

Hermitian Young Operators

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Abstract. Starting from conventional Young operators we construct Hermitian operators which project orthogonally onto irreducible representations of the (special) unitary group.

I Introduction

Young operators play a crucial role in the representation theories of the symmetric group S_n as well as of $GL(N)$ and the classical compact Lie groups. They first appear when decomposing the group algebra $\mathcal{A}(S_n)$ of the symmetric group S_n into a direct sum of minimal (left) ideals, thus completely reducing the regular representation and constructing all irreducible representations. The Young operators are primitive idempotents generating these ideals. They can be written down in a straightforward way from the corresponding (standard) Young tableaux.

When reducing product representations of $GL(N)$ or classical compact Lie groups, Young operators appear again as projectors onto irreducible subspaces, allowing, e.g., the construction of all irreducible representations of classical Lie groups like $U(N)$ or $SU(N)$. These topics are covered by many classical text books.^{1,2,3,4,5}

Young operators can be cast in a diagrammatic form, resembling Feynman diagrams, which is particularly handy in applications to general relativity and non-abelian quantum field theories.⁶ The first treatment of this kind, known to us, is by Penrose.⁷ More elaborate accounts, which we will also refer to in the following, can be found in Refs. 8 & 9. Following Cvitanović⁹ we refer to expressions in this diagrammatic notation as birdtracks.

In applications to quantum chromodynamics (QCD) a Young operator in the group algebra of S_n can be used in order to project from (the color part of) the Hilbert space for n quarks (or for n anti-quarks) onto an irreducible subspace invariant under $SU(3)$. A conventional Young operator, however, is in general not Hermitian, and thus this projection is not orthogonal. But in applications to QCD orthogonality is a desirable property.

In QCD the color of individual quarks or gluons is never observed as they are confined into hadrons, which are color singlets. QCD color space may therefore always be treated by summing over the color degrees of freedom of all incoming and outgoing particles. Consequently, it is useful to expand scattering amplitudes into a basis of overall color singlet states. Projection operators onto irreducible subspaces are color singlets. Thus,

they are a convenient starting set of states which one would like to extend to a basis. If one does so, say for a process with n incoming and n outgoing quarks, using conventional (i.e. non-Hermitian) Young operators, the resulting basis will in general be non-orthogonal – which is a serious drawback for explicit calculations. If, however, Hermitian Young operators are used then the resulting multiplet basis is orthogonal, see, e.g., Appendix B of Ref. 10 or Secs. 2 & 3 of Ref. 11 for the case of 3 quarks going to 3 quarks. Therefore, one would like to replace conventional Young operators with Hermitian alternatives.

Hermitian Young operators for up to 3 quarks, i.e. for $n \leq 3$, are given in Refs. 12 & 10, in diagrammatic notation. Canning¹² also sketches (without proof) the idea underlying the general construction to be discussed below. Cvitanović⁹ describes a general algorithm for constructing Hermitian Young operators, and explicitly gives all birdtrack diagrams for complete sets of Hermitian Young operators with $n \leq 4$, see Fig. 9.1 in Ref. 9. His method requires the solutions to certain characteristic equations, which tend to become more complicated for larger n – whereas conventional Young operators always can be constructed directly from the Young tableaux.

Our main result, as summarized in Theorem 6 below, is a recursive algorithm for directly constructing a Hermitian Young operator corresponding to any given standard Young tableau. This allows for a complete decomposition of the n -quark Hilbert space into an orthogonal sum of irreducible $SU(N)$ -invariant subspaces.

The article is organized as follows. We review some properties of the group algebra of finite groups in Sec. II and of Young operators in Sec. III. In Sec. IV we discuss product representations of $SU(N)$ whereby we introduce the diagrammatic birdtrack notation and recall how the dimensions of irreducible $SU(N)$ representations corresponding to standard Young tableaux are calculated diagrammatically. Our main results are stated and proven in Sec. V, which we conclude with some examples. Some detailed birdtrack calculations have been moved to the Appendix.

II The group algebra of a finite group

We briefly recount some basic properties of the group algebra of a finite group in order to fix our notation; details can be found in standard text books.^{2,3,4,5}

For a finite group G we define its group algebra $\mathcal{A}(G)$ as the \mathbb{C} -vector space spanned by the group elements with multiplication induced from the group multiplication. The group algebra carries the regular representation of the group, which contains all irreducible representations. Minimal left ideals of $\mathcal{A}(G)$ are also irreducible G -invariant subspaces, and hence carry the irreducible representations of G . Left ideals are generated by right multiplication with idempotents $e \in \mathcal{A}(G)$. Primitive idempotents generate minimal left ideals. Below we will make use of the following two statements.

Lemma 1. *An idempotent e is primitive if and only if $\forall r \in \mathcal{A}(G)$ there exists $\lambda_r \in \mathbb{C}$ such that $ere = \lambda_r e$.*

Lemma 2. *Two primitive idempotents e_1 and e_2 generate equivalent irreducible representations if and only if there exists an $r \in \mathcal{A}(G)$ such that $e_1 r e_2 \neq 0$.*

Proofs can be found, e.g., in App. III of Ref. 5. With a set of primitive idempotents e_j , satisfying $e_j e_k = \delta_{jk} e_j$ and $\sum_j e_j = \mathbb{1}$, the regular representation can be reduced completely, and all irreducible representations of G can be constructed. For $G = S_n$, the symmetric group, this reduction of the regular representation is achieved in terms of Young operators.

III Young operators

Young diagrams, Young tableaux and Young operators and their properties are discussed in many excellent text books.^{1,2,3,4,5,9} Nevertheless – as scope, conventions, and notation vary considerably between different presentations – we find it convenient to summarize a few definitions and results in order to keep the presentation reasonably self-contained and to fix our notation.

A Young diagram is an arrangement of n boxes in r rows of lengths $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. We also denote by $s = \lambda_1$ the number of columns of the diagram, and by $\mu_1 \geq \mu_2 \geq \dots \geq \mu_s$ the lengths of its columns. A Young tableau Θ is a Young diagram with each of the numbers $1, \dots, n$ written into one of its boxes. For standard Young tableaux the numbers increase within each row from left to right and within each column from top to bottom. We denote the set of all standard Young tableaux with n boxes by \mathcal{Y}_n , e.g.

$$\mathcal{Y}_2 = \left\{ \begin{array}{|c|} \hline 1\ 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right\}, \quad \mathcal{Y}_3 = \left\{ \begin{array}{|c|c|c|} \hline 1\ 2\ 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1\ 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1\ 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right\}. \quad (1)$$

Removing the box containing the number n from $\Theta \in \mathcal{Y}_n$ one obtains a standard tableau $\Theta' \in \mathcal{Y}_{n-1}$.

For $\Theta \in \mathcal{Y}_n$ let $\{h_\Theta\}$ be the set of all horizontal permutations, i.e. $h_\Theta \in S_n$ leaves the sets of numbers appearing in the same row of Θ invariant. Analogously, vertical permutations v_Θ leave the sets of numbers appearing in the same column of Θ invariant. Then the Young operator Y_Θ is defined in terms of the row symmetrizer, $s_\Theta = \sum_{\{h_\Theta\}} h_\Theta$, and the column anti-symmetrizer, $a_\Theta = \sum_{\{v_\Theta\}} \text{sign}(v_\Theta) v_\Theta$, as

$$Y_\Theta = \frac{1}{|\Theta|} s_\Theta a_\Theta. \quad (2)$$

The normalization factor is given by the product of hook lengths of the boxes of Θ ,

$$|\Theta| = \prod_{j=1}^r \prod_{k=1}^{\lambda_j} (\lambda_j - k + \mu_k - j + 1). \quad (3)$$

For two Young tableaux $\Theta \neq \vartheta$ of the same shape we have $|\Theta| = |\vartheta|$, i.e. the normalization depends only on the corresponding Young diagram. For instance, writing the hook lengths into the boxes of $\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}$ we obtain

$$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 & \\ \hline \end{array} \quad \text{and thus} \quad \left| \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right| = 24. \quad (4)$$

For $\Theta \in \mathcal{Y}_n$ the corresponding Young operator $Y_\Theta \in \mathcal{A}(S_n)$ is a primitive idempotent. Different Young operators $Y_\Theta, Y_\vartheta \in \mathcal{A}(S_n)$ satisfy $Y_\Theta Y_\vartheta = 0$ if the corresponding Young tableaux have different shapes. For small n one even has

$$Y_\Theta Y_\vartheta = \delta_{\Theta\vartheta} Y_\Theta \quad \forall \Theta, \vartheta \in \mathcal{Y}_n \text{ and } \forall n \leq 4, \quad (5)$$

a property to which we refer as *transversality*. For $n \geq 5$, however, transversality no longer holds in general, the standard example^{2,4} being the two 5-box diagrams $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}$ and $\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}$ for which one obtains

$$Y_{\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}} Y_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}} = 0 \quad \text{but} \quad Y_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}} Y_{\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}} \neq 0. \quad (6)$$

Different methods for curing this are in use. One option is to choose a different set of (non-standard) Young tableaux such that property (5) is reestablished; for $n = 5$, see e.g. Sec. II.3.6 in Ref. 4. A different way out, which straightforwardly extends to larger n , is to construct the Young operators corresponding to standard tableaux in a certain order and to subtract multiples of already constructed operators, such that (5) holds again, see e.g. Sec. 5.4 in Ref. 2 or Sec. II.3.6 in Ref. 4. The Hermitian Young operators, which we will introduce below, automatically satisfy (5) for arbitrary n .

IV Tensor products, birdtracks and invariant tensors

Besides their crucial role for constructing the irreducible representations of the symmetric group S_n , Young operators are equally important for the representation theory of the general linear group $GL(N)$ and the classical compact groups $U(N)$, $SU(N)$, $O(N)$, $SO(N)$ and $Sp(N)$. Motivated by applications in QCD, in the following we will always speak about $SU(N)$, although all results hold verbatim for $U(N)$, and similarly for $GL(N)$.

Let $V = \mathbb{C}^N$ be the carrier space of the defining representation of $SU(N)$, and denote by \bar{V} its dual. Tensor products $V^{\otimes n}$ carry a product representation of $SU(N)$ and – by permutation of the contributions in the different factors – a representation D of S_n as well as its group algebra $\mathcal{A}(S_n)$. The same holds for tensor products $\bar{V}^{\otimes n}$ of the dual.

V is naturally endowed with a scalar product $\langle \cdot, \cdot \rangle_V$ which is invariant under the action of $SU(N)$, i.e.

$$\langle gv, gw \rangle_V = \langle v, w \rangle_V \quad \forall v, w \in V \quad \forall g \in SU(N). \quad (7)$$

This also induces scalar products on the dual \bar{V} and on tensor products. Young operators act as projectors onto $SU(N)$ -invariant irreducible subspaces of $V^{\otimes n}$. Irreducible representations of $SU(N)$ corresponding to different Young diagrams are inequivalent. In general, however, Young operators are not Hermitian with respect to the scalar product induced by (7) on the product space $V^{\otimes n}$. Thus, these projections are in general not orthogonal.

In the diagrammatic birdtrack notation, we assign to $P : V^{\otimes n} \rightarrow V^{\otimes n}$, i.e. $P \in V^{\otimes n} \otimes \bar{V}^{\otimes n}$, a diagram with $2n$ external lines, which can be translated to index notation as follows,

$$\begin{array}{c} a_1 \longrightarrow \\ \vdots \\ a_n \longrightarrow \end{array} \boxed{P} \begin{array}{c} \longrightarrow b_1 \\ \vdots \\ \longrightarrow b_n \end{array} = P^{a_1 \dots a_n} b_1 \dots b_n. \quad (8)$$

Correspondingly, we represent an operator $\overline{V}^{\otimes n} \rightarrow \overline{V}^{\otimes n}$ by a diagram with all arrows pointing in the opposite direction. The diagram of the Hermitian conjugate of an operator is obtained from its diagram by mirroring about a vertical axis and inverting all arrows. Index contractions are performed by joining lines. See e.g. App. A of Ref. 11 for a brief summary of our conventions or Cvitanović⁹ for a more detailed account of the birdtrack notation.

Symmetrization or anti-symmetrization over a set of indices is indicated by a white or black bar, respectively, e.g.

$$\begin{aligned} \text{---} \boxed{\text{---}} \text{---} &= \frac{1}{2!} \left(\text{---} \text{---} + \text{---} \text{---} \right), \\ \text{---} \blacksquare \text{---} &= \frac{1}{3!} \left(\text{---} \text{---} - \text{---} \text{---} - \text{---} \text{---} - \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} \right), \end{aligned} \quad (9)$$

whereby we include a factorial as normalization factor. For all diagrams appearing in the following all arrows on external lines will point in the same direction. We therefore omit arrows from now on. The following recursion relations, which are derived in Chap. 6 of Ref. 9, will be used below,

$$\begin{aligned} p \text{ lines } \left\{ \begin{array}{l} \text{---} \boxed{\text{---}} \text{---} \\ \vdots \\ \text{---} \boxed{\text{---}} \text{---} \end{array} \right. &= \frac{1}{p} \begin{array}{l} \text{---} \boxed{\text{---}} \text{---} \\ \vdots \\ \text{---} \boxed{\text{---}} \text{---} \end{array} + \frac{p-1}{p} \begin{array}{l} \text{---} \boxed{\text{---}} \text{---} \\ \vdots \\ \text{---} \boxed{\text{---}} \text{---} \end{array}, \\ q \text{ lines } \left\{ \begin{array}{l} \text{---} \blacksquare \text{---} \\ \vdots \\ \text{---} \blacksquare \text{---} \end{array} \right. &= \frac{1}{q} \begin{array}{l} \text{---} \blacksquare \text{---} \\ \vdots \\ \text{---} \blacksquare \text{---} \end{array} - \frac{q-1}{q} \begin{array}{l} \text{---} \blacksquare \text{---} \\ \vdots \\ \text{---} \blacksquare \text{---} \end{array}. \end{aligned} \quad (10)$$

With this notation the Young operators (2) of Sec. III can be written diagrammatically,^{8,9} e.g. we find

$$Y_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}} = 2 \begin{array}{c} \text{---} \boxed{\text{---}} \text{---} \\ \text{---} \blacksquare \text{---} \\ \text{---} \boxed{\text{---}} \text{---} \\ \text{---} \blacksquare \text{---} \end{array} \quad \text{and} \quad Y_{\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}} = 2 \begin{array}{c} \text{---} \boxed{\text{---}} \text{---} \\ \text{---} \blacksquare \text{---} \\ \text{---} \boxed{\text{---}} \text{---} \\ \text{---} \blacksquare \text{---} \end{array}. \quad (11)$$

Notice that the pre-factor equals $3!(2!)^3/|\square\square\square| = 2$ in agreement with (2) and the normalization (9) of the (anti-)symmetrizers. From the birdtrack diagrams the first equation in (6) is obvious, as in this product the first two lines both join the same anti-symmetrizer and symmetrizer.

Strictly speaking, we should only write Y_{Θ} when considering Young operators as elements of the group algebra $\mathcal{A}(S_n)$. Viewing them as linear maps $V^{\otimes n} \rightarrow V^{\otimes n}$ they should be denoted by $D(Y_{\Theta})$. However, in order to simplify our notation, we omit the D in the following.

We are now in the position to state the dimension formula for the $SU(N)$ -representations.

Using the recursion relations (10), the diagram in Eq. (15) can be readily reduced to

$$\begin{aligned}
 \begin{array}{c} \vdots \\ \text{all other} \\ \text{(anti-)symmetrizers of } Y_\Theta \\ \vdots \end{array} &= \frac{N+p-q}{pq} \overbrace{\begin{array}{c} \vdots \\ \text{all other} \\ \text{(anti-)symmetrizers of } Y_\Theta \\ \vdots \end{array}}^{:=B_{\Theta'}} \\
 &- \frac{(p-1)(q-1)}{pq} \begin{array}{c} \vdots \\ \text{all other} \\ \text{(anti-)symmetrizers of } Y_\Theta \\ \vdots \end{array},
 \end{aligned} \tag{16}$$

where the first term is proportional to $Y_{\Theta'}$, with Θ' being the Young tableau which one obtains by removing from Θ the box with highest number. We temporarily denote the birdtrack part of this term by $B_{\Theta'}$. Elvang et al. show^{8,9} that the second term of Eq. (16) vanishes since on the outside all lines are connected to (anti-)symmetrizers in the same way as in $Y_{\Theta'}$, but internally the leftmost symmetrizer and the rightmost anti-symmetrizer are connected by a line, which automatically leads to a vanishing diagram since there can be no such connection within $Y_{\Theta'}$. Collecting all contributions one finds

$$\begin{aligned}
 \text{tr}' Y_\Theta &= \frac{N+p-q}{pq} \frac{\prod_{j=1}^r \lambda_j! \prod_{k=1}^s \mu_k!}{|\Theta|} B_{\Theta'} = (N+p-q) \frac{|\Theta'|}{|\Theta|} \frac{\prod_{j=1}^{r'} \lambda'_j! \prod_{k=1}^{s'} \mu'_k!}{|\Theta'|} B_{\Theta'} \\
 &= (N+p-q) \frac{|\Theta'|}{|\Theta|} Y_{\Theta'},
 \end{aligned} \tag{17}$$

where we have used that $(\prod_{j=1}^r \lambda_j!)/p = \prod_{j=1}^{r'} \lambda'_j!$ since the row lengths λ'_j of Θ' are the same as those of Θ except for the row from which one removes a box, which has length p in Θ but length $(p-1)$ in Θ' . The analogous statement holds for the column lengths and the factor q . Taking the trace, and observing that $(N+p-q)$ is the contribution to $f_\Theta(N)$ in (13) coming from the box which distinguishes Θ from Θ' , we get

$$\text{tr} Y_\Theta = (N+p-q) \frac{|\Theta'|}{|\Theta|} \text{tr} Y_{\Theta'} = (N+p-q) \frac{|\Theta'|}{|\Theta|} \frac{f_{\Theta'}(N)}{|\Theta'|} = \frac{f_\Theta(N)}{|\Theta|}, \tag{18}$$

which concludes the induction step. \square

We continue this section by discussing properties of invariant tensors⁹ following from Schur's lemma.

Definition. (Invariant tensor)

Let W_1 and W_2 be vector spaces carrying representations Γ_1 and Γ_2 of $SU(N)$ and let $T : W_1 \rightarrow W_2$ be linear. T is called *invariant tensor* if

$$T \circ \Gamma_1(g) = \Gamma_2(g) \circ T \quad \forall g \in SU(N). \tag{19}$$

REMARKS:

1. Invariant tensors are defined in the same way for other compact Lie groups.
2. In the following we are mainly interested in the case $W_1 = W_2 = V^{\otimes n}$ (or $\overline{V}^{\otimes n}$, leading to equivalent results).
3. Invariance of the scalar product (7) is equivalent to invariance of the tensor δ^j_k , $j, k = 1, \dots, N$, (in index notation), or invariance of the quark line $\longrightarrow : V \rightarrow V$ (in birdtrack notation).
4. Since

$$\Gamma(g) \circ D(r) = D(r) \circ \Gamma(g) \quad \forall g \in \text{SU}(N) \text{ and } \forall r \in \mathcal{A}(S_n) \quad (20)$$

every r in the group algebra of S_n corresponds to an invariant tensor $D(r) : V^{\otimes n} \rightarrow V^{\otimes n}$.

Lemma 4. (Schur)

Let $T : W_1 \rightarrow W_2$ be an invariant tensor and $\Gamma_{1,2}$ irreducible representations. If Γ_1 and Γ_2 are non-equivalent then $T = 0$. If Γ_1 and Γ_2 are equivalent and if $W_1 = W_2$ then T is a multiple of the identity.

For the proof see any standard book on group and representation theory. Simon's proof¹³ is particularly concise. Below we will use the following two consequences of Schur's lemma.

Corollary 5. Let $T : W \rightarrow W$ be both, an invariant tensor and a (possibly non-orthogonal) projection onto a subspace $U \subset W$ carrying the irreducible representation Γ of $\text{SU}(N)$.

- (i) Upon decomposing W into an orthogonal sum $\bigoplus_j W_j$ of subspaces W_j carrying irreducible representations Γ_j of $\text{SU}(N)$, with $\Gamma_1, \dots, \Gamma_m$ equivalent to Γ , and Γ_j , $j > m$ not equivalent to Γ , and representing $w \in W$ as

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \\ w_{m+1} \\ \vdots \end{pmatrix}, \quad w_j \in W_j, \quad (21)$$

T assumes the block structure

$$T = \begin{pmatrix} T_{11} & \cdots & T_{1m} & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \\ T_{m1} & \cdots & T_{mm} & & \\ 0 & \cdots & & 0 & \\ \vdots & & & & \ddots \end{pmatrix}. \quad (22)$$

The diagonal blocks T_{jj} , $j \leq m$, are proportional to the identity, and the T_{jk} vanish if $j > m$ or $k > m$ (as already displayed above).

- (ii) Let, furthermore, $P : W \rightarrow W$ be a Hermitian projector, $P^\dagger = P$, onto an invariant subspace $\widetilde{W} \subseteq W$, which contains only one subspace $\widetilde{U} \subseteq \widetilde{W}$ carrying a representation equivalent to Γ , and let $PTP \neq 0$. Then PTP is a multiple of the orthogonal projection onto \widetilde{U} .

PROOF: By applying Schur's lemma to each of the blocks $T_{jk} : W_k \rightarrow W_j$ (i) follows immediately. Since P is Hermitian its image is orthogonal to its kernel, $\text{im } P = (\ker P)^\perp$, i.e. W is an orthogonal sum, $W = \text{im } P \oplus \ker P$. We further decompose $\text{im } P$ into \widetilde{U} and its orthogonal complement $\widetilde{U}^\perp \subseteq \text{im } P$. Writing $w \in W$ as

$$w = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{with} \quad u_1 \in \widetilde{U}, \quad u_2 \in \widetilde{U}^\perp, \quad u_3 \in \ker P, \quad (23)$$

P takes the form $P = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and according to (i) we have

$$T = \begin{pmatrix} T_{11} & 0 & T_{13} \\ 0 & 0 & 0 \\ T_{31} & 0 & T_{33} \end{pmatrix}, \quad (24)$$

with T_{11} proportional to the identity. Now by direct computation $PTP = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is proportional to $\begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the orthogonal projection onto \widetilde{U} . \square

V Hermitian Young operators

As Young operators Y_Θ are elements of the group algebra $\mathcal{A}(S_n)$ they act as invariant tensors $V^{\otimes n} \rightarrow V^{\otimes n}$. Moreover, as already mentioned, each Young operator Y_Θ with $\Theta \in \mathcal{Y}_n$ projects onto an $SU(N)$ -invariant irreducible subspace of $V^{\otimes n}$. As an invariant tensor a Young operator cannot map an invariant subspace to a subspace carrying a non-equivalent representation. In general, however, a tensor product $V^{\otimes n}$ can contain several (irreducible) subspaces carrying equivalent representations. A projector onto such a subspace can then also contain parts mapping one invariant subspace to another one carrying an equivalent representation. Only through this mechanism can projectors onto invariant subspaces be non-Hermitian. This happens with conventional Young operators for $n \geq 3$, e.g.

$$Y_{\begin{smallmatrix} \boxed{12} \\ \boxed{3} \end{smallmatrix}} = \frac{4}{3} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad \text{and} \quad Y_{\begin{smallmatrix} \boxed{13} \\ \boxed{2} \end{smallmatrix}} = \frac{4}{3} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad (25)$$

are both non-Hermitian. One can, however, find the Hermitian operators^{12,10,9}

$$P_{\begin{smallmatrix} \boxed{12} \\ \boxed{3} \end{smallmatrix}} = \frac{4}{3} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad \text{and} \quad P_{\begin{smallmatrix} \boxed{13} \\ \boxed{2} \end{smallmatrix}} = \frac{4}{3} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad (26)$$

which also project onto irreducible subspaces carrying the representation corresponding to the Young diagram $\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}$ and which satisfy

$$P_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + P_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} = Y_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + Y_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}. \quad (27)$$

In the following theorem we show how to construct, for any given standard tableau, a Hermitian Young operator (and thus an orthogonal projection) by suitably projecting the corresponding conventional Young operator.

Theorem 6. *Let $P_\Theta := Y_\Theta$ for $\Theta \in \mathcal{Y}_2$ and*

$$P_\Theta := (P_{\Theta'} \otimes \mathbf{1})Y_\Theta(P_{\Theta'} \otimes \mathbf{1}) \quad \text{for } \Theta \in \mathcal{Y}_n, \quad n \geq 3, \quad (28)$$

where $\Theta' \in \mathcal{Y}_{n-1}$ denotes the standard Young tableaux obtained from Θ by removing the box with the highest number. Then the operators P_Θ are (a complete set of) Hermitian transversal Young projectors, i.e. they satisfy

- (i) $P_\Theta P_\vartheta = \delta_{\Theta\vartheta} P_\Theta \quad \forall \Theta, \vartheta \in \mathcal{Y}_n$ (transversality),
- (ii) $\text{tr} P_\Theta = \text{tr} Y_\Theta \quad \forall \Theta \in \mathcal{Y}_n$ (dimension),
- (iii) $\sum_{\Theta \in \mathcal{Y}_n} P_\Theta = \mathbf{1}^{\otimes n}$ (completeness) and
- (iv) $P_\Theta^\dagger = P_\Theta$ (Hermiticity).

REMARK: In the representation theory of the symmetric group the P_Θ are known as semi-normal idempotents^{14,15,16} and were introduced by Thrall¹⁵ who also proved properties (i) and (iii).

PROOF: As the construction is recursive we prove the properties by induction. For $n = 2$ the conventional Young operators are Hermitian and properties (i)–(iii) are also fulfilled trivially.

We begin with the dimensions of the irreducible subspaces, property (ii). Taking a partial trace of the last factor in P_Θ and using Eq. (17) yields

$$\begin{aligned} \text{tr}' P_\Theta &= \text{tr}' \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_\Theta} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right) = \text{tr}' \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{Y_\Theta} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right) \\ &= (N + p - q) \frac{|\Theta'|}{|\Theta|} \text{tr}' \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{Y_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right). \end{aligned} \quad (29)$$

From the recursive definition (28) it follows that

$$\text{tr}' \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right) = \text{tr}' \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta''}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right) = \text{tr}' \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta''}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{P_{\Theta'}} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right). \quad (30)$$

Therefore, we can insert factors of $P_{\Theta'} \otimes \mathbb{1}$ to the left and to the right of $Y_{\Theta'}$ in Eq. (29), leading to

$$\text{tr}' P_{\Theta} = (N + p - q) \frac{|\Theta'|}{|\Theta|} P_{\Theta'}^3 = (N + p - q) \frac{|\Theta'|}{|\Theta|} P_{\Theta'}, \quad (31)$$

i.e. the Hermitian Young operators P_{Θ} fulfill the same recursion relation as the conventional Young operators Y_{Θ} , cf. Eq. (17). Together with $\text{tr} P_{\Theta} = \text{tr} Y_{\Theta} \forall \Theta \in \mathcal{Y}_2$ this proves property (ii) by induction.

Hermiticity of the operators P_{Θ} follows from Corollary 5, part (ii): According to the induction hypothesis the $P_{\Theta'}$ are Hermitian (and thus project orthogonally), Y_{Θ} is an invariant tensor and a projector onto an irreducible subspace, and due to property (ii) $\text{tr} P_{\Theta} = \text{tr} Y_{\Theta} \neq 0 \Rightarrow P_{\Theta} \neq 0$. Hence, P_{Θ} is proportional to the orthogonal projection onto an irreducible invariant subspace. Moreover, since $\text{tr} P_{\Theta} = \text{tr} Y_{\Theta}$ fixes the dimension of this subspace the proportionality constant is unity, i.e. $P_{\Theta}^{\dagger} = P_{\Theta}$ and $P_{\Theta}^2 = P_{\Theta}$, thereby also establishing property (i) for the case $\Theta = \vartheta$.

In order to show property (i) it only remains to discuss the case $\Theta \neq \vartheta$, for which we have

$$P_{\Theta} P_{\vartheta} = (P_{\Theta'} \otimes \mathbb{1}) Y_{\Theta} (P_{\Theta'} \otimes \mathbb{1}) (P_{\vartheta'} \otimes \mathbb{1}) Y_{\vartheta} (P_{\vartheta'} \otimes \mathbb{1}). \quad (32)$$

If $\Theta' \neq \vartheta'$ then $P_{\Theta'} P_{\vartheta'} = 0$ and thus also $P_{\Theta} P_{\vartheta}$ vanishes. If $\Theta' = \vartheta'$ then Θ and ϑ have different shapes, and thus $Y_{\Theta} \sigma Y_{\vartheta} = 0 \forall \sigma \in S_n$ according to Lemma 2. Since the terms between Y_{Θ} and Y_{ϑ} in (32) are nothing but a linear combination of permutations, $P_{\Theta} P_{\vartheta}$ also vanishes in this case.

The remaining property (iii), completeness, follows straightforwardly from properties (i) and (ii). To this end define

$$P := \sum_{\Theta \in \mathcal{Y}_n} P_{\Theta}. \quad (33)$$

Then

$$P^2 \stackrel{(i)}{=} P \quad \text{and} \quad \text{tr} P \stackrel{(ii)}{=} N^n, \quad (34)$$

which proves property (iii) and thus concludes the proof of Theorem 6. \square

As an illustration consider once more the two 5-box standard diagrams $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}$ from Sec. III, whose conventional Young operators are not transversal, see Eqs. (6) and (11). Using the recursive construction of Theorem 6, the corresponding Hermitian Young operators can be shown to be

$$P_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}} = 2 \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \text{and} \quad P_{\begin{array}{|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}} = 2 \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}. \quad (35)$$

We give a step-by-step derivation in the Appendix. As guaranteed by Theorem 6 these projectors are not only Hermitian – which is manifest from their birdtrack diagrams being mirror symmetric – but also transversal since $\begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \end{array} = 0$.

ends but which can pass through each other. Keeping in mind that we can always rearrange the order in which the lines enter a given symmetrizer we find

$$(42)$$

Taking the average of these two expressions we finally obtain

$$P_{\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 \end{smallmatrix}} = 2 \text{ (diagram)} . \quad (43)$$

The Hermitian Young operator for the tableau $\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 \end{smallmatrix}$ can be obtained analogously. This time we need to consider the sequence $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 \end{smallmatrix}$. The first few projectors read

$$P_{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}} = Y_{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}} = \text{(diagram)} , \quad (44)$$

$$Y_{\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}} = \frac{4}{3} \text{(diagram)} , \quad (45)$$

$$P_{\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}} = \frac{4}{3} \text{(diagram)} , \quad (46)$$

$$Y_{\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}} = \frac{4}{3} \text{(diagram)} . \quad (47)$$

Similarly to the situation in Eq. (41) there is only one subspace of $\left(P_{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}} \otimes \mathbb{1}^{\otimes 2}\right) V^{\otimes 4}$ carrying an irreducible representation corresponding to the Young diagram $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. Invoking again Corollary 5 (ii) we get

$$P_{\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}} = \frac{4}{3} \text{(diagram)} . \quad (48)$$

Using a similar trick as in Eq. (42), this time exchanging the two symmetrizers, the projector can be written in manifestly Hermitian form,

$$P_{\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}} = \frac{4}{3} \text{(diagram)} . \quad (49)$$

For the next step we have to sandwich

$$Y_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}} = 2 \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (50)$$

between two copies of $P_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}}$,

$$P_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}} = \frac{32}{9} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (51)$$

In order to simplify this expression consider all lines passing through the dotted boxes. On one side they connect to two anti-symmetrizers, on the other side they connect to two symmetrizers. In order for this connection to be non-zero each anti-symmetrizer has to be connected to both symmetrizers. Thus, up to re-ordering the lines entering a given symmetrizer, there is a unique non-vanishing connection, i.e.

$$P_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}} \propto \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (52)$$

The normalization can be fixed by considering the square,

$$\left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right)^2 = \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (53)$$

Here the dotted box is equal to

$$\left(\frac{1}{2} Y_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}} \right)^2 = \frac{1}{4} Y_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}}, \quad (54)$$

and hence

$$P_{\begin{matrix} \boxed{135} \\ \boxed{24} \end{matrix}} = 2 \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (55)$$

is the desired projector.

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