boldsymbol{t}|KK_3'jl\rangle = a_K\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle,$$

$$\langle KK_3jl|\boldsymbol{j}|KK_3'jl\rangle = (1-a_K)\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle$$

$$\langle KK_3jl|\frac{\partial\delta M}{\partial\boldsymbol{\omega}}|KK_3'jl\rangle = \zeta_K\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle \quad (41)$$

(the second of these relations follows from $\boldsymbol{j}+\boldsymbol{t}=\boldsymbol{K}$) where a_K and ζ_K are some constants specific for given excited level. Using explicit expressions for wave functions of the levels with given K one can derive the following relation for a_K:

$$a_K = \frac{K+1-c_K(2K+1)}{2K(K+1)},$$

$$c_K = \frac{\int dr\, r^2\left(h^2(r)+j^2(r)\right)}{\int dr\, r^2\left(h^2(r)+j^2(r)+g^2(r)+f^2(r)\right)} \quad (42)$$

where h, j, f, g are radial wave functions — solutions of the Dirac equation (8-11). Coefficient c_K $(0 < c_K < 1)$ is measuring the admixture of the state $j = K+\frac{1}{2}$ in the wave function of the level with given K (complete wave function consists of two states $j = K\pm\frac{1}{2}$) (see Appendix C).

In general case it is not possible to calculate the coefficient ζ_K. In particular, it depends on the form of the meson Lagrangian. The coefficient ζ_K renormalizes the value of a_K. Fortunately, the correction to the mean field δM is zero in the wide class of theories.

Collecting all terms we obtain the collective Lagrangian for 1-quark excitations in sector of u, d-quarks:

$$\mathcal{L}_K[\chi,R,S] = \sum_{K_3}\chi^+_{K_3} i\frac{\partial\chi_{K_3}}{\partial t} + \frac{N_c}{2\sqrt{3}}\tilde{\Omega}_8 +$$

$$+[(1-\tilde{a}_K)\boldsymbol{\omega}+\tilde{a}_K\boldsymbol{\Omega}]\sum_{K_3K_3'}\chi^+_{K_3}\chi_{K_3'}\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle +$$

$$+\sum_{i=1}^{3}\frac{I_1}{2}(\tilde{\Omega}_i-\tilde{\omega}_i)^2 + \sum_{a=4}^{7}\frac{I_2}{2}\tilde{\Omega}_a^2, \qquad \tilde{a}_K = a_K - \zeta \quad (43)$$

Quantization of χ_{K_3} with Lagrangian eq. (43) is trivial. Due to the presence of collective variable χ_{K_3} the quantity:

$$\sum_{K_3K_3'}\chi^+_{K_3}\chi_{K_3'}\langle KK_3jl|\boldsymbol{K}|KK_3'jl\rangle = \hat{\boldsymbol{K}} \quad (44)$$

behaves as quantum operator of the angular momentum K. Differentiating in ω, Ω we obtain the momenta in the body fixed frame:

$$\tilde{\boldsymbol{J}} = I_1(\boldsymbol{\omega}-\boldsymbol{\Omega}) + (1-\tilde{a}_K)\hat{\boldsymbol{K}}, \qquad \tilde{\boldsymbol{T}} = I_1(\boldsymbol{\Omega}-\boldsymbol{\omega}) + \tilde{a}_K\hat{\boldsymbol{K}}$$

$$\tilde{T}_a = I_2\tilde{\Omega}_a,\ (a=4\ldots8), \qquad \tilde{T}_8 = \frac{N_c}{2\sqrt{3}} \quad (45)$$

It leads to the following quantization conditions instead of eq. (33):

$$\tilde{\boldsymbol{T}}+\tilde{\boldsymbol{J}} = \hat{\boldsymbol{K}}, \qquad \tilde{Y} = \frac{N_c}{3} \quad (46)$$

Constructing now the Hamiltonian from the Lagrangian eq. (43) we obtain:

$$\mathcal{H}_K = \frac{1}{2I_2}\sum_{a=4}^{7}(\tilde{T}_a)^2 + \frac{(\tilde{\boldsymbol{T}}-\tilde{a}_K\hat{\boldsymbol{K}})^2}{2I_1} =$$

$$= \frac{1}{2I_2}\sum_{a=4}^{7}(\tilde{T}_a)^2 + \frac{(\tilde{\boldsymbol{T}}-\tilde{a}_K(\tilde{\boldsymbol{J}}+\tilde{\boldsymbol{T}}))^2}{2I_1} \quad (47)$$

Energy levels are:

$$\mathcal{E}_K = \frac{c_2(r)-\tilde{T}(\tilde{T}+1)-\frac{3}{4}\tilde{Y}^2}{2I_2} + \frac{1}{2I_1}\left[\tilde{a}_K J(J+1)+\right.$$

$$\left.+(1-\tilde{a}_K)\tilde{T}(\tilde{T}+1)-\tilde{a}_K(1-\tilde{a}_K)K(K+1)\right] \quad (48)$$

We used here that $\tilde{J} = J$. Available spins are determined by the quantization rule eq. (46): $J = |\tilde{T}-K|\ldots\tilde{T}+K$.

It is easy to construct the collective wave function. For this case it depends on $\mathcal{S}$, R and χ_{K_3}:

$$\Psi_K(R,\mathcal{S},\chi) = \sqrt{\frac{\dim(r)(2J+1)}{2K+1}}\times$$

$$\times\sum_{\tilde{T},\tilde{T}_3,\tilde{J}_3} C^{KK_3}_{\tilde{T}\tilde{T}_3\,J\tilde{J}_3}\mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}(R^+)\mathcal{D}^{J}_{\tilde{J}_3;J_3}(\mathcal{S}^+)\chi_{K_3} \quad (49)$$

This wave function is an eigenfunction of hypercharge Y, isospin T and its projection T_3 as well as spin J and its projection J_3. In fact, it is completely fixed by the symmetry and quantization requirements eq. (46).

C. s-quark excited states

At last let us describe excitations in the sector of s-quarks. Let us assume that we consider the 1-quark excitation where one quark is taken from ground state $K=0$ and put to the level for s-quark with some total angular momentum S. Excited state is $2S+1$ fold degenerate, we take the mixture

$$|\text{excited}\rangle = \sum_{S_3} \chi_{S_3} |S_3\rangle \tag{50}$$

where $|S_3\rangle$ are one quark wave functions with different projections of $\boldsymbol{S}$. The calculation of matrix elements gives now instead of eq. (41)

$$\langle S_3|\boldsymbol{j}|S_3'\rangle = \langle S_3|\boldsymbol{S}|S_3'\rangle, \qquad \langle S_3|\frac{\partial \delta M}{\partial \boldsymbol{\omega}}|S_3'\rangle = \zeta_S \langle S_3|\boldsymbol{S}|S_3'\rangle \tag{51}$$

and matrix elements of $\boldsymbol{t}$ are zero as s-quark does not carry isospin. Thus these matrix elements coincide with eq. (41) for $a_K = 0$. The subsequent steps are straightforward. The quantization rule eq. (46) change to

$$\tilde{\boldsymbol{T}} + \tilde{\boldsymbol{J}} = \hat{\boldsymbol{S}}, \qquad \tilde{Y} = \frac{N_c - 3}{3} \tag{52}$$

The first rule repeats eq. (46) with evident replacement $\boldsymbol{K} \to \boldsymbol{S}$. The second rule appears because we substitute the quark with hypercharge 1/3 (one of u, d-quarks in the ground state) by one s-quark (on excited level) with hypercharge $-2/3$. This rule can be also derived directly by calculating coefficient in front of the Wess-Zumino-Witten term.

Levels of energy for s-quark excitations are given by

$$\mathcal{E}_S = \frac{c_2(r) - \tilde{T}(\tilde{T}+1) - \frac{3}{4}\tilde{Y}^2}{2I_2} + \frac{1}{2I_1}\Big[(1+\zeta_S)\tilde{T}(\tilde{T}+1) - $$

$$-\zeta_S J(J+1) + \zeta_S(1+\zeta_S)S(S+1)\Big] \tag{53}$$

which is eq. (48) with substitution $K \to S$ and $a_K = 0$. Available spins are $J = |\tilde{T} - S| \ldots \tilde{T} + S$; collective wave function is analogue of eq. (49):

$$\Psi_S(R, \mathcal{S}, \chi) = \sqrt{\frac{\dim(r)(2J+1)}{2S+1}} \times$$

$$\times \sum_{\tilde{T}, \tilde{T}_3, S_3} C^{SS_3}_{\tilde{T}\tilde{T}_3\,J\tilde{J}_3} \mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}(R^+) \mathcal{D}^{J}_{\tilde{J}_3;J_3}(\mathcal{S}^+) \chi_{S_3} \tag{54}$$

To summarize: rotational bands around the given excited intrinsic energy should be constructed in the following way. One has to choose $SU(3)$-multiplets which contain states obeying quantization rule for $\tilde{Y}$, read off the corresponding to this $\tilde{Y}$ the value of $\tilde{T}$ and use formulae (34), (48), (53) for their rotational energy.

Quark wave functions in the mean field approximation are the product of one-particle wave functions of the filled levels. One has to rotate them according to eq. (18) and then project to collective wave functions obtained above (see eq. (36), eq. (49), eq. (54)). "Projection" means that one has to multiply rotated quark wave function by conjugated collective wave function and integrate in matrices R and S. This will produce quark wave functions of the excited baryons with given quantum numbers.

V. ONE QUARK EXCITATIONS IN THE MEAN FIELD AND THE QUARK MODEL

In the limit of $N_c \to \infty$ the quark model becomes a particular case of the general soliton considered above. Indeed, the mean field approximation should work for the quark model as well, any inter-quark potential can be considered in the this approximation. As far as we are discussing only symmetry properties of the quark states, details of the potential are not important.

The real difference between the quark model and the picture considered above is a symmetry of the mean field. The quark model insists on the complete spherical symmetry and the only mean field in the quark model is the scalar field $S(\boldsymbol{x})$. The excitations of the quark model arise as $SU(6)$-multiplets (to be more precise, multiplets of the $SU(6) \otimes O(3)$, $O(3)$-group standing for orbital angular momentum). All splittings of such multiplets considered as a small perturbation. This should be confronted to the soliton approach where it is assumed that the mean field has symmetry of hedgehog and departure from the $SU(6)$ is not considered to be small, it is of order of unity even at large N_c. Nevertheless we should be able to reproduce multiplets of the quark model taking the spherically symmetrical mean field.

Due to the Witten quantization rule the $SU(3)$-multiplets for large N_c becomes large as well. The classification of such multiplets was developed in [18].

Every $SU(3)$-multiplet is characterized by two numbers p and q. However, this is inconvenient for our purposes. Let us label multiplets by i) $\tilde{Y}$ — what Witten condition is fulfilled by this multiplet, ii) $\tilde{T}$ — what intrinsic isospin corresponds to this value and iii) *exoticness* X which is defined as a minimal number of quark-antiquark pairs in the wave function of the given baryon. Standard p and q-numbers, dimension of the multiplet and Casimir operator can be expressed in terms of these numbers as follows:

$$p = 2\tilde{T} - X \geq 0, \qquad q = \frac{3}{2}\tilde{Y} + 2X - \tilde{T} \geq 0 \tag{55}$$

and

$$\dim = \frac{2\tilde{T}+1-X}{2}\left(\frac{3}{2}\tilde{Y}+1+2X-\tilde{T}\right)\left(\frac{3}{2}\tilde{Y}+2+X+\tilde{T}\right),$$

$$c_2(r) = \frac{3}{4}\tilde{Y}(\tilde{Y}+2) + \tilde{T}(\tilde{T}+1) + X\left(\frac{3}{2}\tilde{Y}+1-\tilde{T}\right) + X^2 \tag{56}$$

Plugging these expressions into the general formula for the energy eq. (48) we arrive at:

$$\mathcal{M}_K = \mathcal{M}_0 + \Delta\mathcal{E} + \frac{1}{2I_1}\Big[\tilde{a}_K J(J+1) + (1-\tilde{a}_K)\tilde{T}(\tilde{T}+1)$$

$$-\tilde{a}_K(1-\tilde{a}_K)K(K+1)\Big] + \frac{(1+X)(2+3\tilde{Y})}{2I_2} \tag{57}$$

We see that I_2 plays the role of the moment of inertia for exotic states; their spectrum is equidistant and distances between states are of order unity (we remind that $I_2 \sim N_c$ and $\tilde{Y} \sim N_c$). Moment of inertia I_1 governs ordinary excitations splitting [18]. We will not consider the *rotational* exotics here, exotic states have some specifics related to the fact that their widths are $\sim O(1)$ [5, 44]. Anyway, these states are separated from the normal rotational band by the interval $\sim O(1)$.

Every excited state has a restricted number of *non-exotic* states entering the rotational band with definite $\tilde{T}$. They are determined from the condition that $p \geq 0$ and $q \geq 0$. In particular, for excitations in sector of s-quarks at $N_c = 3, \tilde{Y} = 0$ we get only one state — singlet with spins $J = S \pm 1/2$ (where S is the spin of excited states) and other multiplets are exotic. At larger N_c s-quark excitations form non-trivial rotational band with energies given by eq. (57) and $\tilde{a}_K = -\zeta_s$ (see eq. (53)).

Excitations in sectors of u, d-quarks have richer structure. For non-exotic states ($X = 0$) at $N_c = 3$ and $\tilde{Y} = 1$ it follows from eq. (55) that we have only two possibilities: $\tilde{T} = \frac{1}{2}$ and $\tilde{T} = \frac{3}{2}$. In other words they can come only as octets or decuplets. At larger N_c other multiplets become non-exotic. At $K = 0$ we obtain the rotational band of $J = \tilde{T}$ with different spins changing in the limits $\frac{1}{2} < J < \frac{N_c}{2}$. Their energies are given by general formula eq. (57) with $X = 0$, $\tilde{a}_K = 0$.

At $K = 1$ we have 3 possible series of rotational bands $J = \tilde{T} - 1, \tilde{T}, \tilde{T} + 1$ and at $K = 2$ there are 5 series with $J = \tilde{T} - 2, \ldots \tilde{T} + 2$. Possible spins are determined again from eq. (55).

Let us confront this picture to the quark model (see, e.g.,[29]). The lowest state of the quark model is 56-plet with orbital moment $L = 0$. This multiplet can be generalized to arbitrary N_c, its dimension is

$$\dim_{'56'} = \frac{1}{120}(N_c+1)(N_c+2)(N_c+3)(N_c+4)(N_c+5),$$

$$C_2(SU(6)) = \frac{5}{12}N_c(N_c+6) \tag{58}$$

The analogue of '56' in the mean field picture is rotational band around ground state baryons. It is easy to check that dimension of the full rotational band of ground state baryons:

$$\sum_{J=\frac{1}{2}}^{N_c} (2J+1)\dim(J, X=0, \tilde{Y} = \frac{N_c}{3}) = \dim_{'56'}$$

In other words rotational band around ground state completely coincides with '56'-plet. At $N_c = 3$ it reduces to the familiar $(\mathbf{56}) = (\mathbf{8}\tfrac{\mathbf{1}}{\mathbf{2}}) \oplus (\mathbf{10}\tfrac{\mathbf{3}}{\mathbf{2}})$.

The next $SU(6)$ multiplet is **70**. Its dimension at arbitrary N_c is equal to:

$$\dim_{'70'} = \frac{1}{24}(N_c-1)(N_c+1)(N_c+2)(N_c+3)(N_c+4),$$

$$C_2(SU(6)) = \frac{1}{12}N_c(5N_c+18) \tag{59}$$

It consists of the following 5 series of $SU(3)$ multiplets:

1. 3 series of multiplets with $\tilde{Y} = N_c/3$: $\tilde{T} = J - 1$ with $J = \frac{3}{2} \ldots \frac{N_c}{2}$, $\tilde{T} = J$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 1$ and $\tilde{T} = J + 1$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 1$.

2. 2 series of multiplets with $\tilde{Y} = (N_c - 3)/3$: $\tilde{T} = J - \frac{1}{2}$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 1$ and $\tilde{T} = J + \frac{1}{2}$ with $J = \frac{1}{2} \ldots \frac{N_c}{2} - 2$

It is easy to check using eq. (56) that the total dimension of all 5 series is equal to $\dim_{'70'}$.

The contents of the $SU(3)$-multiplets entering '70'-plet implies that in the mean field picture it consists of one-quark excitation in sector of s-quark with spin 1/2 and $K = 1$ excitation in the sector of u, d-quarks. These states become degenerate in the case of spherically symmetrical mean field. However, there are *more* states in the mean field approximation: in this approximation *all* states with $\frac{1}{2} \leq \tilde{T} \leq \frac{N_c}{2}$ are present. We will see that absent states are spurious: they are forbidden by the Pauli principle which is not accounted in the mean field approximation.

'70'-plet is used in the quark model, e.g., in order to describe baryons with negative parity [29]. It is assumed that these baryons are described the representation $(\mathbf{70}, \mathbf{3})$ representation of the $SU(6) \otimes O(3)$: the baryons have relative orbital angular moment of two quarks $L = 1$. This provides negative parity and makes the total baryon wave function symmetrical (or antisymmetrical if one accounts for color) under simultaneous exchange of spin, flavor indices ($SU(6)$) and coordinates. Adding $L = 1$ to the mean field multiplets obtained above we see that these baryons should be described by $K = 0, 1, 2$ excitations in the sector of u, d-quarks and $S = \frac{1}{2}, \frac{3}{2}$ excitations in the sector of s-quarks and their rotational bands. The difference with quark model is that these 5 sets of states are splitted by $O(1)$ owing to the hedgehog (not spherical) symmetry of the mean field from the very beginning.

Other multiplets of the quark model can be also analyzed in the same way and one can find their interpretation in terms of states appearing in the mean field approximation.

Presented above series of five $SU(3)$ multiplets, which become at large N_c '70'-plet of $SU(6)$, were observed first in remarkable paper [32]. In this paper the approach close in spirit to approach of Manohar et al [27]. was used. Additionally, Authors of Ref.[32] claimed a relation between masses of multiplets with different grand spin K which in our notation reads:

$$\Delta\mathcal{E}(0 \to 0) + 3\Delta\mathcal{E}(0 \to 1) + 5\Delta\mathcal{E}(0 \to 2) = 0 \tag{60}$$

This relation looks surprisingly since it cannot be fulfilled for an arbitrary external field. In particular, it is not fulfilled in the exactly solvable model which we consider in Appendix A.

Situation in the sector with positive parity is completely different. The quark model prediction for excited baryons in this sector consists of five $SU(6)\otimes O(3)$ multiplets[30]: $(\mathbf{56},\mathbf{0})$ (radial excitation of the ground multiplet), $(\mathbf{56},\mathbf{2})$, $(\mathbf{70},\mathbf{0})$, $(\mathbf{70},\mathbf{2})$ and $(\mathbf{20},\mathbf{1})$. All these multiplets should be nearly degenerate in the quark model (they are degenerate in the oscillator limit of the model [30]). The vast majority of the states in these multiplets is not observed in nature.

Contrary to the situation with negative parity baryons, some quark model multiplets in the parity-plus sector have no interpretation as 1-quark excitations in the mean field at $N_c\to\infty$. Instead they correspond to, at least, two-quark excitations when 2 quarks from the ground level are transferred to the excited levels (possibly with different K). It is especially clear for $(\mathbf{20{,}1})$ multiplet where $SU(6)$ wave function is completely antisymmetric. This multiplet cannot fit the picture where N_c-1 quarks are sitting on the same level (and thus have completely symmetric wave function) and only last quark carries angular momentum (to be more precise, non-zero K). The same statement is true for $(\mathbf{70},\mathbf{0})$ and $(\mathbf{70},\mathbf{2})$ whose also have mixed symmetry. The 1-quark excited multiple should be symmetric in the $SU(6)$ index of the excited quark and hence antisymmetric in the first two sitting in the ground state.

One can consider the two-particle excitations in the mean field at $N_c\to\infty$ as well, keeping in mind, however, that the **70** and **20** multiplets should be generalized to large N_c in different way than it was done for the negative parity baryons. We will not come in details but we mention that '20'-plet at arbitrary N_c has dimension $\frac{1}{12}(N_c-2)(N_c-1)(N_c+1)(N_c+2)(N_c+3)$ and corresponds to 10 series of $SU(3)$-multiplets with different spins ($SU(6)$ Casimir operator is $N_c(5N_c+6)/12$). Dimension of two-quark excited '70'-plet with positive parity is $\frac{1}{3}(N_c-2)(N_c-1)N_c(N_c+2)(N_c+4)$ and it contains 40 series of $SU(3)$-multiplets ($SU(6)$ Casimir operator is $3+N_c(5N_c+6)/12$). Of course, most of the states are spurious at $N_c=3$. One can decompose these series to the state corresponding to different K and check that they are, indeed, 2-quark excitations (e.g. in the toy model considered in Appendix A).

However, in this paper we would like to insist that even positive parity baryons can be also constructed as one-quark excitations in the mean field. We believe that baryons with two quarks excited are much heavier and have larger width than one-quark excitations. In other words, we think that quark model is misleading for the positive parity excited baryons (in particular, $SU(6)$ group is completely broken) and its obvious success in the sector with negative parity is explained by the fact that negative parity baryons *are* 1-quark excitations. This point of view is supported by the the fact that significant part of the quark model states with positive parity was never observed. We will adopt the idea that they are described by the same set $K=0,1,2$ levels as for the negative parity baryons. One can easily construct the wave functions of these states directly and check that there are only one spurious multiplet among them (for more details, see next Section). Our classification, as was already said, does not correspond to the quark model.

Different situation arises if one puts $N_c=3$ from the very beginning. Then identification of one-particle and two-particle excitations becomes difficult. Indeed, after separation of the center of motion movement, the wave functions are not the product of one-particle states. Moreover, one can impose $SU(6)$ limit (central symmetrical scalar mean field) and then proceed to the oscillator limit of the interaction. Then all constructed $K=0,1,2$ states should fall in one of the five $SU(6)$ multiplets mentioned above (We remind that mean field approximation becomes exact for oscillator model). And they do: one can construct directly wave functions of these states with help of the method described at the end of the previous Section and confront them to the $SU(6)$ wave functions built in [30] (Appendix B). Then one can see that completely antisymmetric $SU(6)$-multiplet **20** remains 2-quark excitation even at $N_c=3$. The contents of $K=0$ and $K=2$ states is completely exhausted by $(\mathbf{56},\mathbf{0})$ and $(\mathbf{56},\mathbf{2})$ multiplets. At last, all one-quark $K=1$ states are inside the multiplets of mixed symmetry $(\mathbf{70},\mathbf{0})$ and $(\mathbf{70},\mathbf{2})$.

To summarize: one-particle excitations in the mean field at large N_c with lowest $K=0,1,2$ is only the small part of quark model multiplets for the positive parity. However, as we shall see in the next Section, it is this part which is observed in Nature (at least below 2 GeV).

VI. COMPARISON WITH THE EXPERIMENTAL SPECTRUM

We shall now look into the experimental spectrum of light baryon resonances up to around 2 GeV, trying to recognize among them the rotational bands about the one-quark excitations from the ground state to the intrinsic quark levels. We shall treat the quantities I_1, $\tilde{a}_K$ and $\Delta\mathcal{E}$ entering eq. (57) as free parameters to be fitted from the known masses, although in principle they are calculable if the (self-consistent) mean field is known. Still, there are much more resonances than free parameters, therefore the rotational bands are severely restricted by eq. (57) . As we shall see these restrictions are well supported by experimental observations, despite that in the real world N_c is only three.

Since at this time we do not take into account the splittings inside $SU(3)$ multiplets as due to the nonzero m_s, eq. (57) should be compared with the *centers* of multiplets. For the octet, the center is defined as $\mathcal{M}_8=(2m_N+2m_\Xi+3m_\Sigma+m_\Lambda)/8$, and for the decuplet it is $\mathcal{M}_{10}=(4m_\Delta+3m_{\Sigma^*}+2m_{\Xi^*}+m_\Omega)/10\approx m_{\Sigma^*}$. We take the phenomenological values for $\mathcal{M}_8$, $\mathcal{M}_{10}$ from the paper by Guzey and Polyakov [22] who have analyzed baryon multiplets up to 2 GeV.

A. Spurious states

Comparing the mean-field predictions (valid at large N_c) with the data, it should be kept in mind that certain rotational states are, in fact, spurious, as they are artifacts of the mean-field approximation where the spatial wave function is a product of one-particle wave functions. When averaging over the center of mass is taken into account (which is an $\mathcal{O}(1/N_c)$ effect) the baryon wave functions depend only on the differences of quark coordinates, which for some states may contradict the Pauli principle. This effect has been long known both in nuclear physics [28] and in the non-relativistic quark model [29]. The simplest way to identify spurious states is to continuously deform the mean field to the non-relativistic oscillator potential where the wave functions are explicit. Again, they can be written directly projecting rotated mean field quark wave functions with collective wave functions constructed here. If some state is absent in that limit, it cannot appear from a continuous deformation. An independent way to check for spurious states is to deform the problem at hand to the exactly solvable (0+1)-dimensional four-fermion interaction model [31] where the large-N_c approximation is also possible and reveals extra states (see Appendix A).

Specifically, in the parity-plus sector, the spurious state is $(\mathbf{10}, 1/2^+)$ arising from the rotational band about the $(0^+ \to 2^+)$ transition. Such state arises also from the $(0^+ \to 1^+)$ transition but then it is not spurious.

In the parity-minus sector there are more spurious states: the multiplets $(\mathbf{10}, 5/2^-)$ and $(\mathbf{10}, 7/2^-)$ stemming from the $(0^+ \to 2^-)$ transition are spurious, two out of three multiplets $(\mathbf{10}, 3/2^-)$ arising from $(0^+ \to 0^-, 1^-, 2^-)$ transitions are spurious, and one out of two multiplets $(\mathbf{10}, 1/2^-)$ stemming from $(0^+ \to 1^-, 2^-)$ transitions is spurious, too. As it was already said, remaining negative parity multiplets exactly coincide with octets and decuplets from $(\mathbf{70}, \mathbf{1})$ multiplet of $SU(6) \otimes O(3)$ of the quark model.

Spurious rotational states should be removed from the consideration.

B. Parity-plus resonances

The two lowest multiplets $(\mathbf{8}, 1/2^+, 1152)$ and $(\mathbf{10}, 3/2^+, 1382)$ (the last number in the parentheses is the center of the multiplet) form the rotational band about the ground-state filling scheme shown in Fig. 2. Fitting these masses by Eq. (57) we find $\mathcal{M}_0 + \frac{3}{4I_2} = 1090\,\mathrm{MeV}$, $1/I_1 = 153\,\mathrm{MeV}$.

Apart from the two lowest multiplets, there is another low-lying pair with the same quantum numbers, $(\mathbf{8}, 1/2^+, 1608)$ and $(\mathbf{10}, 3/2^+, 1732)$. Other parity-plus multiplets are essentially higher. One needs a $0^+ \to 0^+$ transition to explain this pair. Comparing the masses one finds that the second $K^P = 0^+$ intrinsic quark level must be 482 MeV higher than the ground state 0^+ level, $\Delta\mathcal{E}(0^+ \to 0^+) = 482\,\mathrm{MeV}$. The moment of inertia appears to be considerably larger than for the ground-state multiplets, $1/I_1 = 83\,\mathrm{MeV}$. Although the difference is an $\mathcal{O}(1/N_c)$ effect, it may be enhanced if the radially excited 0^+ level has a much larger effective radius.

There is a group of five multiplets, $(\mathbf{8}, 3/2^+, 1865)$, $(\mathbf{8}, 5/2^+, 1873)$, $(\mathbf{10}, 3/2^+, 2087)$, $(\mathbf{10}, 5/2^+, 2071)$, $(\mathbf{10}, 7/2^+, 2038)$ that are good candidates for the rotational band around the $0^+ \to 2^+$ transition. Indeed, this is precisely the content of the rotational band for this transition (the spurious multiplet $(\mathbf{10}, 1/2^+)$ excluded), and a fit to the masses according to eq. (57) gives a small $\sqrt{\chi^2} = 15\,\mathrm{MeV}$. It should be kept in mind, however, that not all members of all multiplets are well established [22], and those that are, have an experimental uncertainty in the masses. It means that the 'experimental' masses for the centers of multiplets are known at best to an accuracy of 20-40 MeV. We find from the fit $1/I_1 = 131\,\mathrm{MeV}$, $\Delta\mathcal{E}(0^+ \to 2^+) = 722\,\mathrm{MeV}$. Therefore, the intrinsic 2^+ level must be higher than the 0^+ one.

The only relatively well established multiplet that is left in the range below 2 GeV, is $(\mathbf{8}, 1/2^+, 1846)$. It prompts that it can arise from the rotational band about the $0^+ \to 1^+$ transition, however, other parts of the band are poorly known. If one looks into non-strange baryons that are left, one finds $N(1/2^+, 1710^{***})$, $N(1/2^+, 1900^{**})$, $\Delta(1/2^+, 1910^{***})$ and $\Delta(5/2^+, 2000^{**})$, with $\Delta(3/2^+)$ missing. The quantum numbers and the masses of these supposed resonances fit rather well the hypothesis that they arise as a rotational band about the $0^+ \to 1^+$ transition, however, their low status prevents a definite conclusion. The intrinsic 1^+ level must be approximately 60 MeV higher than the 2^+ quark level.

C. Parity-minus resonances

The situation here is similar to the parity-plus sector: one needs intrinsic quark levels with $K^P = 0^-, 1^-, 2^-$ to explain the resonances as belonging to rotational bands about these transitions. Given that several rotational states in the parity-minus sector are spurious, one expects to find the following multiplets stemming from these transitions: $(\mathbf{8}, 1/2^-) \times 2$, $(\mathbf{8}, 3/2^-) \times 2$, $(\mathbf{8}, 5/2^-)$, $(\mathbf{10}, 1/2^-)$, $(\mathbf{10}, 3/2^-)$: these are precisely the observed multiplets.

We know that all remaining multiplet are spurious but we do not know the way to assign specific K to the observed one. We attribute them according to eq. (57) requiring that no mixing can happen ($\zeta_K = 0$). There is only one way to do this.

We assign 4 lowest multiplets $(\mathbf{8}, 1/2^-, 1592)$, $(\mathbf{8}, 3/2^-, 1673)$, $(\mathbf{10}, 1/2^-, 1758)$ and $(\mathbf{10}, 3/2^-, 1850)$ to the rotational band of $K = 1^-$ level. The fit tells that corresponding moment of inertia is $1/I_1 = 171\,\mathrm{MeV}$ and energy of the level $\Delta\mathcal{E}(0^+ \to 1^-) = 468\,\mathrm{MeV}$ is close to $\Delta\mathcal{E}(0^+ \to 0^+)$. This does not look impossible.

The multiplet $(\mathbf{8}, 1/2^-, 1716)$ should be ascribed as $0^+ \to 0^-$ transition and two remaining multiplets $(\mathbf{8}, 3/2^-, 1896)$ and $(\mathbf{8}, 5/2^-, 1801)$ to $0^+ \to 2^-$ transition.

These assignments produce reasonable values of mixing coefficients $\tilde{a}_K$ which can be explained without mixing of rotations and other degrees of freedom in the effective meson Lagrangian. Probably, some other information(mass splittings or resonance widths) should be used to fix finally the attribution of multiplets to the rotational bands. If the final scheme would be different from assumed here, it will witness the large role of other than pion mesons in formation of negative parity baryons.

To summarize, all parity-plus and parity-minus baryons around 2 GeV and below can be accommodated by the scheme, assuming they all arise as rotational excitations about the $0^+ \to 0^+, 1^+, 2^+$ and $0^+ \to 0^-, 1^-, 2^-$ transitions, see Table 1. There are no unexplained resonances left, but there appears an extra state $\Delta(3/2^+, \sim 1945)$ stemming from the $0^+ \to 1^+$ transition, which is so far unobserved, so this state is a prediction.

D. *s* quarks

As emphasized in Section 3, s quarks are in a completely different external field than u, d quarks, even in the chiral limit. Only the confining forces which we model by a linear rising scalar field are the same for all quarks. The two excited levels for s quarks are shown in Fig. 3: they are needed to explain the singlet $\Lambda(1/2^-, 1405)$ and $\Lambda(3/2^-, 1520)$ resonances. The corresponding values of $\Delta\mathcal{E}$ presented in the Table 1. No more singlet Λ's are known below 2 GeV, therefore there should be no intrinsic s-quark levels either with positive or negative parity in this range.

Following the standard logic of the quark model we describe in this paper all baryon resonances as excitations of *valence quarks* (see Fig.2). This is not necessary, however. New resonances can appear due to transitions from the levels which belong to the Dirac continuum. The main configuration for such baryons will consist of 5 quarks (3 valence quarks plus quark-antiquark pair) but this does not mean that they should be exotic. Just opposite, most of them would be ordinary octets and decuplets.

For example, the intriguing question is where is the highest *filled* level of s quarks? Presumably, it must be a level with quantum numbers $J^P = \frac{1}{2}^+$ as possessing maximal symmetry. There can be one-quark excitations from that level both to the s-quark excited levels $\frac{1}{2}^-$ and $\frac{3}{2}^-$, and to the u, d excited levels $0^+, 0^-, \ldots$, see Fig.3. Transitions of the first type generate a rotational band consisting of $(\mathbf{8}, 1/2^-) \times 2$, $(\mathbf{8}, 1/2^-)$, $(\mathbf{10}, 1/2^-)$, $(\mathbf{10}, 3/2^-) \times 2$ and $(\mathbf{10}, 5/2^-)$. Transitions of the second type called, in the terminology of nuclear physics, Gamov-Teller transitions, generate the exotic *antidecuplet* $(\overline{\mathbf{10}}, 1/2^+)$, etc. [14] . It turns out that it is difficult, if not impossible, to move the highest filled level of s quarks $\frac{1}{2}^+$, which must satisfy Eq. (6) in a 'realistic' mean field, more than $\sim 700\,\mathrm{MeV}$ below the first excited level $\frac{1}{2}^-$. Therefore, the parity-minus resonances generated by the transition *1* in Fig. 4 must reveal themselves in the spectrum below 2 GeV. We note that such resonances will have a substantial 5-quark component $u(d)u(d)u(d)s\bar{s}$ since they require an s quark to be pulled out of the filled level and put onto an excited level. Probably, the real-world resonances are certain mixtures of these excitations with the u, d excitations described in the previous section. This is a welcome feature as, for example, the well-known resonance $N(1/2^-, 1535)$ has a surprisingly large coupling to the η meson (see also [32, 33]). The 'Gamov-Teller' transition *2* gives a natural explanation of the exotic Θ^+ resonance [34] exactly at the position where it has been claimed by a number of experiments [14] .

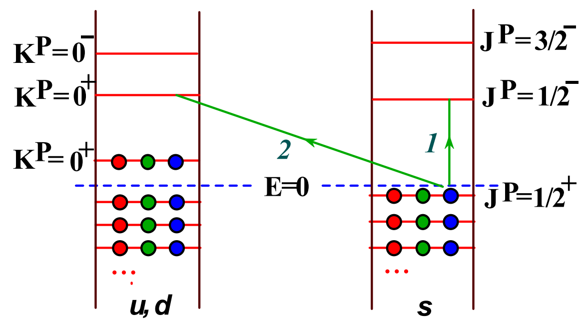

FIG. 3. (Color online) Possible transitions of the s quark from the highest filled s level to excited s levels (*1*), and to excited u, d levels (*2*).

VII. MASS SPLITTINGS

The non-zero mass of the strange quark m_s breaks down $SU(3)$ flavor group and splits $SU(3)$ multiplets. Let us calculate these splittings. Inserting quark mass m (a matrix in flavor) into the quark determinant eq. (21) and expanding up to the first order in m we obtain

$$\delta_m S = -i \sum_c \mathrm{Sp}_{\mathrm{occupied}} \Big\{ R^+ m R \gamma^0 \times$$

$$\times \frac{1}{i\frac{\partial}{\partial t} - \mathcal{H}(M + \delta M) - \tilde{\Omega} t_a - \tilde{\omega}_i j_i} \Big\} \qquad (61)$$

Mass of strange quark has both singlet and octet part $m = m_0\mathbf{1} + m_8\lambda_8$ and $m_8 = m_s/\sqrt{3}$ and splittings are determined only by octet part m_8.

We want to calculate mass splitting in the zeroth and the first order in angular and flavor frequencies $\tilde\omega$ and $\tilde\Omega$. For ground state baryons this calculation was carried out in many papers (see, e.g. [39]). Result reads

$$\delta_m S = \frac{m_s}{\sqrt{3}}\int dt \left[\sum_a \sum_{\rm occup} \mathcal{D}^{(8)}_{8a}(R)\langle n|\lambda_a\gamma^0|n\rangle + \right.$$

$$\left. +2K_1\sum_{i=1}^{3}\mathcal{D}^{(8)}_{8i}(R)(\tilde\Omega_i - \tilde\omega_i) + 2K_2\sum_{a=4}^{7}\mathcal{D}^{(8)}_{8a}(R)\tilde\Omega_a\right] \tag{62}$$

Here the first term is of the zeroth order in frequencies, the second and the third terms represent the first order corrections. K_1 and K_2 are some constants analogous to the moments of inertia eq. (28) (again n runs over occupied levels and m runs over free levels in the mean field):

$$K_{ab} = N_c\sum_{n,m}\frac{\langle n|\gamma^0 t_a|m\rangle\langle m|\gamma^0 t_b|n\rangle + \langle n|\gamma^0 t_b|m\rangle\langle m|\gamma^0 t_a|n\rangle}{\varepsilon_m - \varepsilon_n} \tag{63}$$

(if there is a mixing of rotations with δM this expression should be modified). Tensor K_{ab} has a structure analogous to the structure of the moment of inertia

$$K_{ab} = \begin{cases} K_1\delta_{ab} & a,b = 1\ldots 3 \\ K_2\delta_{ab} & a,b = 4\ldots 7 \\ 0, & a = b = 8 \end{cases} \tag{64}$$

Expression (62) is valid for rotational bands above the ground state and one-particle excitations. First term for ground state baryons is non-zero only at $a = 8$, it can be expressed through the experimentally known quantity — so-called Σ-term:

$$\sum_c \sum_{\rm occup}\langle n|\lambda_8\gamma^0|n\rangle = \frac{1}{3}\frac{m_s}{m_u + m_d}\Sigma,$$

$$\Sigma = (m_u + m_d)\frac{\partial\mathcal{M}}{\partial(m_u + m_d)} \tag{65}$$

It was used here that all valence levels are located in the sector of u,d-quarks. We will imply below only this case. Indeed, we have seen that at $N_c = 3$ one-particle excitations in the sector of s-quarks (from the ground state) are singlets, so there is no mass splitting present for this type of excitations.

In the sector of u,d-quarks there is also another possibility $a = 1,2,3$ for excited level:

$$\langle {\rm excited}|\frac{\lambda_i}{2}\gamma^0|{\rm excited}\rangle = d_K\sum_{K_3,K_3'}\chi^+_{K_3'}\chi_{K_3}\langle K_3'|K_i|K_3\rangle, \tag{66}$$

where d_K is some constant which is determined by wave function

$$d_K = \pm\int dr r^2\left[\frac{g^2(r) - f^2(r)}{2K} - \frac{h^2(r) - j^2(r)}{2(K+1)}\right] \tag{67}$$

where plus sign stands for the states with parity $(-1)^K$ and minus for states with parity $(-1)^{K+1}$. Calculation of d_K is analogous to the calculation of coefficient c_K (see above).

First order terms in frequencies in eq. (62) can be simplified as well. We substitute frequencies by operators $\tilde T_a$ according to eq. (45). Using relation:

$$T_8 = \sum_{a=1}^{8}\mathcal{D}^{(8)}_{8i}(R)\tilde T_a$$

one can express the last term in eq. (62) in terms of the sum with $a = 1,2,3$ and hypercharge $Y = 2T_8/\sqrt{3}$. Proceeding to the Hamiltonian

$$\mathcal{H}_m = \alpha\mathcal{D}^{(8)}_{88}(R) + \beta Y + \sqrt{3}\gamma\sum_{i=1}^{3}\mathcal{D}^{(8)}_{8i}(R)\tilde T_i +$$

$$+\sqrt{3}\delta\sum_{i=1}^{3}\mathcal{D}^{(8)}_{8i}(R)\hat K_i \tag{68}$$

where

$$\alpha = -\frac{2}{3}\frac{m_s}{m_u + m_d}\Sigma + m_s\frac{K_2}{I_2}, \quad \beta = -m_s\frac{K_2}{I_2},$$

$$\gamma = \frac{2m_s}{3}\left(\frac{K_1}{I_1} - \frac{K_2}{I_2}\right), \quad \delta = \frac{2m_s}{3}\left(d_K - \frac{K_1}{I_1}\tilde a_K\right) \tag{69}$$

We see that mass splittings are determined by four possible structures. Only the last term is novel, other three are known for ground state baryons. Moreover, constants α,β,γ up to corrections of order $1/N_c$ are the same for all levels. As to δ it is determined by the properties of the excited level and is individual for given level. Nevertheless, δ is the same for all rotational band of the given level Note also that $\alpha \sim O(N_c)$ while $\beta,\gamma,\delta \sim O(1)$.

Mass splittings are determined by the average of the Hamiltonian eq. (68) in collective wave functions eq. (36) and eq. (49). Resulting expressions, of course, respect Gell-Mann-Okubo formula. We parameterize the masses of particles in the octet as:

$$\mathcal{M}_N = M_8 - \frac{7}{4}\mu_1^{(8)} - \mu_2^{(8)}, \quad \mathcal{M}_\Lambda = M_8 - \mu_1^{(8)},$$

$$\mathcal{M}_\Sigma = M_8 + \mu_1^{(8)} \quad \mathcal{M}_\Xi = M_8 + \frac{3}{4}\mu_1^{(8)} + \mu_2^{(8)} \tag{70}$$

and masses of decuplet particles as

$$\mathcal{M}_\Delta = M_{10} - \mu^{(10)}, \quad \mathcal{M}_\Sigma = M_{10}, \quad \mathcal{M}_\Xi = M_{10} + \mu^{(10)}$$

$$\mathcal{M}_\Omega = M_{10} + 2\mu^{(10)} \tag{71}$$

This parametrization obeys Gell-Mann-Okubo formula automatically. In the Table 2 we give the values of μ in terms of $\alpha, \beta, \gamma, \delta$ for different values of K and the spin of the multiplet J.

Expressions for mass splittings give rise to the number of relations between masses of the particles entering the same rotational band. These relations are similar to well-known Guadagnini relation which is valid for the ground state octet and decuplet (see, e.g., [34]). Let us itemize these relations for $K = 2$ rotational band of the baryons with positive parity, for octets and decuplets:

$$5\mu_2^{(8)}\left(\frac{3}{2}\right) + 9\mu^{(10)}\left(\frac{5}{2}\right) = 14\mu^{(10)}\left(\frac{3}{2}\right),$$

$$5\mu_2^{(8)}\left(\frac{5}{2}\right) + 11\mu^{(10)}\left(\frac{3}{2}\right) = 16\mu^{(10)}\left(\frac{5}{2}\right), \tag{72}$$

and for decuplets only

$$5\mu^{(10)}\left(\frac{7}{2}\right) + 7\mu^{(10)}\left(\frac{3}{2}\right) = 12\mu^{(10)}\left(\frac{5}{2}\right),$$

$$3\mu^{(10)}\left(\frac{5}{2}\right) + 5\mu^{(10)}\left(\frac{1}{2}\right) = 8\mu^{(10)}\left(\frac{3}{2}\right) \tag{73}$$

(we put in parenthesis the spin of the particles). All these relations work with accuracy better than 10% and some even with accuracy 1-2%.

For $K = 1$ (negative parity) we get two relations:

$$7\mu^{(10)}\left(\frac{1}{2}\right) + 3\mu_2^{(8)}\left(\frac{3}{2}\right) = 10\mu^{(10)}\left(\frac{3}{2}\right),$$

$$5\mu^{(10)}\left(\frac{3}{2}\right) + 3\mu_2^{(8)}\left(\frac{1}{2}\right) = 8\mu^{(10)}\left(\frac{1}{2}\right) \tag{74}$$

While the first is fulfilled with accuracy 2%, the second is fulfilled only at 10% level.

Last relation for $K = 0$ (which is precisely Guadagnini's one but for excited baryons) reads: $\mu^{(10)}\left(\frac{3}{2}\right) = \mu_2^{(8)}\left(\frac{1}{2}\right)$. This relation which is working rather good for ground state octet and decuplet is broken surprisingly strong for $K = 0^+$ excited state.

The situation changes in the strict limit $N_c \to \infty$, in the approach advocated in [26]. According to the last approach one should consider Clebsch-Gordan coefficients in the same limit $N_c \to \infty$. The required isoscalar factors are collected in Appendix E, so calculations are straightforward.

The recalculated results demonstrate different N_c counting. It appears that mass splittings are not $O(m_s N_c)$ but only $O(m_s)$. Both constants α and β enter the leading term, while γ and δ appear in corrections $O(m_s/N_c)$. Probably, this picture is more satisfactory from the general point of view. Let us note that it coincides with N_c counting developed in [26, 27] and all derived there mass relations are also automatically fulfilled.

Gell-Mann Okubo relations appear to be still valid. This is not trivial, especially for "decuplets", where not one but two final states at arbitrary N_c are available (so it is possible to talk about F- and D-scheme for "decuplets"). However, at large N_c Gell-Mann-Okubo relations are restored, up to the order $O(1/N_c)$ inclusive (they are not exact in N_c!). To save space we will not fill up the complete table of masses analogous to the table at $N_c = 3$. Instead we write down only mass relations which are independent on the concrete model (some part of them were already known). For $K = 0$

$$\mu^{(10)}\left(\frac{3}{2}\right) = \mu_2^{(8)}\left(\frac{1}{2}\right) - \frac{1}{4}\mu_1^{(8)}\left(\frac{1}{2}\right) \tag{75}$$

which substitutes Guadagnini's relation derived at $N_c = 3$ (see above). We see that accuracy of this relation is less than of original one. It is not surprising, as the continuation of the Clebsch-Gordan coefficient introduces a new source of inaccuracy. At $K = 1$ there are following relations:

$$12\mu_2^{(8)}\left(\frac{3}{2}\right) - 3\mu_1^{(8)}\left(\frac{3}{2}\right) + 14\mu^{(10)}\left(\frac{3}{2}\right) = 26\mu^{(10)}\left(\frac{1}{2}\right),$$

$$12\mu_2^{(8)}\left(\frac{1}{2}\right) - 3\mu_1^{(8)}\left(\frac{1}{2}\right) + 20\mu^{(10)}\left(\frac{3}{2}\right) = 32\mu^{(10)}\left(\frac{1}{2}\right) \tag{76}$$

And at last for $K = 2$

$$20\mu_2^{(8)}\left(\frac{5}{2}\right) - 5\mu_1^{(8)}\left(\frac{5}{2}\right) + 44\mu^{(10)}\left(\frac{3}{2}\right) = 64\mu^{(10)}\left(\frac{5}{2}\right)$$

$$20\mu_2^{(8)}\left(\frac{3}{2}\right) - 5\mu_1^{(8)}\left(\frac{3}{2}\right) + 34\mu^{(10)}\left(\frac{3}{2}\right) = 54\mu^{(10)}\left(\frac{5}{2}\right) \tag{77}$$

(relation for decuplets is the same as at $N_c = 3$). These relations are valid in the linear order in m_s and up to the order $O(1/N_c)$ inclusive. In general, they are obeyed worse than original ones at $N_c = 3$.

VIII. CONCLUSIONS

If the number of colors N_c is treated as a free algebraic parameter, baryon resonances are classified in a simple way. At large N_c all baryon resonances are basically determined by the intrinsic quark spectrum which takes certain limiting shape at $N_c \to \infty$. This spectrum is the same for light baryons ($q \ldots qq$ with N_c light quarks q)

and for heavy baryons ($q\dots qQ$ with N_c-1 light quarks and one heavy quark Q), since the difference is a $1/N_c$ effect [13].

One can excite quark levels in various ways called either one-particle or particle-hole excitations; in both cases the excitation energy is $\mathcal{O}(1)$. On top of each one-quark or quark-antiquark excitation there is generically a band of $SU(3)$ multiplets of baryon resonances, that are rotational states of a baryon as a whole. Therefore, the splitting between multiplets is $\mathcal{O}(1/N_c)$. The rotational band is terminated when the rotational energy reaches $\mathcal{O}(1)$.

In reality N_c is only 3, and the above idealistic hierarchy of scales is somewhat blurred. Nevertheless, an inspection of the spectrum of baryon resonances reveals certain hierarchy schematically summarized as follows:

- Baryon mass: $\mathcal{O}(N_c)$, numerically 1200 MeV, the average mass of the ground-state octet
- One-quark and particle-hole excitations in the intrinsic spectrum: $\mathcal{O}(1)$, typically 400 MeV, for example the excitation of the Roper resonance
- Splitting between the centers of $SU(3)$ multiplets arising as rotational excitations of a given intrinsic state: $\mathcal{O}(1/N_c)$, typically 133 MeV
- Splitting between the centers of rotational multiplets differing by spin, that are degenerate in the leading order: $\mathcal{O}(1/N_c^2)$, typically 44 MeV
- Splitting inside a given multiplet owing to the nonzero strange quark mass: $\mathcal{O}(m_sN_c)$, typically 140 MeV.

In practical terms, we have shown that all baryon resonances up to 2 GeV made of light quarks can be understood as rotational excitations about certain transitions between intrinsic quark levels. The quantum numbers of the resonances and the splittings between multiplets belonging to the same rotational band are dictated by the quantum numbers of the intrinsic quark levels, and appear to be in good accordance with the data. The content and the splitting of the lowest charmed (and bottom) baryon multiplets are also in accordance with their interpretation as a rotational band about the ground-state filling scheme.

In this paper, we have concentrated on the algebraic aspect of the problem leaving aside the dynamical aspects. Dynamical models should answer the question why the intrinsic quark levels for u,d quarks with $K^P=0^\pm,1^\pm,2^\pm$ and the s quark levels with $J^P=\frac{1}{2}^\pm,\frac{3}{2}^\pm$, *etc.*, have the particular energies summarized in Table 1. However, we feel that it is anyway a step forward: Instead of explaining two hundreds resonances one needs now to explain the positions of only a few intrinsic quark levels. Fig. 1 illustrates that approximately the needed intrinsic spectrum can be achieved from a reasonable set of mean fields [46].

The proposed scheme for understanding baryon resonances has numerous phenomenological consequences that can be investigated even before real dynamics is considered. Namely, the fact that certain groups of $SU(3)$ multiplets belong to the same rotational band related to one and the same one-quark transition implies relations between their couplings, form factors, splittings inside multiplets owing to the nonzero m_s, and so on.

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Appendix A: Toy model

Our approach can be illustrated by the simple but still instructive model suggested in [31]. The model is one of the class considered first in context of the nuclear physics in [36] and is exactly solvable. It describes 0-dimensional quarks with spin, flavor and color which interact by means of 4-fermion color blind potential. We consider the specific case of potential and write the Lagrangian of the model in already bosonized form:

$$\mathcal{L}=\frac{\gamma}{2N_c}\left(\rho_i^a\right)+\psi^+(i\partial_t-\gamma\rho_i^a\lambda^a\sigma_i)\psi \qquad \text{(A1)}$$

Here σ_i are Pauli matrices acting on the spin indices of quarks,λ^a — Gell-Mann matrices from $SU(3)_{flavor}$ group; the sum in color indices is implied. This model is of the type considered in the main text, in the limit $N_c\to\infty$ it can be considered in the mean field approximation. The symmetry of ρ_i^a is analogous to the one of the octet of vector mesons.

Integrating over ρ_i^a we arrive at the Lagrangian with four-fermion interaction:

$$\mathcal{L}_f=\psi^+i\partial_t\psi+\frac{\gamma}{2N_c}\left(\psi^+\lambda^a\sigma_i\psi\right)\left(\psi^+\lambda^a\sigma_i\psi\right),$$

$$\mathcal{H}=-\frac{\gamma}{2N_c}\left(\psi^+\lambda^a\sigma_i\psi\right)\left(\psi^+\lambda^a\sigma_i\psi\right) \qquad \text{(A2)}$$

which has $SU(3)_{flavor}\otimes SU(2)_{spin}$ symmetry.

It is convenient to unite spin and flavor indices in the one $SU(6)$ index and classify states of the model as

$SU(6)$-multiplets. However, these multiplets are split by the potential which is not $SU(6)$ symmetric. Let us introduce generators $\mathcal{T}_a$ of $SU(6)$ group and generators T_a (S) for $SU(3)_{flavor}$ ($SU(2)_{spin}$) group

$$\mathcal{T}_a = \tfrac{1}{2}\psi^+\Lambda_a\psi, \qquad T_a = \tfrac{1}{2}\psi^+\lambda_a\psi, \qquad S_i = \tfrac{1}{2}\psi^+\sigma_i\psi \tag{A3}$$

where Λ_a ($a = 1\dots35$) are Gell-Mann matrices for $SU(6)$. Using the Fierz identities for $SU(6), SU(3), SU(2)$ group the Hamitonian eq. (A2) can be identically rewritten as:

$$\mathcal{H}_f = \frac{\gamma}{N_c}\left[4(\mathcal{T}_a)^2 - 2(T_a)^2 - \frac{4}{3}(S_i)^2\right] \tag{A4}$$

The first term contains the Casimir operator for $SU(6)$ group, second for $SU(3)_{flavor}$ and third for $SU(2)_{spin}$ one.

According to Pauli principle the allowed colorless states for N_c quarks should be completely symmetric under exchange of $SU(6)$ indices. There is only one $SU(6)$-multiplet which obeys this condition — symmetric spinor of N_c-th rank. Its dimension is given by eq. (58) of the main text (56-plet at $N_c = 3$). The $SU(3)_{flavor}\otimes SU(2)_{spin}$ contents of '56'-plet is discussed in the text and consists of the series of multiplets with $S = 1/2\dots N_c/2$. Their energies are given by eq. (A4)

$$\mathcal{E}_{`56`} = \gamma\left(\frac{3N_c}{2} + 9 - \frac{10}{3N_c}S(S+1)\right) \tag{A5}$$

(we used here expression eq. (58) for C_2 and eq. (56) for c_2 with $\tilde{Y} = N_c/3, X = 0, \tilde{T} = J$). The first term here is the classical energy, the second is the quantum correction and the third is the rotational energy. These formulae coincide with ones obtained in [31] for more general case.

The states eq. (A5) exhaust the spectrum of the model [31] only because this model is too poor. One can easily generalize the model adding to quarks some internal parameters (indices) and assuming that potential remains the same and does not depend on this "hidden" parameters. In this case Pauli principle does not dictate unique symmetry wave function. In generalized model flavor-spin wave function can have any symmetry in such a way that its product with wave function of "hidden" parameters is totally symmetric as required.

The first excited state of the model corresponds to the $SU(6)$-multiplet with all $SU(6)$-indices completely symmetric except one pair which is antisymmetric. Its dimension is determined by eq. (59) of the text; at $N_c = 3$ it corresponds to 70-plet. Using once more expressions for $SU(6)$ and $SU(3)$ Casimir operators we obtain from eq. (A4)

$$\mathcal{E}^{(1-3)}_{`70`} = \gamma\left[\frac{3N_c}{2} + 5 - \frac{4}{3N_c}S(S+1) - \frac{2}{N_c}\tilde{T}(\tilde{T}+1)\right] \tag{A6}$$

where for 3 different series of rotational excitations $\tilde{T} = S-1, S, S+1$. These 3 series correspond, as we shall see, excitations in the sector of u,d-quarks. There are also two series with energies:

$$\mathcal{E}^{(4-5)}_{`70`} = \gamma\left[\frac{3N_c}{2} + 6 - \frac{4}{3N_c}S(S+1) - \frac{2}{N_c}\tilde{T}(\tilde{T}+1) + \frac{3}{2N_c}\right] \tag{A7}$$

where $\tilde{T} = S\pm\frac{1}{2}$. We recognize exactly those 5 series of $SU(3)\otimes SU(2)$ which were described in the text. Other excited states of the model can be also constructed.

Now we are going to reproduce eqs.(A6,A7) in the mean field approximation. We solve Dirac equation and the consistency equation following from the Lagrangian eq. (A1)

$$\rho^a_i\sigma_i\lambda^a\phi = \varepsilon\phi, \qquad \rho^a_i = \phi^+_{\rm ground}\sigma_i\lambda^a\phi_{\rm ground} \tag{A8}$$

We are looking for the mean field as $SU(2)$ "hedgehog": $\rho^a_i = \bar{\rho}\delta^a_i$ for $a = 1\dots3$ and zero otherwise. There are 3 solutions of the Dirac equation:

$$\phi^{\alpha i}_0 = \begin{pmatrix}\varepsilon^{\alpha i}\\ 0\end{pmatrix}, \qquad \phi^{\alpha i}_{1(a)} = \frac{1}{\sqrt{2}}\begin{pmatrix}\varepsilon^{\alpha j}(\sigma_a)^i_j\\ 0\end{pmatrix},$$

$$\phi^{\alpha i}_{s(a)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \delta^{ia}\end{pmatrix}, \tag{A9}$$

Here $\phi^{\alpha i}_0$ (α -isospinor, i — spinor index) is the wave function with $K = 0$ and energy $\varepsilon_0 = -3\gamma\bar{\rho}$, $\phi^{\alpha i}_{1(a)}$ are (three degenerate)wave function of the states with $K = 1$ and energy $\varepsilon_1 = \gamma\bar{\rho}$ in sector of u,d quarks. At last $\phi^{\alpha i}_{s(a)}$ is wave function of (doubly degenerate) level in the sector of s-quarks with energy $\varepsilon_s = 0$.

We obtain the ground state filling level ε_0 by N_c quarks, then consistency equation gives $\bar{\rho} = -1$. The classical part of the mass (proportional to N_c) is obtained substituting the ground state wave function to the Hamiltonian eq. (A2). The $O(1)$ part of energy implies the calculation of quantum corrections (it comes from one-loop diagram in the quantum field ρ^a_i) and is not interesting for us (see [31]). However the difference between energy of the ground state and excited ones is calculable and indeed is determined by the difference of 1-quark levels:

$$\mathcal{E}^{(1-3)}_{`70`} - \mathcal{E}_{`56`} = \varepsilon_1 - \varepsilon_0 + O(\frac{1}{N_c}) = -4\gamma$$
$$\mathcal{E}^{(4-5)}_{`70`} - \mathcal{E}_{`56`} = \varepsilon_s - \varepsilon_0 + O(\frac{1}{N_c}) = -3\gamma \tag{A10}$$

(we have to choose $\gamma < 0$).

Turning to the rotational energy ($O(1/N_c)$) we see that in all cases it determined by the eq. (57) of the text with moment of inertia:

$$I_1 = -\frac{3}{20\gamma}N_c \tag{A11}$$

and mixing coefficient $\tilde{a}_K = 2/5$ both for the excitations in the sector of u,d-quarks, eq. (A6), and s-quarks, eq. (A7).

The moment of inertia I_1 does not coincide with the Inglis [37] moment of inertia $I_{\rm Inglis}$ obtained in the cranking approximation:

$$\mathcal{E}_{\rm rot}=\frac{N_c}{2}\sum_{m={\rm free}}\frac{<0|\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|m><m|\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|0>}{\varepsilon_m-\varepsilon_0}=\frac{1}{2}I_{Inglis}(\tilde{\boldsymbol\Omega}-\tilde{\boldsymbol\omega})^2,\qquad I_{\rm Inglis}=-\frac{1}{8\gamma}N_c \tag{A12}$$

This was noticed for the first time in [31]. Hence in this model there is non-zero mixing of rotations with other quantum fluctuations of ρ_i^a field.

To account for effect of this mixing in the rotational energy we have to find the correction $\delta\rho_i^a$ to the mean field. We solve the Dirac equation with rotational correction and self-consistency equation:

$$\varepsilon\phi^{\alpha i}=\rho_l^a(\lambda^a)^\alpha_\beta(\sigma^l)^i_j\phi^{\alpha' i'}+\frac{1}{2}\tilde\omega_l(\sigma_l)^i_{i'}\phi^{\alpha i'}+\frac{1}{2}\tilde\Omega_l(\lambda_l)^\alpha_{\alpha'}\phi^{\alpha' i},$$

$$\rho_i^a=\phi^+\sigma_i\lambda^a\phi \tag{A13}$$

in the leading order in spin frequency ω_l and flavor frequency Ω^a (we restrict ourselves to the $a=1\dots3$ which are only relevant at the moment) and obtain

$$\delta\psi^{\alpha i}=-\frac{3}{20\sqrt{2}}\left(\tilde\omega_l(\sigma_l)^i_{i'}\varepsilon^{\alpha i'}+\tilde\Omega_a(\lambda_a)^\alpha_{\alpha'}\varepsilon^{\alpha' i}\right),$$

$$\delta\rho_i^8=\frac{\sqrt3}{10\gamma}(\tilde\Omega_i-\tilde\omega_i) \tag{A14}$$

The last equation here gives the change of the mean field we are looking for. Due to the hedgehog symmetry $\delta\rho$ depends only on difference of flavor and spin frequencies. The correction to energy due to rotation is (here the first term is 2nd order expansion of quark determinant in rotation and change of mean field $\delta\rho$ and the second is change of pure meson Lagrangian.)

$$\mathcal{E}_{\rm rot}=\frac{N_c}{2}\sum_{m=free}\frac{\langle 0|\gamma\delta\rho_i^a\lambda^a\sigma_i+\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|m\rangle\langle m|\gamma\delta\rho_i^a\lambda^a\sigma_i+\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|0\rangle}{\varepsilon_m-\varepsilon_0}+\frac{\gamma N_c}{2}(\delta\rho_i^a)^2=-\frac{3N_c}{40\gamma}(\tilde{\boldsymbol\Omega}-\tilde{\boldsymbol\omega})^2 \tag{A15}$$

which is, indeed, rotational energy with moment of inertia I_1. Let us stress again that the fact that this expression is different from the cranking approximation eq. (A12) is completely due to the mixing of rotational degrees of freedom with other quantum fluctuations. Let us note that this mixing is absent if the flavor group is $SU(2)$. In general, the mixing appears because the model is non-relativistic (see Appendix B).

There are two types of one-quark excitations: in the sector with u,d-quarks (wave function ϕ_1) and in the sector of s-quarks (wave function ϕ_s). Mixing coefficients a_K are determined by linear terms in the frequencies (see Section IV). We have:

$$\delta\mathcal{S}_{\rm rot}=-\int dt\sum_{m=\rm excited}\Big[\langle m|\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|m\rangle+$$

$$+\langle m|\delta\rho_i^a\lambda^a\sigma_i|m\rangle\Big]\equiv-\int dt\left[(1-a_K)\tilde{\boldsymbol\omega}_K\cdot\boldsymbol K+a_K\tilde\Omega_K\cdot\boldsymbol K\right] \tag{A16}$$

For $K=1$ level in the sector of u,d-quarks we introduce wave function of excited level as $\phi^{\alpha i}_{\rm excited}=\sum\chi_{K_3}\phi^{\alpha i}_{1(K_3)}$, then

$$\langle{\rm excited}|\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|{\rm excited}\rangle=$$

$$=\frac{1}{2}\sum_{K_3,K_3'}\langle K_3'|(\tilde{\boldsymbol\Omega}+\tilde{\boldsymbol\omega})\cdot\boldsymbol K|K_3\rangle\chi^+_{K_3'}\chi_{K_3}$$

and

$$\langle{\rm excited}|\delta\rho_i^a\sigma_i\lambda_a|{\rm excited}\rangle=$$

$$-\frac{1}{10}\sum_{K_3,K_3'}\langle K_3'|(\tilde{\boldsymbol\Omega}-\tilde{\boldsymbol\omega})\cdot\boldsymbol K|K_3\rangle\chi^+_{K_3'}\chi_{K_3} \tag{A17}$$

Comparing with the eq. (A16) we see that in this case $\tilde a_K=\frac{1}{2}-\frac{1}{10}=\frac{2}{5}$. In terms of Section IV we can say that $c_K=0$ (this is the consequence of the fact that in our theory there is no orbital momenta) and mixing coefficient $\zeta=1/10$.

For excitations of s-quark wave function is $\phi^{\alpha i}_{\rm excited}=\sum\chi_{j_3}\phi^{\alpha i}_{s(j_3)}$ where $j=\frac{1}{2}$ is total momentum of s-quark excitation. Then:

$$\langle{\rm excited}|\frac{1}{2}(\tilde{\boldsymbol\Omega}\cdot\boldsymbol\lambda+\tilde{\boldsymbol\omega}\cdot\boldsymbol\sigma)|{\rm excited}\rangle=\sum_{j_3,j_3'}\langle j_3'|\tilde{\boldsymbol\omega}\cdot\boldsymbol j|j_3\rangle\chi^+_{j_3'}\chi_{j_3}$$

$$\langle{\rm excited}|\delta\rho_i^a\sigma_i\lambda_a|{\rm excited}\rangle=\frac{2}{5}\sum_{j_3,j_3'}\langle j_3'|(\tilde{\boldsymbol\Omega}-\tilde{\boldsymbol\omega})\cdot\boldsymbol j|j_3\rangle\chi^+_{j_3'}\chi_{j_3} \tag{A18}$$

From the second of these expressions we obtain again the mixing coefficient $\tilde{a}_K = \frac{2}{5}$ (the role of $\boldsymbol{K}$ is played now by $\boldsymbol{j}$).

Thus the mean field approximation correctly reproduces exact formulae for energy eq. (A6) and eq. (A7) in all cases.

Appendix B: Mixing of the slow rotations in relativistic theories

Rotational degrees of freedom can mix with other quantum fluctuations. This phenomenon is known well in nuclear physics [38]. As we show below this phenomenon is absent for soliton constructed within the mean field approximation (at $N_c \to \infty$) with help of relativistically invariant meson Lagrangian. This statement is true, at least, for solitons made of pions , e.g. in the Skyrme model [2] or quark-soliton chiral model [10].

Indeed, let us consider in the Skyrme model flavor rotation $R(t)$ together with some other quantum fluctuations $\delta\pi^a$. The general pion field is presented as:

$$\pi_a(\boldsymbol{x},t)\lambda_a = R(t)\left[\bar{\pi}^a(\boldsymbol{x}) + \delta\pi(\boldsymbol{x},t)\right]\lambda_a R^+(t) \qquad \text{(B1)}$$

where $\bar{\pi}_a(\boldsymbol{x})$ is the time independent mean field. We have to substitute eq. (B1) into the Skyrme action or action obtained by integration over quarks in quark-soliton model and analyze the contribution of fluctuations $\delta\pi$ to effective Lagrangian.

Mixing of the rotations with other quantum fluctuations implies terms of the form $\delta\pi(x,t)\mathcal{K}\Omega$ where Ω is the flavor frequency and $\mathcal{K}$ is some operator. The appearance of Ω means that the mixing can arise only from those terms of effective chiral Lagrangian which contain time derivatives. However, in the relativistic theory there is usually an even number of time derivative and ar least two of them. From the other hand, in the leading order in N_c we can consider frequency Ω to be constant in time. Therefore, the second derivative should be applied to $\delta\pi$, so $\mathcal{K}$ is at least linear in time derivatives. However, all such terms are full derivatives and can be omitted.

The noticable exclusion from this rule is Wess-Zumino-Witten term which is linear in time derivative. We have to apply this derivative to $R(t)$ in order to obtain flavor frequency. Using independence of Ω on coordinates we arrive at (see, e.g., [10]):

$$\delta L = \frac{N_c}{48\sqrt{3}\pi^2}\int dt\, \Omega_8 \int d^3x\, \varepsilon_{ijk} \times$$

$$\times\, \mathrm{Tr}\left[\lambda_8\left(\left(U^+\partial_i U\right)\left(U^+\partial_j U\right)\left(U^+\partial_k U\right)\right]\right. \qquad \text{(B2)}$$

where $U = \exp i(\bar{\pi}^a + \delta\pi^a(x,t))\lambda^a$ is the pion mean field together with quantum fluctuations. The quantity:

$$Q_t = \frac{1}{24\pi}\int d^3x\, \varepsilon_{ijk}\mathrm{Tr}\left[\lambda_8\left(U^+\partial_i U\right)\left(U^+\partial_j U\right)\left(U^+\partial_k U\right)\right] \qquad \text{(B3)}$$

is topological charge of the field U and cannot be changed by the small fluctuations of the pion field. In other words:

$$\frac{\delta Q}{\delta\pi(x,t)} = 0 \qquad \text{(B4)}$$

Hence mixing of rotations with quantum fluctuations is absent in the Skyrme model.

Situation is similar in the quark-soliton model [10]. Mixing can appear only from so-called "imaginary part" of the effective π-meson action. This part of the action starts by WZW term but in principle is an infinite series in gradients of pion field. However all these terms are full derivatives and the sum reduce to the complete baryon charge of the state. This quantity is determined by the number of valence quarks and cannot be changed by the small fluctuations of the pion field.

This does not mean that mixing is always zero. First, it can appear in more general meson Lagrangians. Second, it can arise if the frequencies of rotations are not small. For example, properties of rotational exotic states (at $\Omega_{4,5,6,7} \sim O(1)$) can be described as mixing of rotations and quantum K-meson states belonging to the continuum spectrum. This leads to the width of these states $\sim O(1)$. However, the rotational theory in this case should be modified anyway, as due to the WZW term rotations in the strange directions turn into small oscillations (see, e.g. [24, 42] and [5]).

Appendix C: Matrix elements of one-particle operators

Dirac equation in the sector of u,d-quarks conserves the grand spin K. The angular part of the wave function of the state with given K is spherical spinor-isospinor:

$$\Xi^{\alpha i}_{KK_3jl}(\boldsymbol{n}) = \sum_{j_3} C^{KK_3}_{jj_3;\frac{1}{2}\alpha}\Omega^i_{jj_3l}(\boldsymbol{n}) \qquad \text{(C1)}$$

Here α is isospinor and i is spinor indices, Ω is a spherical spinor with total angular moment j (projection j_3) and orbital momentum l; Clebsh-Gordan coefficient $C^{\cdots}_{\cdots}$ joins j and isospin $\boldsymbol{t}$ ($t = \frac{1}{2}$) into the grand spin $\boldsymbol{K}$. Total angular momentum can be $j = K \pm \frac{1}{2}$. Spherical spinor Ω^i is constructed according to the same rule out of spin of the quark $\boldsymbol{s}$ ($s = \frac{1}{2}$) and orbital momentum $\boldsymbol{l}$

$$\Omega^i_{jj_3l}(\boldsymbol{n}) = \sum_{j_3} C^{jj_3}_{ll_3;\frac{1}{2}i} Y_{ll_3}(\boldsymbol{n}) \qquad \text{(C2)}$$

where $Y_{ll_3}(\boldsymbol{n})$ are usual spherical harmonics.

We are looking for the solution of the Dirac equation: $\varepsilon\Psi = \mathcal{H}\Psi$ with the Dirac Hamiltonian eq. (1) which is a bispinor $\{\varphi,\chi\}$ in the form:

$$\Psi^{\alpha i} = \begin{pmatrix} g(r)\Xi^{\alpha i}_{KK_3,K-\frac{1}{2},K} + h(r)\Xi^{\alpha i}_{KK_3,K+\frac{1}{2},K} \\ f(r)\Xi^{\alpha i}_{KK_3,K-\frac{1}{2},K-1} + j(r)\Xi^{\alpha i}_{KK_3,K+\frac{1}{2},K+1} \end{pmatrix} \qquad \text{(C3)}$$

This state has a "natural parity" $(-1)^K$. Indeed, parity transformation is $\varphi(\boldsymbol{r}) \to \varphi(-\boldsymbol{r})$, $\chi(\boldsymbol{r}) \to -\chi(-\boldsymbol{r})$ and hence parity is determined by the value l. The state with parity $(-1)^{K+1}$ corresponds by exchange of φ and χ in this expression. At $K = 0$ wave function is determined only by 2 functions $h(r)$ and $j(r)$.

We want to calculate the matrix elements of $\boldsymbol{t}$. Since it acts only on isospinor indices α and spherical spinors Ω are orthonormal, we obtain:

$$\langle K_3'|\boldsymbol{t}|K_3\rangle = \sum_{\alpha\beta,j,j_3} \left(C^{KK_3'}_{jj_3\frac{1}{2}\alpha}\right)^* \langle\alpha|\boldsymbol{t}|\beta\rangle C^{KK_3}_{jj_3\frac{1}{2}\beta} \times$$

$$\left[(1-c_K)\delta_{jK-\frac{1}{2}} + c_K\delta_{jK+\frac{1}{2}}\right] \qquad \text{(C4)}$$

with c_K defined by eq. (42) and $\langle\alpha|\boldsymbol{t}|\beta\rangle$ are usual generators of isospin $\frac{1}{2}$.

Substituting Clebsh-Gordan coeffients we arrive at eq. (41). In particular, if $\boldsymbol{t} \to t_3$ the matrix elements are diagonal in α, β and hence diagonal in K_3, K_3'. The matrix element:

$$\langle K_3|t_3|K_3\rangle = \tfrac{1}{2}\left(\left|C^{KK_3}_{j,K_3-\frac{1}{2};\frac{1}{2},\frac{1}{2}}\right|^2 - \left|C^{KK_3}_{j,K_3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}}\right|^2\right) \times$$

$$\times\left[(1-c_K)\delta_{jK-\frac{1}{2}} + c_K\delta_{jK+\frac{1}{2}}\right] \qquad \text{(C5)}$$

and using expressions for Clebsch-Gordan coefficients we obtain

$$\langle K_3|t_3|K_3\rangle = K_3\left(\frac{1-c_K}{2K} - \frac{c_K}{2(1+K)}\right) \qquad \text{(C6)}$$

It is a particular case of eq. (41).

Appendix D: Decays of excited baryons

Calculations of the widths of excited baryons is outside the scope of this paper, however, in this appendix we present only general discussion of the baryon decays. For the ground state baryons the procedure of calculation is known: was constructed for the Skyrme model already in [2], for the quark-soliton model see, e.g. [10, 19, 39]. At last, strictly in the limit $N_c \to \infty$ decay constants in the approach [26] were calculated, e.g., in [40, 41] and in other, already cited papers.

Typical decays of excited baryons below 2 GeV are of the type $B_i \to B_f M$ with one emitted meson, at least such decays always give essential part of the width. To be specific, we will talk about decays into π-mesons. Let us estimate the width in the limit of large N_c. (This estimate is already known: we are close here to J.L.Goity in [8] (see also [41])). The one pion decays of the excited baryons are described by the effective Lagrangian of the type:

$$\mathcal{L}_{eff} = \frac{g_a}{F_\pi}\int d^3x\ \bar{\Psi}^{(f)}_B\gamma_\mu\gamma_5\frac{\lambda^a}{2}\Psi^{(i)}_B\partial_\mu\pi \qquad \text{(D1)}$$

Here $\Psi^{(i)}_B$ and $\Psi^{(f)}_B$ are fields of initial and final baryon, π is the π-meson field with flavor a, λ_a is the corresponding Gell-Mann matrix. At last g_a is the transitional axial coupling constant. The width Γ_{fi} of partial decay to $B_f\pi$ is proportional to coupling constant squared and phase volume:

$$\Gamma_{fi} \sim \frac{g_a^2}{8\pi F_\pi^2}\Delta^3 \qquad \text{(D2)}$$

(see, e.g. [5]) where $\Delta = M_i - M_f$ is the difference of mass of the initial and final baryon.

The coupling constant can be calculated as a matrix element of the corresponding quark operator between mean field initial and final state:

$$g_a(k) \sim \int d^3x\ \langle\text{fin}|\bar{\psi}\gamma_5\gamma_\mu\psi(x)|\text{in}\rangle e^{ikx} \qquad \text{(D3)}$$

The role of quark operator is played by axial current for decays with π-mesons, vector current for decays into ρ-mesons, etc. Expression eq. (D3) already implies the $N_c \to \infty$ limit, as baryons are considered to be heavy (mass $O(N_c)$) non-relativistic objects. (Expression eq. (D2) is also written in this limit). Plane wave e^{ikx} represents wave function of emitted meson, with k being its momentum. At last $|\text{in}\rangle$ and $|\text{fin}\rangle$ are mean field approximations for initial and final baryon quark wave functions. They are product of all 1-quark wave functions — solutions of Dirac equation in the mean field — for all filled levels. In general, one has to write here wave functions rotated by matrices R and S in order to take into account degeneracy of the mean field. After the calculation of matrix element eq. (D3) we obtain some operator depending on collective coordinates. Averaging this operator with collective wave functions of initial and final baryon we obtain the coupling constant for some specific decay.

In fact, eq. (D3) is only the first term of expansion in the time derivatives of collective coordinates. Next terms can be obtained in the same manner as it was done for corrections in m_s in the main text. Due to the limit $N_c \to \infty$ all collective coordinates are slowly varying functions of coordinates, so the expansion in time derivatives is an expansion in $1/N_c$, with eq. (D3) being its leading term. For ground state baryons and for decays into pions the corresponding formulae were presented in [39].

Decays of excited baryons are possible either to the baryons belonging to the same rotational band or to the baryons which have the different filling of intrinsic quark levels (e.g., to ground state baryons). In the first case the coupling constant is large, $O(N_c)$. Example is the transitional axial constant $g_a(\Delta \to N\pi)$ [2]. In the second case the coupling constant is always smaller. This difference is clearly seen from eq. (D3).

Indeed, for the configuration of levels being the same for the initial and final state, the coupling constant is a sum of N_c 1-particle matrix elements corresponding to all N_c quarks. If excited quark changes its intrinsic state then only one of N_c contributions would survive, which is the overlap 1-particle matrix element between initial and final states of this quark (all other are zero due to orthogonality of wave functions). However, if the final state is the ground state additional factor $\sqrt{N_c}$ appears which is due to the different normalization of initial and final wave functions:

$$g_a(R,S) \sim \int d^3x \phi_f^*(\mathbf{x}) S^+ \gamma_3 \gamma_5 S R^+ \frac{\lambda^a}{2} R \phi_i(\mathbf{x}) j_l(k|\mathbf{x}|) \times$$

$$\times \mathcal{D}^l_{m_1,m_2}(S) Y_{lm_2}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \begin{cases} N_c & i,f = \text{same band} \\ \sqrt{N_c} & i = \text{excited}, f = \text{ground} \\ 1 & i = \text{excited}, f = \text{excited}' \end{cases} \tag{D4}$$

This expression is written a bit schematically. Wave functions ψ_i and ψ_f are initial and final wave functions of excited quark, R is a rotational matrix in flavor and S in ordinary space, $\mathcal{D}^l_{m_1 m_2}(S)$ is Wigner function and Y_{lm} is an ordinary spherical harmonics (summation in all possible m_2 is implied). At last $j_l(kr)$ is a spherical Bessel function. It appears (together with spherical harmonics) as a result of expansion of a plane wave in eq. (D3) into the set of spherical waves. If the momentum of emitted meson is small $ka \ll 1$ (a is the scale of wave functions $\psi_{i,f}$ which coincides with the characteristic size of the baryon) it is sufficient to account only for the lowest angular momentum $l=0$ (angular momentum of emitted pion is 1).

Axial constant eq. (D4) is an operator in the space of collective coordinates (derived in the leading order in N_c). To obtain coupling constant responsible for decay of concrete baryon to another one, we have to average expression eq. (D4) with collective wave functions:

$$g_a(i \to f) = \int dR dS \; \psi_f^{(rot)*}(R,S) g_a(R,S) \psi_i^{(rot)}(R,S) \tag{D5}$$

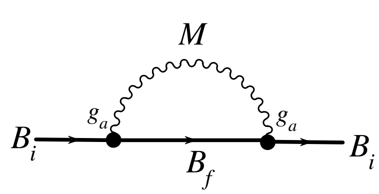

FIG. 4. Self energy correction to the excited baryon mass

In spite of the fact that coupling constants are smaller, the widths of decays to the different quark levels are typically larger in N_c. The reason is that phase volume in this case is always larger. The mass differences are $O(1/N_c)$ for decays inside the same rotational band but $O(1)$ for transitions with change intrinsic state of excited quark. As a result the widths of decays inside the rotational band are suppressed as $O(1/N_c^2)$ while decays of excited baryons with discharge of the excitation are always $O(1)$ (and decays to the the other levels are suppressed). In particular, the total width of ground state baryons (decuplet with spin $\frac{3}{2}$) is only $O(1/N_c^2)$, while all remaining baryons have total width of $O(1)$.

In practical terms only decays to the ground octet or decuplet are observable. For all baryons they have partial widths independent on N_c up to corrections in $1/N_c$ which can be still essential at $N_c = 3$. Let us prove the theorem: widths of all baryons belonging to the same rotational band are the same in the leading order in N_c.

Indeed, mass differences of all baryons entering the same rotational band are the same in the leading order in N_c. Hence

$$\Gamma^{(i)}_{tot} = \sum_f \Gamma(i \to f) = \frac{\Delta^3}{8\pi} \sum_f g_a^2(i \to f) = \Gamma_{level} \tag{D6}$$

However, the sum of axial constants squared over all possible final states does not depend on the initial state of the band. According to eq. (D5) axial constant squared contains two integrals in R,S and R',S'. Due to completeness of final baryon rotational functions:

$$\sum_f \psi_f^{(rot)*}(R,S) \psi_f^{(rot)}(R',S') = \delta(R-R')\delta(S-S')$$

leads to $R=R'$ and $S=S'$. Then the sum in all possible flavors of pseudoscalar mesons and directions of axial current gives the expression which does not depend on R and S due to Fiertz identities. Dependence on matrices remains only in the initial wave function. Integral over R and S is becoming the normalizing integral for initial collective wave function and the dependence on initial state disappears completely. Obtained total width has a sense of the complete width of the intrinsic quark level and is universal for the whole rotational band around it.

The proved theorem is broken strongly in nature. There are many reasons for that: corrections in N_c and mass of the strange quark m_s to the coupling constants, mixing of multiplets, etc. Perhaps, the strongest source of the deviations is simply the difference in the phase volumes (which is $O(1/N_c)$ effect) for different baryons entering the same rotational band.

The fact that widths of excited baryons are not suppressed in the large N_c limit, as it was mentioned in the Introduction, makes these baryons not well-defined. One can count only on numerical smallness of the width not related to N_c. In such a situation baryon resonances can be defined only as poles in the complex plane of meson-nucleon scattering amplitude. This approach was applied in [5] to the problem of pentaquark (which also has width independent on N_c) for the Skyrme model but in general case it looks too complicated. If the width is small one returns to the self-consistent field description presented here.

It seems that width of the baryon which is not suppressed at $N_c \to \infty$ possesses the danger to our approach in general. Indeed, due to unitarity non-zero width implies not only imaginary part of the pole but also a shift in the real part, i.e. leads to the change of the baryon mass. It can be small numerically but it is $O(1)$ in N_c. If it is different for the baryons entering the same rotational band, our formulae for mass splittings inside the rotational band would become senseless. Fortunately, it is not the case.

Corrections to the mass due to decays into the π-mesons are presented by self-energy diagram Fig.4. The imaginary part of this diagram gives the width of $B_i \to B_f M$ decay, real part gives the shift of mass. The point is that the mass shift does not depend on the baryon B_i on the same rotational band:

$$\Delta M \sim \sum_f \int \frac{d^4k}{(2\pi)^4} \frac{g_a^2(i\to f)}{k^2(\Delta + k_0)}$$

as it was proved above. Hence we arrive at conclusion that mass shifts are universal for all baryons inside the rotational bands. It can be included to the general shift of the intrinsic level and does not break down the mass relations in the $O(1/N_c)$ order derived in the text. Next order corrections in N_c due to the finite width of the resonance also do not destroy these relations. However, they can renormalize the moment of inertia I_1. Example of such a situation is given by pentaquark in the Skyrme model [5].

Appendix E: Isoscalar factors for $N_c \to \infty$

Clebsch-Gordan coefficients for large N_c-baryon multiplets were calculated in a number of References[26, 43, 44]. Two methods can be used for this calculation: either based on decomposition $SU(3)$ spinor with large number of indices[26, 43] or applying lowering and rising generators in the given representation to the state with highest weight[44].

However, the tables of CG-coefficients in above references are usually incomplete and do not correspond to the conventions of the [45] (which serves as a common standard for $SU(3)$-group) but differ from this standard by unitary transformation. This is inconvenient and for this reason we give here the complete tables for isoscalar factors at arbitrary N_c.

During the refereeing we have been informed that the complete tables of isoscalar coefficients were presented in [47]. The tables presented here are analogous and differs only method of calculation. Nonetheless, we leave here the tables for the completeness of description. We thank referee for taking our attention for this article.

Consulting with tables one can see that the change of Clebsch-Gordan coefficients from $N_c \to \infty$ to $N_c = 3$ is, indeed, rather large. In particular, one can note the cases when isoscalar factor changes the sign during this transition. Moreover, for "decuplet" there are *two* possible final multiplets at $N_c \geq 5$, so one can discuss F/D ratio for "decuplet". Corresponding multiplet dies out as $N_c = 3$.

From the other hand, Clebsch-Gordan coefficients can be easily taken into account at any N_c and this source of inaccuracy can be avoided. For this reason we prefer to use isoscalar factors at $N_c = 3$.

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quark levels	rotational bands	$(I_1)^{-1}$, MeV	$\tilde{a}_K$
$K^P = 0^+$, ground state	$(\mathbf{8}, 1/2^+, 1152)$ $(\mathbf{10}, 3/2^+, 1382)$	153	
$0^+ \to 0^+$ 482 MeV	$(\mathbf{8}, 1/2^+, 1608)$ $(\mathbf{10}, 3/2^+, 1732)$	83	
$0^+ \to 2^+$ 722 MeV	$(\mathbf{8}, 3/2^+, 1865)$ $(\mathbf{8}, 5/2^+, 1873)$ $(\mathbf{10}, 3/2^+, 2087)$ $(\mathbf{10}, 5/2^+, 2071)$ $(\mathbf{10}, 7/2^+, 2038)$	131	-0.050
$0^+ \to 1^+$ $\sim$780 MeV	$N(1/2^+, 1710)$ $N(3/2^+, 1900)$ $\Delta(1/2^+, 1910)$ $\Delta(3/2^+, \sim 1945)$? $\Delta(5/2^+, 2000)$		
$0^+ \to 1^-$ 468 MeV	$(\mathbf{8}, 1/2^-, 1592)$ $(\mathbf{8}, 3/2^-, 1673)$ $(\mathbf{10}, 1/2^-, 1758)$ $(\mathbf{10}, 3/2^-, 1850)$	171	0.336
$0^+ \to 0^-$ 563 MeV	$(\mathbf{8}, 1/2^-, 1716)$	155(fit)	
$0^+ \to 2^-$ 730 MeV	$(\mathbf{8}, 3/2^-, 1896)$ $(\mathbf{8}, 5/2^-, 1801)$	155(fit)	-0.244
$0^+ \to \frac{1}{2}^-$ 254 MeV	$(\mathbf{1}, 1/2^-, 1405)$		
$0^+ \to \frac{3}{2}^-$ 379 MeV	$(\mathbf{1}, 1/2^-, 1520)$		

TABLE I. Interpretation of all baryon resonances below 2 GeV, as rotational excitations on top of intrinsic quark states.

K	Rep	J	$\mu_1^{(8)}$	$\mu_2^{(8)}$	$\mu^{(10)}$
0	8	$\frac{1}{2}$	$-\frac{\alpha}{10} - \frac{3\gamma}{20}$	$-\frac{\alpha}{8} - \beta + \frac{5\gamma}{16}$	
	10	$\frac{3}{2}$			$-\frac{\alpha}{8} - \beta + \frac{5\gamma}{16}$
1	8	$\frac{1}{2}$	$-\frac{\alpha}{10} - \frac{11\gamma}{20} - \frac{3\delta}{5}$	$-\frac{\alpha}{8} - \beta + \frac{55\gamma}{48} - \frac{5\delta}{4}$	
		$\frac{3}{2}$	$-\frac{\alpha}{10} + \frac{\gamma}{20} + \frac{3\delta}{10}$	$-\frac{\alpha}{8} - \beta - \frac{5\gamma}{48} - \frac{5\delta}{8}$	
	10	$\frac{1}{2}$			$-\frac{\alpha}{8} - \beta + \frac{35\gamma}{48} + \frac{5\delta}{8}$
		$\frac{3}{2}$			$-\frac{\alpha}{8} - \beta + \frac{23\gamma}{48} + \frac{\delta}{4}$
		$\frac{5}{2}$			$-\frac{\alpha}{8} - \beta + \frac{\gamma}{16} - \frac{3\delta}{8}$
2	8	$\frac{3}{2}$	$-\frac{\alpha}{10} - \frac{3\gamma}{4} - \frac{9\delta}{10}$	$-\frac{\alpha}{8} - \beta + \frac{25\gamma}{16} + \frac{15\delta}{8}$	
		$\frac{5}{2}$	$-\frac{\alpha}{10} + \frac{\gamma}{4} + \frac{3\delta}{5}$	$-\frac{\alpha}{8} - \beta - \frac{25\gamma}{48} - \frac{5\delta}{4}$	
	10	$\frac{1}{2}$			$-\frac{\alpha}{8} - \beta + \frac{17\gamma}{16} + \frac{9\delta}{8}$
		$\frac{3}{2}$			$-\frac{\alpha}{8} - \beta + \frac{13\gamma}{16} + \frac{3\delta}{4}$
		$\frac{5}{2}$			$-\frac{\alpha}{8} - \beta + \frac{19\gamma}{48} + \frac{\delta}{8}$
		$\frac{7}{2}$			$-\frac{\alpha}{8} - \beta - \frac{3\gamma}{16} - \frac{3\delta}{4}$

TABLE II. Mass splittings for octet and decuplet particles for different K.

	$\eta N \to N$	$\pi N \to N$	$K\Sigma \to N$	$K\Lambda \to N$	$KN \to \Sigma$	$\eta\Sigma \to \Sigma$
F	$-\frac{3(\nu^2+3\nu+1)}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{\nu^2+5\nu+9}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{(11+4\nu)\sqrt{\nu}}{\sqrt{2(\nu+2)(\nu+4))}}$	$\frac{\sqrt{3}(2\nu+3)}{\sqrt{2(\nu+4)}}$	$-\frac{\sqrt{\nu}(11+4\nu)}{\sqrt{3(\nu+2)(\nu+4)}}$	$\frac{(1-\nu)(8+3\nu)}{\sqrt{3(\nu+2)(\nu+4)}}$
D	$\sqrt{\frac{\nu}{2}(\nu+2)}$	$-3\sqrt{\frac{\nu}{2}(\nu+2)}$	$3\sqrt{\frac{\nu}{2}+1}$	$-\sqrt{\frac{3\nu}{2}}$	$-\sqrt{3(2+\nu)}$	$(3-\nu)\sqrt{\frac{2\nu+3}{2\nu}}$
	$\pi\Sigma \to \Sigma$	$\pi\Lambda \to \Sigma$	$K\Xi \to \Sigma$	$KN \to \Lambda$	$\eta\Lambda \to \Lambda$	$\pi\Sigma \to \Lambda$
F	$\frac{\sqrt{\nu+4}(\nu+5)}{\sqrt{3(\nu+2)}}$	$\frac{(1-\nu)\sqrt{\nu}}{\sqrt{6(\nu+4)}}$	$\frac{7\nu+8}{3\sqrt{\nu+4}}$	$\frac{\sqrt{3}(2\nu+3)}{\sqrt{\nu+4}}$	$\frac{3(2-\nu-\nu^2)}{\sqrt{2(\nu+2)(\nu+4)}}$	$\frac{(\nu-1)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$
D	$(1-\nu)\sqrt{\frac{3(2+\nu)}{\nu}}$	$\sqrt{\frac{3}{2}}(\nu+1)$	$-\frac{2\nu+1}{\sqrt{\nu}}$	$-\sqrt{3\nu}$	$-\frac{(\nu+5)\sqrt{\nu}}{\sqrt{2(\nu+2)}}$	$-\sqrt{\frac{3}{2}}(\nu+1)$
	$K\Xi \to \Lambda$	$K\Sigma \to \Xi$	$K\Lambda \to \Xi$	$\eta\Xi \to \Xi$	$\pi\Xi \to \Xi$	
F	$\frac{5\sqrt{\nu(\nu+2)}}{\sqrt{\nu+4}}$	$\frac{7\nu+8}{\sqrt{6(\nu+4)}}$	$-\frac{5\sqrt{\nu(\nu+2)}}{\sqrt{2(\nu+4)}}$	$\frac{(8-3\nu)\sqrt{\nu+2}}{2\sqrt{\nu+4}}$	$\frac{(16-\nu)\sqrt{\nu+2})}{3\sqrt{2(\nu+4)}}$	
D	$\frac{3}{\sqrt{\nu+2}}$	$-\frac{(2\nu+1)\sqrt{3}}{\sqrt{2\nu}}$	$\frac{3}{\sqrt{2(\nu+2)}}$	$\frac{3-5\nu-\nu^2}{\sqrt{2\nu(\nu+2)}}$	$\frac{\nu^2+3\nu+5}{\sqrt{2\nu(\nu+2)}}$	

TABLE III. Isoscalar factors for $8_M \otimes "8"_B \to "8"_B$ at arbitrary $N_c = 2\nu+1$. In the table two isoscalar factor for antisymmetrical (F) and symmetrical (D) case are presented. Any value from the table should be divided by universal factor $\sqrt{5\nu^2+16\nu+9}$ which we omit for brevity. Definition of isoscalar factors correspond to conventions of [45] and reduce to the usual ones at $N_c = 3$

	$K\Sigma^* \to \Delta$	$\eta\Delta \to \Delta$	$\pi\Delta \to \Delta$	$K\Xi^* \to \Sigma^*$	$\pi\Sigma^* \to \Sigma$	$\eta\Sigma^* \to \Sigma^*$
F	$\frac{\sqrt{5(\nu+4)(3\nu+2)}}{2\sqrt{(\nu+2)(\nu+6)}}$	$-\frac{\sqrt{5}(\nu^2+5\nu+3)}{\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{\nu^2+7\nu+15}{\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{2\sqrt{5(\nu+3)(2\nu+3)}}{3\sqrt{(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(\nu^2+5\nu+12)}{2\sqrt{3(\nu+2)(\nu+6)}}$	$-\frac{\sqrt{5}\nu(\nu+3)}{\sqrt{2(\nu+2)(\nu+6)}}$
D	$-\frac{\sqrt{\nu}}{2}$	$-\sqrt{\frac{\nu(\nu+4)}{2}}$	$\sqrt{\frac{5\nu(\nu+4)}{2}}$	$-\frac{2\sqrt{\nu(\nu+3)}}{3\sqrt{\nu+4}}$	$-\frac{(19+5\nu)\sqrt{\nu}}{2\sqrt{\nu+4}}$	$-\frac{(\nu+5)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$
	$K\Delta \to \Sigma^*$	$\pi\Xi^* \to \Xi^*$	$\eta\Xi^* \to \Xi^*$	$K\Omega \to \Xi^*$	$K\Sigma^* \to \Xi^*$	$K\Xi^* \to \Omega$
F	$\frac{\sqrt{5(\nu+4)(2\nu+3)}}{\sqrt{3(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(\nu^2+3\nu+9)}{3\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(3-\nu-\nu^2)}{\sqrt{2(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(2\nu+3)}{\sqrt{2(\nu+6)}}$	$\frac{\sqrt{10(\nu+3)(2\nu+3)}}{\sqrt{3(\nu+2)(\nu+6)}}$	$\frac{\sqrt{5}(2\nu+3)}{\sqrt{\nu+6}}$
D	$-\sqrt{\frac{\nu}{3}}$	$-\frac{(5\nu+18)\sqrt{\nu}}{3\sqrt{2(\nu+4)}}$	$-\frac{(\nu+6)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$	$-\sqrt{\frac{\nu(\nu+2)}{2(\nu+4)}}$	$-\sqrt{\frac{2\nu(\nu+3)}{3(\nu+4)}}$	$-\sqrt{\frac{\nu(\nu+2)}{\nu+4}}$
	$\eta\Omega \to \Omega$					
F	$\frac{(3-\nu)\sqrt{5(\nu+2)}}{\sqrt{2(\nu+6)}}$					
D	$-\frac{(\nu+7)\sqrt{\nu}}{\sqrt{2(\nu+4)}}$					

TABLE IV. Isoscalar factors for $8_M \otimes$ "10"$_B$ $\to$ "10"$_B$ at arbitrary $N_c = 2\nu + 3$. Two factors for antisymmetrical (F) and symmetrical (D) case are given. The table value should be divided by factor $\sqrt{3\nu^2 + 16\nu + 15}$. Definition of isoscalar factors correspond to conventions of [45] and reduce to the usual ones at $N_c = 3$. In particular, at $N_c = 3$ only symmetrical representation survives