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SUNSET INTEGRALS AT FINITE VOLUME

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Abstract

In this thesis we present a method for the calculation of finite volume corrections to the sunset diagram, for arbitrary masses and powers of the propagators. We also present numerical results showing the dependence of these corrections on the volume, which is an exponential decay for large volumes. The results are of high relevance since they can be used in Chiral Perturbation Theory to asses the precision of computations done within the framework of lattice Quantum Chromodynamics, for properties like particle masses and form factors, as well as to correct for the effects of a finite volume.

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Chapter 1

Introduction

The current theory describing the interactions between elementary particles, known as the Standard Model, has so far been very successful. Since its formulation in the 1960's it has been tested with higher and higher accuracy in all ways imaginable, and so far it agrees with every experiment performed. Despite its success it is however widely believed that it is not the correct description of nature but merely an effective theory that works well at the energies reached so far. This belief stems from theoretical considerations such as the large number of free parameters, the *ad hoc* inclusion of fermion masses, the fact that is does not include gravity and the problems with hierarchy and grand unification.

By its mathematical formulation the Standard Model is a renormalizable quantum field theory with a Lagrangian density invariant under the gauge group $U(1) \times SU(2) \times SU(3)$, which is spontaneously broken by the inclusion of a Higgs field. This has many consequences and in particular the renormalizability implies that the strength of the interactions, as measured by their respective coupling constants, vary with energy. As of yet there exists no analytical solution of the Standard Model equations of motion and for calculations one instead has to rely on perturbative methods. In the case of the electroweak interaction the couplings are at current experimental energies still small enough to treat interactions as a perturbation on the free particle Hamiltonian. Starting from this observation one can show that it is possible to expand the matrix element of any elementary process in powers of the coupling, which is the foundation of the Feynman diagram formalism that has become standard practice for calculations of scattering amplitudes and decay rates. This approach has been extremely successful as demonstrated for example by the agreement between experiment and calculation of the anomalous magnetic moment of the electron to more than ten significant figures.

For the strong interaction the situation is more complicated since the coupling grows rapidly with decreasing energy, and in fact it is by some people believed to have a Landau pole at low energies. Because of this the theory does no longer allow for a perturbative approach, because the expansion in powers of the coupling diverges for low energies and the Feynman diagram formalism breaks down. This is unfortunate since many of the properties that would be desirable to compute, such as the masses and internal workings of baryons, are in this non-perturbative region of the theory. One then has to try and find other ways of performing calculations.

The success of the Feynman diagram formalism is a strong motivation for trying to come up with another type of perturbative expansion, using some other small parameter instead of the coupling constant to assign importance to the different terms. In the model known as Chiral Perturbation Theory (χPT) the starting point is the symmetries of QCD in the limit of vanishing quark masses, which are used to construct an effective Lagrangian where the dynamical degrees of freedom are the particles of the pseudo-scalar octet (π, K, η). For momenta of less than about 1 GeV it is possible to define a new perturbative expansion using powers of mass and momentum as expansion parameters, and in this way one can recover much of the features of the ordinary Feynman diagram formalism. One difference is however that the theory no longer will be renormalizable, which allows for a great number of possible interaction terms that grow fast with the order of the expansion. Including all allowed terms it is still possible to renormalize the Lagrangian order by order, so that at each fixed order the theory is free from divergences. The fields of the χPT Lagrangian can also be coupled to the Standard Model gauge bosons such as W_μ or A_μ by introducing a covariant derivative, after which the procedure of computing matrix elements for decay or scattering processes follow ordinary Feynman rules only with different couplings.

Another possible way to treat the perturbation expansion breakdown is to abandon perturbation theory altogether, and try to solve the theory exactly. Although this cannot be done analytically there is a numerical approach known as lattice Quantum Chromodynamics (lQCD) where the functional integral of QCD is evaluated on a discrete set of space-time points, the lattice. This method has been used to compute many low-energy properties of QCD such as particle masses and form factors, and it can with some assumptions be used to support numerically the hypothesis of color confinement in the low energy limit. Due to its numerical character and the restrictions in computing power the lattice is forced to contain only a finite number of points and the computations have to be carried out in a finite volume. Since the underlying theory is formulated in infinite volume it is reasonable to expect effects due to this difference, which cannot however be addressed from within the theory itself.

With the increasing precision of lQCD computations it becomes more and more relevant to address the question of finite volume corrections to the results, and fortunately this can be done using the methods of χPT. Since what is usually of interest is the deviation from the infinite volume result it is desirable to use a framework in which this contribution can easily be identified, and how this can be done will be shown in more detail in Chapter 3.

The structure of this thesis is as follows: The first chapter will give a brief introduction to χPT starting with the symmetries of the QCD Lagrangian in the zero-mass limit, and how these give rise to Goldstone bosons by spontaneous symmetry breaking. This will be used to motivate an appropriate form of the effective Lagrangian and it will be explained how a successful scheme for power counting can be constructed. In the following chapters we successively develop a method for computing non-factorizable sunset integrals in finite volume. This is done by first considering some one-loop integrals in Chapter 3 which will serve as building blocks for the more complex integrals, and will also illustrate the methods in a simpler framework. We will present numerical values for these integrals in Chapter 4, and show that the finite volume corrections decay exponentially at large volumes. In Chapter 5 it will be shown that the sunset integrals can be divided into parts containing one or two quantized loop momenta respectively, and these parts will then be computed one at a time. The first type of integrals will turn out to contain non-local divergences which have to be identified and isolated from the finite part. In Chapter 6 some numerical results for the sunset integrals will be presented as well as a comparison of the different methods used.

Chapter 2

Chiral Perturbation Theory

2.1 QCD and chiral symmetry

It has been found through theoretical considerations and experimental verification that at an energy scale of about 1 GeV the strong coupling constant becomes of the order one and continues to increase towards lower energies. An unfortunate consequence of this is that the perturbative approach to QCD using Feynman diagrams breaks down, and in order to restore any predictive power to the theory it is necessary to find another way of performing computations. A general method for doing this is to try and identify those degrees of freedom that are of importance at the energy scale of interest, and which symmetries the theory exhibits there. That information can then be used to construct an effective theory including the right symmetries and particles. If we follow this procedure we are led to a description of low energy QCD phenomena in terms of light hadrons, which is known as Chiral Perturbation Theory (χPT). It is the basic principles of this theory and the computational formalism it leads to that is the interest of this chapter, and we will see at the end of it how the sunset diagram appears naturally in calculations of e.g. the pion mass. The presentation here will be along the lines of [1] and [2], where a more detailed discussion of these topics can be found. We have included this section since it is the motivation for the main part of this work, the finite volume integrals.

As a first step towards an effective theory we will show that in the limit of zero quark masses, the QCD Lagrangian exhibits symmetries additional to those of the Standard Model. Due to the natural hierarchy of the quark masses expressed through the relation $m_u, m_d, m_s \ll 1$ GeV $\ll m_c, m_b, m_t$, it is sufficient at low energies to consider the restriction of QCD to the three lightest flavors of quarks, since the contributions to quantum fluctuations from heavy quarks will be negligible. With these assumptions the Lagrangian becomes

$$\mathcal{L}_{QCD} = \sum_f \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}G_{\mu\nu,a}G^{\mu\nu,a} \tag{2.1}$$

where the sum is over the quark flavors u, d and s. In this equation the covariant derivative is given by

$$D_\mu = \partial_\mu - ig\sum_{a=1}^{8}\frac{\lambda_a}{2}A_{\mu,a} \tag{2.2}$$

where $A_{\mu,a}$ are the eight gluon fields described by the field strength tensor $G_{\mu\nu,a} = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} + gf_{abc}A_{\mu,b}A_{\nu,c}$. They have to be introduced in order for the Lagrangian to be invariant under local $SU(3)_C$ transformations of the color indices, which are generated by the Hermitian trace-less matrices $\lambda_a/2$. These matrices satisfy the commutation relations $[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}] = if_{abc}\frac{\lambda_c}{2}$ with structure constants f_{abc} showing that they form a closed Lie algebra. We will assume throughout that the λ_a are given in the Gell-Mann representation.

In the so-called chiral limit where the quark masses $m_f \to 0$, the Lagrangian becomes

$$\mathcal{L}^0_{QCD} = \sum_f i\bar{q}_f\not{D}q_f - \frac{1}{4}G_{\mu\nu,a}G^{\mu\nu,a}. \tag{2.3}$$

Making use of the chirality matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ we define the projections operators $P_R = (1+\gamma^5)/2$ and $P_L = (1-\gamma^5)/2$, that applied to the quark fields give a separation into left- and right-handed fields $q_R = P_Rq$ and $q_L = P_Lq$. With some use of the relation $\{\gamma^\mu, \gamma^5\} = 0$ it is easy to show that the left- and right-handed fields decouple in the chiral limit because

$$\begin{aligned}\bar{q}\not{D}q &= \bar{q}(P_R + P_L)\gamma^\mu D_\mu(P_R + P_L)q \\ &= \bar{q}P_R\gamma^\mu D_\mu P_Lq + \bar{q}P_L\gamma^\mu D_\mu P_Rq \\ &= \bar{q}_R\not{D}q_R + \bar{q}_L\not{D}q_L\end{aligned} \tag{2.4}$$

and so the Lagrangian can be rewritten as

$$\mathcal{L}^0_{QCD} = \sum_f i\bar{q}_{f,R}\not{D}q_{f,R} + i\bar{q}_{f,L}\not{D}q_{f,L} - \frac{1}{4}G_{\mu\nu,a}G^{\mu\nu,a}. \tag{2.5}$$

If we now perform independent global $U(1)$- and $SU(3)$-transformations of the flavor indices of the right- and left-handed fields, parametrized by Θ_R, Θ_L, Θ^a_R and Θ^a_L respectively, we find that

$$q_R \to \exp\left(-i\Theta^a_R\frac{\lambda_a}{2}\right)e^{-\Theta_R}q_R \tag{2.6}$$

$$i\bar{q}_R\not{D}q_R \to i\bar{q}_R\exp\left(+i\Theta^a_R\frac{\lambda_a}{2}\right)e^{+\Theta_R}\not{D}e^{-\Theta_R}\exp\left(-i\Theta^a_R\frac{\lambda_a}{2}\right)q_R = i\bar{q}_R\not{D}q_R$$

and similarly for the left-handed fields, so the Lagrangian remains invariant.

With this we have shown that the chiral Lagrangian is invariant under the group $G = SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$. When going from a classical to a quantum theory it however turns out that the part $U(1)_L \times U(1)_R$ is broken to the group $U(1)_V$ corresponding to performing the same transformation of the left- and right-handed fields, so the that the Lagrangian then only is invariant under the group $G = SU(3)_V \times SU(3)_A \times U(1)_V$.

In a more realistic theory where the quark masses are small but non-zero the symmetries discussed above will no longer be exact but only approximate. In fact a mass term of the form $-\bar{q}Mq$ with mass matrix

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{2.7}$$

will mix left- and right-handed fields according to $-\bar{q}_L M q_R - \bar{q}_R M q_L$ and so explicitly break the $SU(3)_{L,R}$-symmetry. This will lead to non-zero terms in the divergence of both the left- and right-handed octet Noether currents, and only the three singlet vector currents $V^\mu = \bar{q}\gamma^\mu q$ will still be conserved corresponding to the conservation of flavor.

2.2 Spontaneous symmetry breaking

Apart from the explicit symmetry breaking caused by the non-zero quark masses, low energy QCD is also believed to exhibit a spontaneous symmetry breaking of the group $G = SU(3)_L \times SU(3)_R \times U(1)_V$ to the smaller group $H = SU(3)_V \times U(1)_V$. This can be achieved dynamically and it has been shown that a sufficient condition for it to happen is that the vacuum expectation value $\langle 0|\bar{q}q|0\rangle$ of the scalar quark operator is non-zero. A symmetry is said to be spontaneously broken when the Lagrangian is invariant under a transformation but the corresponding conserved charges don't annihilate the vacuum; more technically, if $\phi(x) \to e^{-i\theta_a\lambda^a}\phi(x)$ implies $\mathcal{L}(\phi) \to \mathcal{L}(\phi)$ and the Noether charges Q_a associated with the transformation satisfies $Q_a|0\rangle \neq 0$, then the symmetry is spontaneously broken.

A theorem by Goldstone says that for each spontaneously broken global symmetry there will be a massless boson field $\phi(x)$ of spin zero, having the same transformation properties as the corresponding generators. In the case of low energy QCD there will be eigth pseudo-scalars fields $\phi_a(x)$ originating from the broken axial vector symmetry, whom have been shown to transform as an octet under $SU(3)_V$.

It is possible to establish an isomorphism between the space of Goldstone field configurations and the $SU(3)$-matrices. When we write down the Lagrangian we can therefore use such a matrix U to represent the Goldstone fields, which has the definite transformation behavior $U \to RUL^\dagger$ under G

and $U \to VUV^\dagger$ under H. Since every $SU(3)$-matrix is generated by some linear combination of the matrices λ_a we can collect the Goldstone fields in a matrix according to

$$\phi(x) = \sum \lambda_a \phi^a(x) = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}, \tag{2.8}$$

and then get the matrix U by simply exponentiating the above. Including the constant F_0 to make the argument dimensionless this can be written

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right). \tag{2.9}$$

We can now identify the components of the field ϕ with the particles in the light pseudo-scalar octet (π, K, η), so that Eq. 2.8 becomes

$$\phi(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{2.10}$$

2.3 The effective Lagrangian

With the dynamical degrees of freedom of the theory conveniently collected in the matrix U, and the symmetries of the underlying theory all identified, what remains is to construct an effective Lagrangian describing their interaction. It has been argued by Weinberg in [4] that to produce the most general S-matrix element the effective Lagrangian has to include all terms that are consistent with the symmetries of the underlying theory, which in our case means Lorentz symmetry and the chiral symmetry discussed above. This however allows for an infinite number of terms and in order for the theory to have any practical usage it is also necessary to construct a scheme for assessing the importance of each term. A natural way of doing this is to look for a small parameter that can be used to perform an expansion around the free Lagrangian, which in the case of χPT will be the external momenta since they are assumed to be much smaller than the chiral symmetry breaking scale of $\simeq$ 1 GeV. It can be shown that terms with quark masses in this scheme are of the order of squared momenta, and since all momenta are proportional to a derivative of some field the Lagrangian can be expanded according to

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \tag{2.11}$$

where the subscripts denote the maximum number of derivatives included in that term.

Applying the above principles to find and order all allowed terms the lowest order Lagrangian can be explicitly written as

$$\mathcal{L}_2 = \frac{F_0^2}{4}\,\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] + \frac{F_0^2}{4}\,\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger] \tag{2.12}$$

where the covariant derivative is $D_\mu U = \partial_\mu U - i r_\mu U + i l_\mu U$ and $\chi = 2B_0(M + s + ip)$. The fields s, p, r_μ and l_μ appearing in these equations are external fields introduced into the generating functional of QCD in order to allow for Green's functions to be obtained through functional derivatives. In order for the Green's functions of χPT to be as close as possible to the ones from QCD we have to couple the theory to the same external fields.

Letting the powers of p increase the number of allowed terms in the Lagrangian grows rapidly, so that $\mathcal{L}_4$ and $\mathcal{L}_6$ consists of 10 and 90 different terms respectively. These have to be taken into account in all calculations beyond tree level, with the first contributions from $\mathcal{L}_4$ at one-loop and from $\mathcal{L}_6$ at two-loop order.

As an application of the formalism developed so far, consider the computation of the pion two-point Green's function. From the general theory of quantum fields we know that the pole of this function defines the physical pion mass, which will be different from the bare mass parameter that appears in the Lagrangian. More explicitly

$$\langle 0|T\{\pi(x)\pi(y)^\dagger\}|0\rangle = \frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \tag{2.13}$$

where $\Sigma(p^2)$ is the pion self-energy and its mass is given by $m_\pi^2 = m^2 + \Sigma(p^2)$. If we want to find the pion mass to $\mathcal{O}(p^6)$ we have to evaluate the diagrams in Figure 2.1 (and many more) whom are all of the same chiral order, but with vertices coming from the different Lagrangians $\mathcal{L}_2$, $\mathcal{L}_4$ and $\mathcal{L}_6$ respectively. Once the vertex factors have been found the matrix elements for the three last diagrams are rather straightforward to write down, while the first sunset diagram is more complicated to compute due to the entangled propagators.

The interest in the sunset diagram comes from the fact that it is the only two-loop integral in this type of calculation that cannot be factorized into a product of one-loop integrals. Compare e.g. to the second diagram in Figure 2.1 which can be taken apart and computed as two separate integrals.

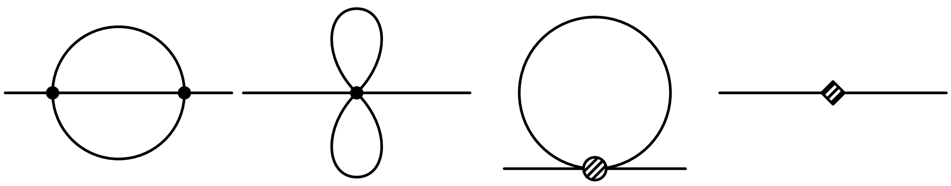

Figure 2.1: Feynman diagrams contributing to the pion self-energy

Chapter 3

Finite Volume One-Loop Integrals

3.1 Finite volume methods

In this chapter we will start the development of a method that will later be used to calculate sunset integrals at finite volume, for arbitrary values of particle masses and powers of the propagators. These kinds of calculations are of importance to assess the precision of lattice QCD (lQCD) calculations, which by their nature are set in a finite volume, and correct for these effects. There is of yet no finite volume formulation of lQCD in which it is possible to perform calculations, so if we want to know how the results depend on the volume we must try to address the problem within another framework, where the finite volume correction is easy to identify. This is possible to do using Chiral Perturbation Theory (χPT), and we will show how to find the volume dependence of the theory by looking at the deviation from the infinite volume limit. We will develop the method in steps in order to make it as transparent as possible, starting with a calculation of two types of one-loop integrals containing one and two propagators respectively. These will serve to illustrate the formalism but will also appear as building blocks in the more complicated integrals of Chapter 5. First however we need to find a way to separate the infinite and finite volume contributions to the integrals from each other.

Instead of moving in four-dimensional Minkowski space we now assume our particles to be confined to a box where the length of the spatial dimensions are L but the time dimension is much larger. We will also work with a Euclidean metric $\delta_{\mu\nu}$ since this is necessary in lQCD in order to change the functional integral from $\int e^{iS}$ to $\int e^{-S}$, making sure that it converges.

From traditional quantum mechanics it is known that the momentum of a particle confined to move in a box of side-length L will become quantized, and if the boundary conditions are chosen to be periodic it will take the values $p_n = 2\pi n/L$ for $n \in \mathbb{Z}$. These boundary conditions are good since they don't give special importance to any particular space-time point and don't require physical properties to go to zero at the boundary.

For the integral of a function $F(p)$ over all possible momenta the quantization means that we will have to replace the integral by a sum according to the rule

$$\int \frac{dp}{2\pi} F(p) \to \frac{1}{L} \sum_{n\in\mathbb{Z}} F(p_n) \equiv \int_V \frac{dp}{2\pi} F(p), \tag{3.1}$$

which also serves to define the symbol $\int_V$. This follows easily when going from a finite to an infinite volume, by noting that the distance between consecutive values in momentum space is $2\pi/L$ so that we can use a Riemann sum to approximate the integral according to

$$\frac{1}{L} \sum_{n\in\mathbb{Z}} F(p_n) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} F(p_n)\Delta p \to \int \frac{dp}{2\pi} F(p) \tag{3.2}$$

when $L \to \infty$. To identify the contribution from the infinite volume we can rewrite the sum in Eq. (3.1) above using Poisson's resummation formula $\sum_{n\in\mathbb{Z}} F(n) = \sum_{k\in\mathbb{Z}} \hat{F}(k)$ and the scaling property $F(ax) = \frac{1}{|a|}\hat{F}(\frac{x}{a})$ of the Fourier transform to find

$$\frac{1}{L} \sum_{n\in\mathbb{Z}} F(p_n) = \frac{1}{L} \sum_{n\in\mathbb{Z}} F(\frac{2\pi n}{L}) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \hat{F}(\frac{nL}{2\pi}) = \sum_{l} \int \frac{dp}{2\pi} e^{ilp} F(p) \tag{3.3}$$

where $l = kL$ for $k \in \mathbb{Z}$. When generalizing this to higher dimensions it becomes necessary to replace the integrals over all finite dimensions with sums over integrals according to the above.

Assuming the time-direction to be so large that it is effectively infinite, we will obtain a sum over each of the components of the four-vector $l_\mu = (0, jL, mL, nL)$. It is immediate to identify the infinite volume contribution as the term with $l_\mu = 0$, and in the following a summation over all values for which $l_\mu \neq 0$ will be denoted by a primed summation sign. In the case where the integral only depends on the length of the vector l_μ, the triple sum over (j, m, n) can be replaced with a single sum according to

$$\sum_{l}{}' F(l^2) = \sum_{k>0} x(k) F(k), \tag{3.4}$$

where $x(k)$ indicates how many times the value $l^2 = kL^2$ shows up during the summation. For functions depending on the components of l_μ no such simplification is possible, but on the other hand they will in many cases vanish due to symmetry arguments.

Some other relations that will often be used concern summations over the tensor structure $l_\mu l_\nu$ and generalized Gaussian integration formulas in arbitrary dimensions. If the product $l_\mu l_\nu$ appears multiplied with a function dependent only on l^2, we can show that

$$\sum l_\mu l_\nu F(l^2) = \frac{1}{3} t_{\mu\nu} \sum l^2 F(l^2) \tag{3.5}$$

where $t_{\mu\nu} = \mathrm{diag}(0,1,1,1)$ is an abbreviation introduced for notational convenience. This follows from the observation that for fixed μ the sum over l_ν is non-zero only when $\mu = \nu$. The Gaussian formulas that will be used are given by

$$\begin{aligned} \int \frac{d^d r}{(2\pi)^d} e^{-r^2} &= \frac{1}{(4\pi)^{d/2}} \\ \int \frac{d^d r}{(2\pi)^d} r_\mu r_\nu e^{-r^2} &= \frac{1}{(4\pi)^{d/2}} \frac{\delta_{\mu\nu}}{2} \end{aligned} \tag{3.6}$$

and can be derived by considering the tensor structure of the integrals and the volume of the unit sphere in d dimensions.

3.2 One propagator integrals

To illustrate the method above, we will apply it to a one-loop integral containing one propagator and of the general form

$$\lfloor X \rfloor = \int_V \frac{d^d r}{(2\pi)^d} \frac{X}{(r^2+m^2)^n} = \sum_{l_\mu} \int \frac{d^d r}{(2\pi)^d} \frac{X e^{il\cdot r}}{(r^2+m^2)^n}. \tag{3.7}$$

Here X is a structure in the set $\{1, r_\mu, r_\mu r_\nu\}$, dimensional regularization has been used with the convention $d = 4 - 2\epsilon$, and we have applied Eq. (3.3) to move the sum from r to l. The integral can be split into a sum of an infinite and a finite volume contribution according to $\lfloor X \rfloor = \lfloor X \rfloor^\infty + \lfloor X \rfloor^V$, where the infinite volume term corresponds to putting $l_\mu = 0$ in the expression above. In the following we will often use Schwinger's parametrization of the propagator

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\lambda\, \lambda^{n-1} e^{-a\lambda}, \tag{3.8}$$

and considering for the moment only the finite volume part and the case where $X = 1$ we find

$$\begin{aligned} \lfloor 1 \rfloor^V &= \sum_l{}' \int \frac{d^d r}{(2\pi)^d} \frac{e^{il\cdot r}}{(r^2+m^2)^n} \\ &= \sum_l{}' \int_0^\infty d\lambda \int \frac{d^d r}{\Gamma(n)(2\pi)^d} e^{il\cdot r} \lambda^{n-1} e^{-\lambda(r^2+m^2)}. \end{aligned} \tag{3.9}$$

Completing the square in the exponent and using the change of variables $r_\mu \to \sqrt{\lambda}(\tilde{r}_\mu - il_\mu/2\lambda)$ this becomes

$$\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)} \sum_l{}' \int_0^\infty d\lambda \lambda^{n-1-\frac{d}{2}} e^{-\frac{l^2}{4\lambda}-m^2\lambda} \int \frac{d^d\tilde{r}}{(2\pi)^d} e^{-\tilde{r}^2} \tag{3.10}$$

which can be evaluated with the help of the Gaussian integration formulas in Eq. (3.6) to give

$$\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)(4\pi)^{d/2}} \sum_l{}' \int_0^\infty d\lambda \lambda^{n-1-\frac{d}{2}} e^{-\frac{l^2}{4\lambda}-m^2\lambda}. \tag{3.11}$$

At this point we have a choice between performing the integral over λ or computing the sum over l_μ, resulting in a sum over modified Bessel functions or an integral over Jacobi theta functions respectively [9]. The first choice leads to

$$\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)(4\pi)^{d/2}} \sum_l{}' \mathcal{K}_{n-d/2}\left(\frac{l^2}{4}, m^2\right) \tag{3.12}$$

where the function $\mathcal{K}$ is used as a compact notation for the Bessel function and is defined in Appendix A. Using Eq. (3.4) and the full expression for the Bessel function the above becomes

$$\lfloor 1 \rfloor^V = \frac{2}{\Gamma(n)(4\pi)^{d/2}} \sum_{k>0} x(k) K_{n-\frac{d}{2}}(\sqrt{kL^2m^2}) \left(\frac{kL^2}{4m^2}\right)^{\frac{n}{2}-\frac{d}{4}}. \tag{3.13}$$

If we instead choose to evaluate the sum analytically using the rescaling $\lambda = L^2\bar{\lambda}/4$, the resulting integral is

$$\begin{aligned} \lfloor 1 \rfloor^V &= \frac{1}{\Gamma(n)(4\pi)^{d/2}} \left(\frac{L^2}{4}\right)^{n-\frac{d}{2}} \int_0^\infty d\lambda \lambda^{n-1-\frac{d}{2}} e^{-\bar{\lambda}\frac{m^2L^2}{4}} \left[\sum_{\mathbf{k}} e^{-\mathbf{k}^2/\bar{\lambda}} - 1\right] \\ &= \frac{1}{\Gamma(n)(4\pi)^{d/2}} \left(\frac{L^2}{4}\right)^{n-\frac{d}{2}} \int_0^\infty d\lambda \lambda^{n-1-\frac{d}{2}} e^{-\bar{\lambda}\frac{m^2L^2}{4}} \left[\theta_{30}(1/\bar{\lambda})^3 - 1\right]. \end{aligned} \tag{3.14}$$

where again the definition of θ_{30} can be found in Appendix A. Here the sum over the vector $\mathbf{k}$ has be split into a product of three sums over each of its components, which can be done since $\mathbf{k}^2 = k_1^2 + k_3^2 + k_3^2$. We have included a factor -1 in order to remove the term with $l_\mu = 0$ from the summation contained in the theta function.

It is good to note here that as long as we're interested only in the one-loop integrals the above expressions can be expanded without ambiguity around the reduced dimension, since all the divergences will be contained in a single term proportional to $1/\epsilon$. We will however see in the case of the sunset integrals that terms of order ϵ will combine with other terms of order $1/\epsilon$ to create finite terms, and due to this it is necessary to specify a convention for which terms are included in which order. Unfortunately this convention is not the same everywhere in the literature.

From Eq. (3.14) we see that as L grows larger only the values of $\bar{\lambda}$ close to zero will contribute to the integral, since the exponential factor in front the theta function will be very small otherwise. Because the largest term in the theta function is $e^{-1/\bar{\lambda}}$ the integral will then decay exponentially with increasing L, as will also be seen numerically in Chapter 4. This conclusion can be reached using the Bessel functions too, as will be demonstrated in Appendix A.

Consider now the integral where $X = r_\mu$, which can be written in general as

$$\lfloor r_\mu \rfloor^V = \sum_l{}' \int_0^\infty \frac{d\lambda d^d r}{\Gamma(n)(2\pi)^d} e^{il\cdot r} \lambda^{n-1} e^{-\lambda(r^2+m^2)} r_\mu. \tag{3.15}$$

Using the same change of variables as we did in Eq. (3.9) we immediately see that the integral consists of two terms, one proportional to $\tilde{r}_\mu$ and one to l_μ; more explicitly

$$\lfloor r_\mu \rfloor^V = \frac{1}{\Gamma(n)} \sum_l{}' \int_0^\infty d\lambda \lambda^{n-1-\frac{d}{2}} e^{-\frac{l^2}{4\lambda}-m^2\lambda} \int \frac{d^d\tilde{r}}{(2\pi)^d} e^{-\tilde{r}^2} \left(\frac{\tilde{r}_\mu}{\sqrt{\lambda}} + \frac{il_\mu}{2\lambda} \right) \tag{3.16}$$

from which we see that the first term is odd in $\tilde{r}_\mu$ and will vanish when the integral is performed. The other term will be zero since the integrand is invariant under the replacement $l_\mu \to -l_\mu$, so when we evaluate the sum all terms will cancel pairwise and give the final result

$$\lfloor r_\mu \rfloor^V = 0. \tag{3.17}$$

The last integral containing one propagator that will be calculated here is for $X = r_\mu r_\nu$, which can be written

$$\lfloor r_\mu r_\nu \rfloor^V = \sum_l{}' \int_0^\infty d\lambda \int \frac{d^d r}{(2\pi)^d \Gamma(n)} e^{il\cdot r} \lambda^{n-1} e^{-\lambda(r^2+m^2)} r_\mu r_\nu. \tag{3.18}$$

Once again the change of variable from Eq. (3.9) can be used to rewrite the last factor as

$$r_\mu r_\nu = \frac{\tilde{r}_\mu \tilde{r}_\nu}{\lambda} + i\frac{\tilde{r}_\mu l_\nu + \tilde{r}_\nu l_\mu}{2\lambda^{3/2}} - \frac{l_\mu l_\nu}{4\lambda^2}. \tag{3.19}$$

Upon integration the middle term will vanish since it is odd in the variable $\tilde{r}_\mu$, and the remaining parts can be evaluated using the relation

$$\int \frac{d^d\tilde{r}}{(2\pi)^d} \tilde{r}_\mu \tilde{r}_\nu F(\tilde{r}^2) = \frac{1}{d}\delta_{\mu\nu} \int \frac{d^d\tilde{r}}{(2\pi)^d} \tilde{r}^2 F(\tilde{r}^2) \tag{3.20}$$

together with Eq. (3.5) and (3.6). The full integral can then be written as

$$\lfloor r_\mu r_\nu \rfloor^V = \frac{1}{\Gamma(n)} \sum_l{}' \int_0^\infty d\lambda \int \frac{d^d\tilde{r}}{(2\pi)^d} \lambda^{n-1-\frac{d}{2}} e^{-\frac{l^2}{4\lambda} - m^2\lambda} e^{-\tilde{r}^2} \left(\frac{\tilde{r}^2}{d\lambda} \delta_{\mu\nu} - \frac{t_{\mu\nu}}{12\lambda^2} l^2 \right)$$
$$= \frac{1}{\Gamma(n)(4\pi)^{d/2}} \sum_l{}' \int_0^\infty d\lambda \lambda^{n-1-\frac{d}{2}} e^{-\frac{l^2}{4\lambda} - m^2\lambda} \left(\frac{\delta_{\mu\nu}}{2\lambda} - \frac{t_{\mu\nu}}{12\lambda^2} l^2 \right). \tag{3.21}$$

As in the case of the $\lfloor 1 \rfloor^V$ integral there is now a choice between integrating with respect to λ or summing over l_μ. With the first choice the final expression is

$$\lfloor r_\mu r_\nu \rfloor^V = \frac{1}{\Gamma(n)(4\pi)^{d/2}} \sum_{k>0} x(k) \left[\mathcal{K}_{n-\frac{d}{2}-1}(\sqrt{kL^2m^2}) \left(\frac{kL^2}{4m^2} \right)^{\frac{n-1}{2}-\frac{d}{4}} \delta_{\mu\nu} \right.$$
$$\left. - \mathcal{K}_{n-\frac{d}{2}-2}(\sqrt{kL^2m^2}) \left(\frac{kL^2}{4m^2} \right)^{\frac{n}{2}-1-\frac{d}{4}} \frac{t_{\mu\nu}}{6} l^2 \right] \tag{3.22}$$

using Eq. (3.4) on the second term. If we instead choose to evaluate the sum first the resulting integral becomes

$$\lfloor r_\mu r_\nu \rfloor^V = \frac{1}{\Gamma(n)(4\pi)^{d/2}} \left(\frac{L^2}{4} \right)^{n-1-\frac{d}{2}} \int_0^\infty d\bar{\lambda} \bar{\lambda}^{n-2-\frac{d}{2}} e^{-\bar{\lambda}\frac{m^2L^2}{4}} \times$$
$$\left\{ \frac{\delta_{\mu\nu}}{2} \left[\theta_{30}(1/\bar{\lambda})^3 - 1 \right] - \frac{t_{\mu\nu}}{\lambda} \theta_{31}(1/\bar{\lambda}) \theta_{30}(1/\bar{\lambda})^2 \right\}. \tag{3.23}$$

where the new function $\theta_{31}(x)$ is defined in Appendix A, and can be viewed as the derivative of the Jacobi theta function.

3.3 Two propagator integrals

Consider now the slightly more general case of a one-loop integral containing two different propagators, which can be written using Eq. (3.3) as

$$\langle X \rangle = \int \frac{d^dr}{(2\pi)^d} \frac{X}{(r^2+m_1^2)^{n_1}((r-p)^2+m_2^2)^{n_2}} \tag{3.24}$$
$$= \sum_l \int \frac{d^dr}{(2\pi)^d} \frac{Xe^{il\cdot r}}{(r^2+m_1^2)^{n_1}((r-p)^2+m_2^2)^{n_2}}.$$

As in the previous section we can divide the integral into two parts according to $\langle X \rangle = \langle X \rangle^\infty + \langle X \rangle^V$, where the first term corresponds to the infinite volume contribution where $l_\mu = 0$ and the second to the finite volume correction. The infinite volume part has be computed before and analytic expressions can be found e.g. in [10]. Considering only the finite volume part and making use of Schwinger's parametrization method we find

$$\begin{aligned}\langle X\rangle^V &= \sum_l{}' \int \frac{d^d r}{(2\pi)^d} \frac{X e^{il\cdot r}}{(r^2+m_1^2)^{n_1}((r-p)^2+m_2^2)^{n_2}} \\ &= \sum_l{}' \int \frac{d^d r}{(2\pi)^d} \int_0^\infty d\lambda_1 d\lambda_2 \frac{\lambda_1^{n_1-1}\lambda_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)} X e^{il\cdot r} e^{-\lambda_1(r^2+m_1^2)} e^{-\lambda_2((r-p)^2+m_2^2)}.\end{aligned} \tag{3.25}$$

It is now convenient to use the change of variables $\lambda_1 = x\lambda$ and $\lambda_2 = (1-x)\lambda$ and to introduce $y = 1-x$ to simplify the notation. We can then complete the square in the exponent with respect to r_μ, which together with the change of variables $r_\mu \to \tilde{r}_\mu/\sqrt{\lambda} + il_\mu/2\lambda + yp_\mu/\lambda$ gives the result

$$\begin{aligned}\langle X\rangle^V = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \sum_l{}' \int_0^\infty d\lambda \int_0^1 dx \int \frac{d^d\tilde{r}}{(2\pi)^d}\, x^{n_1-1} y^{n_2-1} \lambda^{n_1+n_2-1-\frac{d}{2}} \times \\ Z e^{-\frac{l^2}{4\lambda}+iyl\cdot p-\lambda\tilde{m}^2} e^{-\tilde{r}^2}\end{aligned} \tag{3.26}$$

where $\tilde{m}^2 = xm_1^2 + ym_2^2 + xyp^2$ and Z is X expressed in the new variable $\tilde{r}$. In the simplest case where $X = 1$ this can be immediately integrated using the formula of Eq. (3.6) to find

$$\langle 1\rangle^V = \sum_l{}' \int_0^1 dx \int_0^\infty d\lambda \frac{x^{n_1-1}y^{n_2-1}\lambda^{n_1+n_2-1-\frac{d}{2}}}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} e^{-\frac{l^2}{4\lambda}+iyl\cdot p-\lambda\tilde{m}^2}. \tag{3.27}$$

As in the previous section we can now either integrate with respect to λ or perform the sum over l_μ, but in distinction it is no longer possible to find a simple expression valid in an arbitrary frame of reference. In order to proceed we therefore assume that we are in the center-of-mass system defined by $l\cdot p = 0$ where the Bessel function representation is given by

$$\begin{aligned}\langle 1\rangle^V =& \frac{1}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} \sum_{k>0} \int_0^1 dx\, x^{n_1-1}y^{n_2-1} \times \\ & x(k) K_{n_1+n_2-\frac{d}{2}}(\sqrt{kL^2\tilde{m}^2}) \left(\frac{kL^2}{4\tilde{m}^2}\right)^{\frac{n_1+n_2}{2}-\frac{d}{4}},\end{aligned} \tag{3.28}$$

making use of Eq. (3.4). To express the integral in terms of theta functions it is best to make the rescaling $\lambda = L^2\bar{\lambda}/4$, after which the integral becomes

$$\begin{aligned}\langle 1\rangle^V =& \frac{1}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} \left(\frac{L^2}{4}\right)^{n_1+n_2-\frac{d}{2}} \int_0^1 dx \int_0^\infty d\bar{\lambda} \times \\ & x^{n_1-1}y^{n_2-1}\bar{\lambda}^{n_1+n_2-1-\frac{d}{2}} e^{-\bar{\lambda}\frac{L^2\tilde{m}^2}{4}} \left[\theta_{30}(1/\bar{\lambda})^3 - 1\right].\end{aligned} \tag{3.29}$$

It is not difficult to find results for another frame of reference, and since they are so similar we will not bother to write them out explicitly. In Chapter 4 however we will present numerical results for this integral both in and out of the center-of-mass frame.

There is nothing conceptually new introduced during the computation of the $X = r_\mu, r_\mu r_\nu$ integrals, so we will only state the results here. For the case where $X = r_\mu$ the integral becomes in the center-of-mass frame

$$\begin{aligned}\langle r_\mu \rangle^V =& \frac{1}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} \sum_{k>0} \int_0^1 dx\, x^{n_1-1} y^{n_2-1} \times \\ & x(k) y p_\mu K_{n_1+n_2-\frac{d}{2}}(\sqrt{kL^2\tilde{m}^2}) \left(\frac{kL^2}{4\tilde{m}^2}\right)^{\frac{n_1+n_2}{2}-\frac{d}{4}} \\ \langle r_\mu \rangle^V =& \frac{1}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} \left(\frac{L^2}{4}\right)^{n_1+n_2-\frac{d}{2}} \int_0^1 dx \int_0^\infty d\bar{\lambda} \times \\ & x^{n_1-1} y^{n_2-1} \bar{\lambda}^{n_1+n_2-1-\frac{d}{2}} e^{-\bar{\lambda}\frac{L^2\tilde{m}^2}{4}} y p_\mu \left[\theta_{30}(1/\bar{\lambda})^3 - 1\right],\end{aligned} \tag{3.30}$$

in terms of Bessel and Jacobi functions respectively. In the case where $X = r_\mu r_\nu$ we find in the center-of-mass frame that

$$\begin{aligned}\langle r_\mu r_\nu \rangle^V =& \frac{1}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} \left(\frac{L^2}{4}\right)^{n_1+n_2-\frac{d}{2}} \int_0^1 dx \int_0^\infty d\bar{\lambda} \times \\ & x^{n_1-1} y^{n_2-1} \bar{\lambda}^{n_1+n_2-1-\frac{d}{2}} e^{-\bar{\lambda}\frac{L^2\tilde{m}^2}{4}} \left[y^2 p_\mu p_\nu \left[\theta_{30}(1/\bar{\lambda})^3 - 1\right]\right. \\ & \left. + \frac{4\delta_{\mu\nu}}{2\bar{\lambda}L^2}\left[\theta_{30}(1/\bar{\lambda})^3 - 1\right] - \frac{t_{\mu\nu}}{\bar{\lambda}^2 L^2}\theta_{31}(1/\bar{\lambda})\theta_{30}(1/\bar{\lambda})^2\right]\end{aligned} \tag{3.31}$$

using Jacobi theta functions, while the same integral in terms of Bessel functions is given by

$$\begin{aligned}\langle r_\mu r_\nu \rangle^V =& \frac{1}{\Gamma(n_1)\Gamma(n_2)(4\pi)^{d/2}} \sum_{k>0} \int_0^1 dx\, x^{n_1-1} y^{n_2-1} \times \\ & x(k) \left[y^2 p_\mu p_\nu K_{n_1+n_2-\frac{d}{2}}(\sqrt{kL^2\tilde{m}^2}) \left(\frac{kL^2}{4\tilde{m}^2}\right)^{\frac{n_1+n_2}{2}-\frac{d}{4}} \right. \\ & + \frac{\delta_{\mu\nu}}{2} K_{n_1+n_2-1-\frac{d}{2}}(\sqrt{kL^2\tilde{m}^2}) \left(\frac{kL^2}{4\tilde{m}^2}\right)^{\frac{n_1+n_2-1}{2}-\frac{d}{4}} \\ & \left. + k\frac{t_{\mu\nu}}{12} K_{n_1+n_2-2-\frac{d}{2}}(\sqrt{kL^2\tilde{m}^2}) \left(\frac{kL^2}{4\tilde{m}^2}\right)^{\frac{n_1+n_2}{2}-1-\frac{d}{4}} \right].\end{aligned} \tag{3.32}$$

As in the case with the simpler integrals there is no simple expression that holds in a general frame, but we can still evaluate the integrals numerically once a direction for the momentum has been chosen. These numerical results will be presented in the next chapter.

Chapter 4

One-Loop Numerical Results

In the previous chapter it was shown that the finite volume corrections to the one-loop integrals can be given formal expressions in terms of either modified Bessel functions or Jacobi theta functions. To estimate the importance of these corrections it remains to numerically evaluate the integrals and relate them to the infinite volume contribution, which will be done in detail in this chapter.

In the center-of-mass system the evaluation of the integrals is straightforward using the Bessel function representation, since we can apply Eq. (3.4) to reduce the sum over the components of l_μ to a single sum, and then compute the integrals using the MATHEMATICA function `BesselK`. This was done using the maximum value $k = 200$ for the summation variable, which is more than enough to make sure that the sum converges. In a moving frame no simplification similar to Eq. (3.4) is possible, and we have to sum over each component of l_μ separately. To keep the computation time reasonable we had to reduce the maximum value of the summation variables to $|l| \simeq 10$, which means that for values of $L \lesssim 2$ fm the sum will not fully converge.

When using theta functions we can evaluate the integrals using the built-in functions `EllipticTheta` and `EllipticThetaPrime`. We then have to make sure to remove the contribution from the term with $l_\mu = 0$ corresponding to infinite volume, which is otherwise included in these functions. This is done by replacing θ_{30}^3 by $\theta_{30}^3 - 1$ while leaving the derivatives untouched, since in the later case the $l_\mu = 0$ term will be zero due to the extra factors of k. The integrals can then be computed to arbitrary precision both in the center-of-mass and moving frames, without any convergence problems.

In the figures that follow the integrals are given as functions of L for different directions of the external momentum p and values of the particle masses m_1 and m_2, with the powers of the propagators as $n_1 = n_2 = 1$ and $p^2 = 0.015\ \mathrm{GeV}^2$. In a physical calculation only momenta with the values $p = 2\pi n/L$ of the components would be allowed, but here we have used the same momentum for all values of L to simplify the presentation.

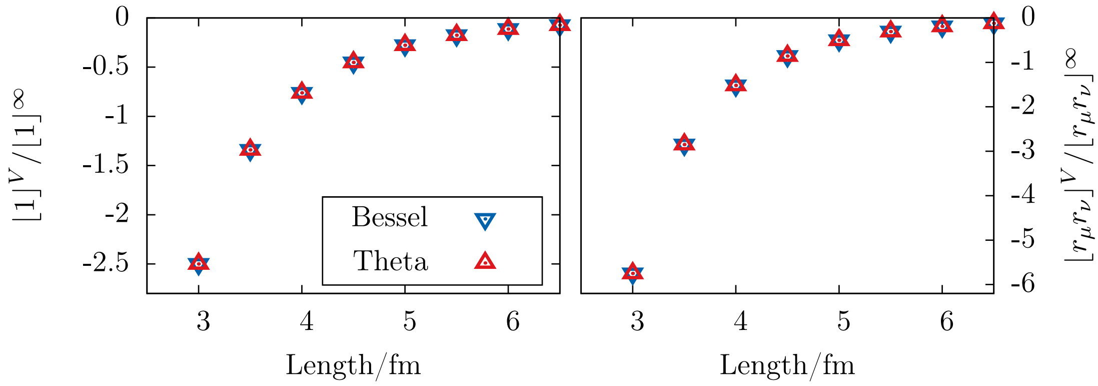

Figure 4.1: Relative correction to the integrals $\lfloor 1 \rfloor$ and $\lfloor r_\mu \rfloor$ for $m = 0.1$ GeV in the center-of-mass frame.

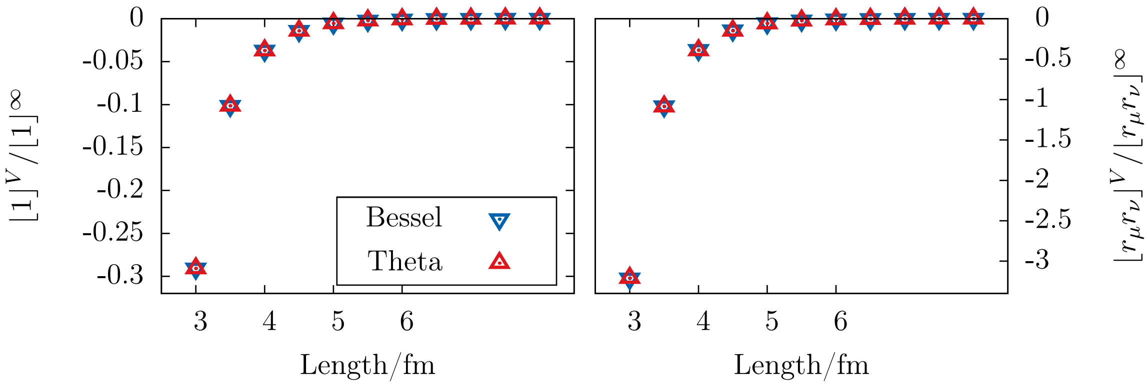

Figure 4.2: Relative correction to the integrals $\lfloor 1 \rfloor$ and $\lfloor r_\mu \rfloor$ for $m = 0.3$ GeV in the center-of-mass frame.

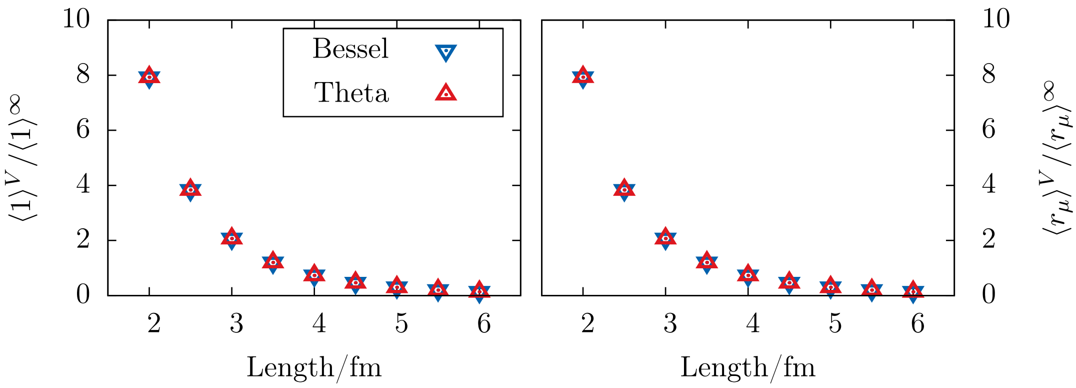

Figure 4.3: Relative correction to the integrals $\langle 1 \rangle$ and $\langle r_\mu \rangle$ for $m_1 = m_2 = 0.1$ GeV in the center-of-mass frame.

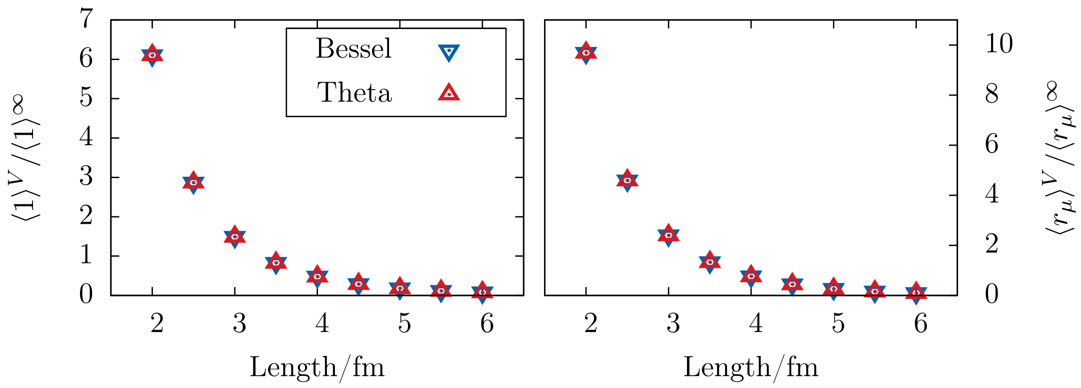

Figure 4.4: Relative correction to the integrals $\langle 1\rangle$ and $\langle r_\mu\rangle$ for $m_1 = m_2 =$ 0.1 GeV and with momentum in the x-direction.

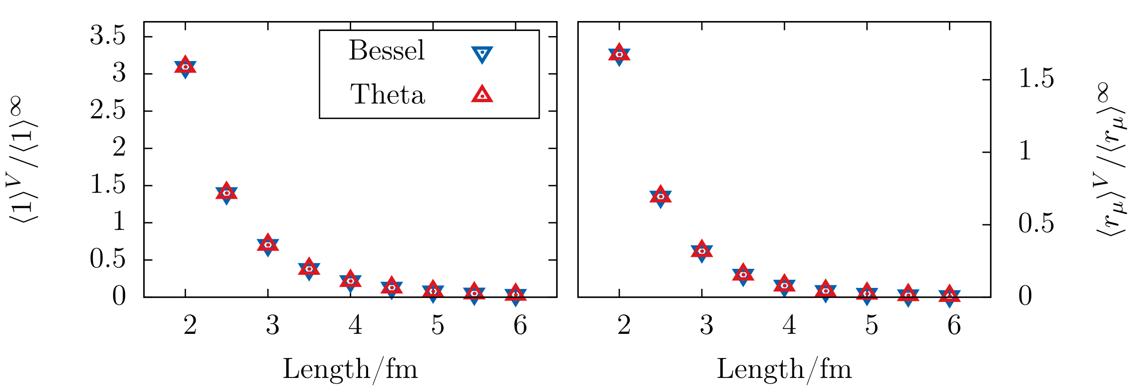

Figure 4.5: Relative correction to the integrals $\langle 1\rangle$ and $\langle r_\mu\rangle$ for $m_1 = 0.1$ GeV and $m_2 = 3m_1$ in the center-of-mass frame.

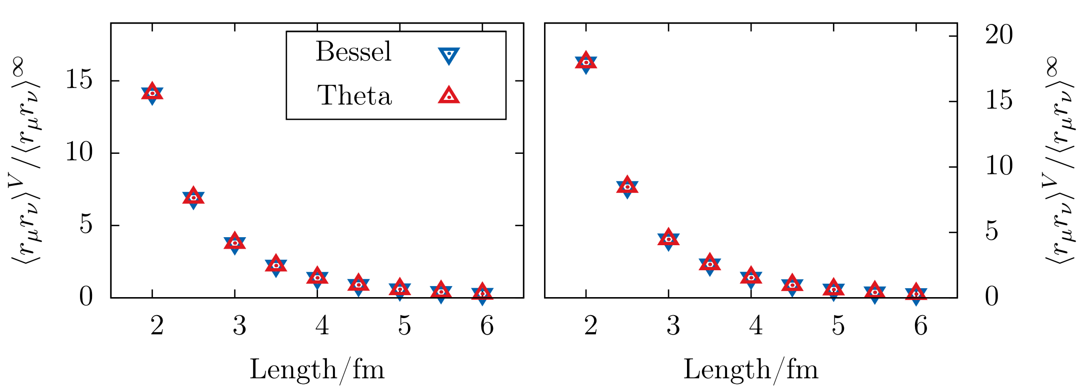

Figure 4.6: Relative correction to the parts of the integral $\langle r_\mu r_\nu\rangle$ proportional to $p_\mu p_\nu$ and $\delta_{\mu\nu}$ for $m_1 = m_2 = 0.1$ GeV in the center-of-mass frame.

Chapter 5

Finite Volume Sunset Integrals

5.1 Structure of the sunset integrals

Most two-loop diagrams can be factorized into products of diagrams containing one loop, the exception being those in which the loops share a propagator. The factorizable diagrams can then be rewritten in terms of one-loop diagrams, and computed using the results of Chapter 3. The simplest example of a case when factorization is not possible is the sunset diagram, the computation of which is the purpose of this chapter. The task of evaluating this integral is in the present case further complicated by the discrete loop momenta induced by the finite volume boundary conditions.

It will be shown below that the sunset integral can be split into parts containing one and two discrete loop momenta respectively. As it turns out, in the case of one discrete loop momenta there will arise non-local divergences from the expansion around the reduced dimension, which have to be identified and handled separately. This will be the first order of business, after which we can continue the computation of the convergent parts without serious problems. In the case of two discrete loop momenta the integrals will always be convergent, so the computation there will be more straightforward. Like in Chapter 3 it is not possible to perform a full calculation analytically,

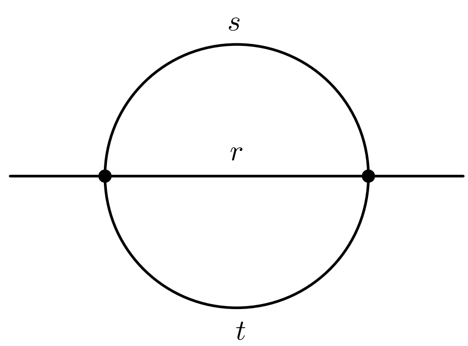

Figure 5.1: Sunset diagram of scalar field theory

and in the end we will have to choose a representation either in terms of modified Bessel functions or in terms of Jacobi and Riemann theta functions. Both these approaches will be considered below, and through the numerical evaluation that will follow in Chapter 6 we will see that they agree for all but very small volumes.

The purpose of this chapter is to carry out, for general masses and powers of the propagators, a calculation of the integral

$$\begin{aligned}\langle\langle X\rangle\rangle &= \int_V \frac{d^dr}{(2\pi)^d}\frac{d^ds}{(2\pi)^d}\frac{X}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}} \\ &= \sum_{l_r,l_s}\int \frac{d^dr}{(2\pi)^d}\frac{d^ds}{(2\pi)^d}\frac{X\,e^{il_r\cdot r}e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}\end{aligned} \tag{5.1}$$

where X is a structure in the set $\{1, r_\mu, s_\mu, r_\mu s_\nu, r_\mu r_\nu\}$ and we have used Eq. (3.3) twice on r and s. The subscripts on l_r and l_s have been introduced to clarify from which quantized momentum the summation originates, and we will in this chapter often suppress the vector indices on the l's in order to keep the notation more clean. The contribution from the infinite volume limit can be easily identified as the term with $l_r = l_s = 0$, so the integral above can be written as the sum of that term and a finite volume correction according to $\langle\langle X\rangle\rangle = \langle\langle X\rangle\rangle^\infty + \langle\langle X\rangle\rangle^V$. The infinite volume integrals have been computed elsewhere and results can be found e.g. in [10], so here we will focus only on the finite volume part.

It turns out that this integral can be further split into four parts, where the first three correspond to the cases where only one of the momenta in Figure 5.1 is quantized and the last to when r, s and $r+s$ are. Formally this can be written as

$$\langle\langle X\rangle\rangle^V = \langle\langle X\rangle\rangle_r + \langle\langle X\rangle\rangle_s + \langle\langle X\rangle\rangle_t + \langle\langle X\rangle\rangle_{rs} \tag{5.2}$$

where the notation $l_t = l_r - l_s$ has been introduced to keep track of the case when $l_r = l_s$ and the subscripts indicate which l is non-zero. More explicitly the integrals are

$$\langle\langle X\rangle\rangle_r = \sum_{l_r}{}' \int \frac{d^dr}{(2\pi)^d}\frac{d^ds}{(2\pi)^d}\frac{X\,e^{il_r\cdot r}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}$$

$$\langle\langle X\rangle\rangle_s = \sum_{l_s}{}' \int \frac{d^dr}{(2\pi)^d}\frac{d^ds}{(2\pi)^d}\frac{X\,e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}$$

$$\langle\langle X\rangle\rangle_t = \sum_{l_t}{}' \int \frac{d^dr}{(2\pi)^d}\frac{d^ds}{(2\pi)^d}\frac{X\,e^{il_t\cdot(p-r-s)}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}$$

$$\langle\langle X\rangle\rangle_{rs} = \sum_{l_r,l_s}'' \int \frac{d^dr}{(2\pi)^d}\frac{d^ds}{(2\pi)^d}\frac{X\, e^{il_r\cdot r}e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}, \tag{5.3}$$

which keeping in mind that $p = 2\pi n/L$ due to the finite volume gives back the full finite volume correction $\langle\langle X\rangle\rangle^V$ when summed. The doubly primed summation sign will in the following be used to denote a sum where the terms with $l_r = 0$, $l_s = 0$ and $l_r = l_s$ have been removed. As can be seen from the expressions above the integrals $\langle\langle X\rangle\rangle_s$ and $\langle\langle X\rangle\rangle_t$ are equivalent to $\langle\langle X\rangle\rangle_r$ since they can be obtained from the latter by a suitable change of variables, which means we only have to compute the parts $\langle\langle X\rangle\rangle_r$ and $\langle\langle X\rangle\rangle_{rs}$ to find $\langle\langle X\rangle\rangle^V$.

When evaluating the $\langle\langle X\rangle\rangle_{rs}$ integrals it will sometimes be possible to use an extension of Eq. (3.4) to two variables in order to simplify the summation. This relation is given by

$$\sum_{l_r,l_s}'' F(l_r^2,l_s^2,l_t^2) = \sum_{k_r,k_s,k_n=1}^{\infty} x(k_r,k_s,k_n)F(k_rL^2,k_sL^2,k_nL^2) \tag{5.4}$$

where the factor $x(k_r,k_s,k_n)$ denotes how many times how many times a vector with a given length appears when l_r and l_s are summed over all non-zero values in $\mathbb{Z}$.

5.2 Sunset integrals with one discrete loop momentum

To start with consider the integral $\langle\langle X\rangle\rangle_r$, which we will now show can be divided into two parts. The first one is convergent and can be computed without serious problems, while the second one exhibits a non-local divergence when letting the reduced dimension $d \to 4$. It is sufficient to consider the case where $n_2 = n_3 = 1$, since all other cases can be obtained from this by linear variable substitutions involving r, s and p, or by differentiating the integral with respect to m_2^2 or m_3^2. Introducing a Feynman parameter x to combine the last two propagators and denoting the part of $\langle\langle X\rangle\rangle_r$ dependent on s by $\langle\langle X\rangle\rangle_r^s$ we find

$$\begin{aligned}\langle\langle X\rangle\rangle_r^s &= \int \frac{d^ds}{(2\pi)^d}\frac{X}{(s^2+m_2^2)((r+s-p)^2+m_3^2)} \\ &= \int_0^1 dx \int \frac{d^ds}{(2\pi)^d}\frac{X}{\left[(1-x)(s^2+m_2^2)+x((r+s-p)^2+m_3^2)\right]^2} \\ &= \int_0^1 dx \int \frac{d^d\tilde{s}}{(2\pi)^d}\frac{Z}{(\tilde{s}^2+\bar{m}^2)^2}\end{aligned} \tag{5.5}$$

where we have shifted the integration variable to $\tilde{s}_\mu = s_\mu + x(r-p)_\mu$, introduced $\bar{m}^2 = m_2^2(1-x) + m_3^2 x + x(1-x)(r-p)^2$ for notational convenience and written Z for X in the new variables. When $X \in \{1, r_\mu, r_\mu r_\nu\}$ and thus independent of s it is straightforward to continue the integration using Eq. (3.6) which gives

$$\langle\langle X \rangle\rangle_r^s = \int_0^1 dx \int \frac{d^d\tilde{s}}{(2\pi)^d} \frac{1}{(\tilde{s}^2 + \bar{m}^2)^2} = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx (\bar{m}^2)^{d/2-2}. \quad (5.6)$$

If on the other hand $X \in \{s_\mu, r_\mu s_\nu\}$ we use the relation $s_\mu = \tilde{s}_\mu - x(r-p)_\mu$ to replace s_μ and to find

$$\begin{aligned} \langle\langle X \rangle\rangle_r^s &= \int_0^1 dx \int \frac{d^d\tilde{s}}{(2\pi)^d} \frac{\tilde{s}_\mu - x(r-p)_\mu}{(\tilde{s}^2 + \bar{m}^2)^2} Z \\ &= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx (\bar{m}^2)^{d/2-2} (-x)(r-p)_\mu Z \end{aligned} \quad (5.7)$$

where $Z \in \{1, r_\mu\}$. Finally for $X = s_\mu s_\nu$ the integral becomes more involved but can be done using the change of variables $s_\mu s_\nu = \tilde{s}_\mu \tilde{s}_\nu - x\tilde{s}_\mu (r-p)_\nu - x\tilde{s}_\nu (r-p)_\mu + x^2 (r-p)_\mu (r-p)_\nu$ which gives

$$\begin{aligned} \langle\langle X \rangle\rangle_r^s &= \int_0^1 dx \int \frac{d^d\tilde{s}}{(2\pi)^d} \frac{s_\mu s_\nu}{(\tilde{s}^2 + \bar{m}^2)^2} \qquad (5.8) \\ &= \frac{1}{(4\pi)^d} \int_0^1 dx \left[\frac{\delta_{\mu\nu}}{2} \Gamma(1-\frac{d}{2}) (\bar{m}^2)^{d/2-1} \right. \\ &\quad \left. + \Gamma(2-\frac{d}{2}) (\bar{m}^2)^{d/2-2} x^2 (r-p)_\mu (r-p)_\nu \right] \end{aligned}$$

with the help of Eq. (3.6) and the fact that terms odd in $\tilde{s}_\mu$ vanish. In the limit $\epsilon \to 0$ where the dimension $d = 4 - 2\epsilon$ approaches 4, these expressions have to be expanded to $\mathcal{O}(\epsilon)$ in order to find the divergences. Carrying out the expansion for the s-dependent parts gives the following expressions

$$\begin{aligned} \langle\langle X_1 \rangle\rangle_r &= \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}} \int_0^1 dx \frac{1}{16\pi^2} (\lambda_0 - 1 - \ln \bar{m}^2) \qquad (5.9) \\ \langle\langle X_2 \rangle\rangle_r &= \int \frac{d^d r}{(2\pi)^d} \frac{Z\, e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}} \int_0^1 dx \frac{(-x)(r-p)_\mu}{16\pi^2} (\lambda_0 - 1 - \ln \bar{m}^2) \\ \langle\langle X_3 \rangle\rangle_r &= \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}} \int_0^1 dx \frac{1}{16\pi^2} \left[\lambda_0 \left(x^2 (r-p)_\mu (r-p)_\nu - \frac{\delta_{\mu\nu}}{2} \bar{m}^2 \right) \right. \\ &\quad \left. + \ln \bar{m}^2 \left(\frac{\delta_{\mu\nu}}{2} \bar{m}^2 - x^2 (r-p)_\mu (r-p)_\nu \right) - x^2 (r-p)_\mu (r-p)_\nu \right] \end{aligned}$$

where $X_1 \in \{1, r_\mu, r_\mu r_\nu\}$, $X_2 \in \{s_\mu, r_\mu s_\nu\}$ and $X_3 \in \{s_\mu s_\nu\}$. In all three cases the divergence is contained in the term proportional to λ_0 which is explicitly given by

$$\lambda_0 = \frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi \quad (5.10)$$

where γ is Euler's constant and we have followed the usual conventions in χPT. This gives the desired separation of the integrals into a divergent and a convergent part.

5.2.1 Divergent parts

Since λ_0 is independent of x it is in most cases not difficult to carry out the integration over x in the expressions above, and once this is done it is straightforward to write the integrals in terms of the simpler integrals $\lfloor X \rfloor^V$ from Chapter 3. To find an explicit expression it is however necessary to expand the $\lfloor X \rfloor^V$ integrals to $\mathcal{O}(\epsilon)$, which we will do now.

In the case where $\lfloor 1 \rfloor^V$ and $\lfloor r_\mu r_\nu \rfloor^V$ are expressed in terms of Bessel functions the lowered dimension d shows up only in the factors $(4\pi)^{-d/2}$ and $\mathcal{K}_{\nu_d}(l_r^2/4, m^2)$, where ν_d has been introduced as a generic expression for an index depending on d. Denoting the derivative of $\mathcal{K}_\nu$ with respect to the order ν by $\tilde{\mathcal{K}}_\nu$ we find the following expansions

$$\frac{1}{(4\pi)^{d/2}} = \frac{1}{16\pi^2} + \epsilon\frac{\ln 4\pi}{16\pi^2} + \mathcal{O}(\epsilon^2) \tag{5.11}$$
$$\mathcal{K}_{\nu_d}\left(\frac{l_r^2}{4}, m^2\right) = \mathcal{K}_\nu\left(\frac{l_r^2}{4}, m^2\right) + \epsilon\tilde{\mathcal{K}}_\nu\left(\frac{l_r^2}{4}, m^2\right) + \mathcal{O}(\epsilon^2),$$

and an explicit expression for $\tilde{\mathcal{K}}_\nu$ can be found in Appendix A. If on the other hand we consider the representation in terms of Jacobi functions the factors that need to be expanded are $(4\pi)^{-d/2}$, $\bar{\lambda}^{\nu_d}$ and $\left(\frac{L^2}{4}\right)^{\nu_d}$, for which we find

$$\bar{\lambda}^{\nu_d} = \bar{\lambda}^\nu + \epsilon\bar{\lambda}^\nu \ln\bar{\lambda} + \mathcal{O}(\epsilon^2) \tag{5.12}$$
$$\left(\frac{L^2}{4}\right)^{\nu_d} = \left(\frac{L^2}{4}\right)^\nu + \epsilon\left(\frac{L^2}{4}\right)^\nu \ln\frac{L^2}{4} + \mathcal{O}(\epsilon^2).$$

Using these results it is easy to show that the terms of $\mathcal{O}(\epsilon)$ in the expansion of $\lfloor X \rfloor^V$ can be found from the zeroth order terms, by simply making the replacements

$$\frac{\mathcal{K}_\nu}{16\pi^2} \to \epsilon\left(\frac{\mathcal{K}_\nu \ln 4\pi}{16\pi^2} + \frac{\tilde{\mathcal{K}}_\nu}{16\pi^2}\right) \tag{5.13}$$

$$\frac{\bar{\lambda}^\nu}{16\pi^2}\left(\frac{L^2}{4}\right)^{\nu-1} \to \epsilon\frac{\bar{\lambda}^\nu}{16\pi^2}\left(\frac{L^2}{4}\right)^{\nu-1} \ln\frac{4\pi\bar{\lambda}L^2}{4}. \tag{5.14}$$

Introducing the notation $\langle\langle X \rangle\rangle_r^D$ for the divergent parts of the integrals in Eq. 5.9 and $\lfloor X \rfloor_\epsilon^V$ for the $\mathcal{O}(\epsilon)$ term in the expansion of $\lfloor X \rfloor^V$ we can immediately write down

$$\langle\langle 1 \rangle\rangle_r^D = \frac{\lambda_0}{16\pi^2}\left(\lfloor 1 \rfloor^V + \lfloor 1 \rfloor_\epsilon^V\right) \tag{5.15}$$

$$\langle\langle r_\mu \rangle\rangle_r^D = 0$$
$$\langle\langle s_\mu \rangle\rangle_r^D = \frac{\lambda_0}{32\pi^2}\left(\lfloor 1 \rfloor^V + \lfloor 1 \rfloor_\epsilon^V\right) p_\mu$$
$$\langle\langle r_\mu r_\nu \rangle\rangle_r^D = \frac{\lambda_0}{16\pi^2}\left(\lfloor r_\mu r_\nu \rfloor^V + \lfloor r_\mu r_\nu \rfloor_\epsilon^V\right)$$
$$\langle\langle r_\mu s_\nu \rangle\rangle_r^D = -\frac{\lambda_0}{32\pi^2}\left(\lfloor r_\mu r_\nu \rfloor^V + \lfloor r_\mu r_\nu \rfloor_\epsilon^V\right).$$

For the $\langle\langle s_\mu s_\nu \rangle\rangle_r$ case the integration is a bit more involved, since its divergent part contains $\bar{m}^2$ which is dependent on x. Defining the new variable $\tilde{m}^2 = \frac{1}{4}\left(m_2^2 + m_3^2 + \frac{p^2 - m_1^2}{3}\right)$ for what remains after the x integration we find the following expression

$$\begin{aligned}\langle\langle s_\mu s_\nu \rangle\rangle_r^D = &\frac{\lambda_0}{48\pi^2}\left(\lfloor r_\mu r_\nu \rfloor^V + \lfloor r_\mu r_\nu \rfloor_\epsilon^V + \lfloor 1 \rfloor^V + \lfloor 1 \rfloor_\epsilon^V\right) \\ &- \frac{\tilde{m}^2\lambda_0}{16\pi^2}\left(\lfloor 1 \rfloor^V + \lfloor 1 \rfloor_\epsilon^V\right)\delta_{\mu\nu} - \frac{\lambda_0}{192\pi^2}\left(\lfloor 1 \rfloor_{n_1-1}^V + \lfloor 1 \rfloor_{n_1-1,\epsilon}^V\right)\delta_{\mu\nu}.\end{aligned} \tag{5.16}$$

Since the $\lfloor X \rfloor^V$ integrals all are independent of m_2 and m_3 we see that the derivative of $\langle\langle X \rangle\rangle^D$ with respect to m_2^2 or m_3^2 vanishes for all X except $X = s_\mu s_\nu$. Taking the derivative of $\langle\langle s_\mu s_\nu \rangle\rangle^D$ we then find that the divergences also in the case vanishes except for when $n_2 = 2$ and $n_3 = 1$ or $n_3 = 2$ and $n_2 = 1$ when we have

$$-\frac{\partial}{\partial m_2^2}\langle\langle s_\mu s_\nu \rangle\rangle^D = -\frac{\partial}{\partial m_3^2}\langle\langle s_\mu s_\nu \rangle\rangle^D = \frac{\lambda_0}{64\pi^2}\left(\lfloor 1 \rfloor^V + \lfloor 1 \rfloor_\epsilon^V\right)\delta_{\mu\nu}. \tag{5.17}$$

5.2.2 Convergent parts

To compute the convergent parts of the $\langle\langle X \rangle\rangle_r$ integrals we have to start again from the expressions in Eq. (5.9). For the s-independent variables denoted by X_1 the integrals have the generic form

$$\langle\langle X_1 \rangle\rangle_r = \int \frac{d^d r}{(2\pi)^d}\frac{X_1\, e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}}\int_0^1 dx \frac{1}{16\pi^2}(-1 - \ln \bar{m}^2) \tag{5.18}$$

This equation can be simplified using partial integration with respect to x, by taking the primitive of the constant factor $\frac{1}{16\pi^2}$ and the derivative of the logarithm, which gives

$$\begin{aligned}\langle\langle X_1 \rangle\rangle_r = -\frac{1}{16\pi^2}\int \frac{d^d r}{(2\pi)^d}\frac{X_1\, e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}}&\Bigg[(1 + \ln m_3^2) \\ &- \int_0^1 dx \frac{x(m_3^2 - m_2^2 + (1-2x)(r_\mu - p_\mu)^2)}{\bar{m}^2}\Bigg].\end{aligned} \tag{5.19}$$

The first term can be found immediately using the results of Chapter 3 since it is just a constant multiple of $\lfloor X_1 \rfloor$, and so it is enough to continue with the second part. Again using Schwinger's method to bring up the denominator we find

$$\langle\langle X_1\rangle\rangle_r = \frac{1}{\Gamma(n_1)16\pi^2}\int_0^1 dx \int \frac{d^d r}{(2\pi)^d} d\lambda_1 d\lambda_4 e^{il_r\cdot r} e^{-\lambda_1(r^2+m_1^2)} e^{-\lambda_4\bar{m}^2} \times$$
$$X_1\lambda_1^{n_1-1} x(m_3^2 - m_2^2 + (1-2x)(r_\mu - p_\mu)^2). \quad (5.20)$$

This can be simplified by completing the square in the exponent with respect to r and making the change of variables $r_\mu = \tilde{r}_\mu/\sqrt{\lambda_5} + il_{r\mu}/2\lambda_5 + x(1-x)\lambda_4 p_\mu/\lambda_5$. If we also make use of the explicit expression for $\bar{m}^2$ and introduce the expressions

$$\lambda_5 = \lambda_1 + x(1-x)\lambda_4 \quad (5.21)$$
$$Y_1 = \lambda_1 m_1^2 + \lambda_4(1-x)m_2^2 + \lambda_4 x m_3^2 + \frac{l_r^2}{4\lambda_5}$$
$$- \frac{ix(1-x)\lambda_4}{\lambda_5} l_r \cdot p + \frac{\lambda_1\lambda_4 x(1-x)p^2}{\lambda_5}.$$

for notational convenience, this type of integral can be given the general expression

$$\langle\langle X_1\rangle\rangle_r = \frac{1}{\Gamma(n_1)16\pi^2}\int_0^1 dx \int \frac{d^d\tilde{r}}{(2\pi)^d}\frac{d\lambda_1 d\lambda_4}{(\lambda_5)^{d/2}} e^{-\tilde{r}^2}\lambda_1^{n_1-1} e^{-Y_1}\times$$
$$xX_1\left[m_3^2 - m_2^2 + (1-2x)\left(\frac{\tilde{r}_\mu}{\sqrt{\lambda_5}} + \frac{il_{r\mu} - 2\lambda_1 p_\mu}{2\lambda_5}\right)^2\right]. \quad (5.22)$$

We now specialize to the case $X = 1$ and continue the integration with respect to $\tilde{r}$, and note that the integral vanishes as soon as the integrand is odd in $\tilde{r}$. The equation above then simplifies to

$$\langle\langle 1\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int_0^1 dx \int d\lambda_1 d\lambda_4 \frac{\lambda_1^{n_1-1}}{\lambda_5^2} e^{-Y_1}$$
$$x\left[m_3^2 - m_2^2 + (1-2x)\left(\frac{2}{\lambda_5} + \frac{\tilde{p}^2}{\lambda_5^2}\right)\right] \quad (5.23)$$

where $\tilde{p}_\mu = il_{r\mu}/2 - \lambda_1 p_\mu$ has been introduced to simplify the notation. Changing variables to $\lambda_2 = (1-x)\lambda_4$ and $\lambda_3 = x\lambda_4$ the above can be put into a more symmetric form

$$\langle\langle 1\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int d\lambda_1 d\lambda_2 d\lambda_3 \frac{\lambda_1^{n_1-1}}{\tilde{\lambda}^2} e^{-Y_2}\lambda_3 \times$$
$$\left[m_3^2 - m_2^2 + \frac{\lambda_2-\lambda_3}{\tilde{\lambda}}\left(2 + \frac{\lambda_2+\lambda_3}{\tilde{\lambda}}\tilde{p}^2\right)\right] \quad (5.24)$$

where $\tilde{\lambda} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$. This representation is compact but not very well suited for numerical evaluations since for that we would like to work with dimensionless quantities and have the contributions from p_μ and $l_{r\mu}$ separated. This can be achieved by another change of variables that has the additional advantage that it is possible to perform one more integration analytically. Letting $\lambda_1 = x\lambda$, $\lambda_2 = y\lambda$ and $\lambda_3 = z\lambda$ where $z = 1 - x - y$ and introducing the variables

$$\begin{aligned}\rho &= \frac{y+z}{\sigma} \\ \delta &= \frac{y-z}{\sigma} \\ \tau &= \frac{yz}{y+z}\end{aligned} \tag{5.25}$$

since the following expressions are quite lengthy, the integral can be rewritten as

$$\begin{aligned}\langle\langle 1\rangle\rangle_r = &\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdyd\lambda \frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2}\, e^{-Y_3}\times \\ &z\left[m_3^2 - m_2^2 + \frac{y-z}{\sigma\lambda}\left(2 + \frac{y+z}{\sigma\lambda}\tilde{p}^2\right)\right] \\ = &\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdyd\lambda \frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2}\, e^{-Y_3}\times \\ &z\left[m_3^2 - m_2^2 + \delta\rho x^2 p^2 + \frac{2\delta}{\lambda} - \frac{ix\delta\rho}{\lambda} l_r\cdot p - \frac{\delta\rho}{4\lambda^2} l_r^2\right]\end{aligned} \tag{5.26}$$

where $\sigma = xy + xz + yz$ and all the variables in the exponent have been collected into

$$Y_3 = \lambda\left(xm_1^2 + ym_2^2 + zm_3^2 + \frac{xyz}{\sigma}p^2\right) - \frac{iyz}{\sigma} l_r\cdot p + \frac{\rho}{4\lambda} l_r^2. \tag{5.27}$$

As in Chapter 3 we can now choose between computing the integral over λ or performing the sum explicitly. Both these options will be investigated later in this chapter, after first deriving similar expressions for the other $\langle\langle X\rangle\rangle_r$ integrals.

In the case where $X_1 = r_\mu, r_\mu r_\nu$ we could proceed along the same line as above, starting from Eq. (5.9) and work through all the steps leading up to Eq. (5.26). An easier and more elegant way is however to define and apply the differential operator $\Delta_\mu = -i\frac{\partial}{\partial l_r}$, whose effect on the integral $\langle\langle 1\rangle\rangle_r$ is to bring down an extra factor of r_μ and so transform it into $\langle\langle r_\mu\rangle\rangle_r$. Using Δ_μ on the l_r-dependent factors of the integral in Eq. (5.26) we find

$$\begin{aligned}\Delta_\mu e^{-Y_3} &= \frac{y+z}{\sigma\lambda}\left(\tau\lambda p_\mu + \frac{il_{r\mu}}{2}\right)e^{-Y_3} \\ \Delta_\mu \tilde{p}_\mu^2 &= \tilde{p}_\mu.\end{aligned} \tag{5.28}$$

With the help of these relations it is not hard to find an expression for the integral $\langle\langle r_\mu\rangle\rangle_r$, which in fact can be written as

$$\langle\langle r_\mu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdyd\lambda \frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2}e^{-Y_3}z\rho\times \tag{5.29}$$
$$\left[\left(\tau A + \frac{\delta(2\tau - x) - \tau B}{\lambda} - \frac{\tau C}{\lambda^2}\right)p_\mu + \frac{i}{2\lambda}\left(A + \frac{3\delta - B}{\lambda} - \frac{C}{\lambda^2}\right)l_{r\mu}\right].$$

Here the contributions from p_μ and $l_{r\mu}$ have been separated making the expression more lengthy, so it makes sense to introduce the scalar quantities

$$\begin{aligned} A &= m_3^2 - m_2^2 + \delta\rho x^2 p^2 \\ B &= ix\delta\rho l_r\cdot p \\ C &= \frac{\delta\rho}{4}l_r^2 \end{aligned} \tag{5.30}$$

in order to keep the notation relatively compact.

The same principle can of course be applied to find the $\langle\langle r_\mu r_\nu\rangle\rangle_r$ integral by once again applying the differential operator to $\langle\langle 1\rangle\rangle_r$, giving the result

$$\langle\langle r_\mu r_\nu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdyd\lambda \frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2}e^{-Y_3}z\rho\times \tag{5.31}$$
$$\left[\frac{1}{2\lambda}\left(A + \frac{3\delta - B}{\lambda} - \frac{C}{\lambda^2}\right)\delta_{\mu\nu} + \rho\tau\left(\tau A + \frac{2\delta(\tau - x) - \tau B}{\lambda} - \frac{\tau C}{\lambda^2}\right)p_\mu p_\nu + \frac{i\rho}{2\lambda}\left(\tau A + \frac{\delta(3\tau - x) - \tau B}{\lambda} - \frac{\tau C}{\lambda^2}\right)\{p, l_r\}_{\mu\nu} - \frac{\rho}{4\lambda^2}\left(A + \frac{4\delta - B}{\lambda} - \frac{C}{\lambda^2}\right)l_{r\mu}l_{r\nu}\right].$$

We now want to evaluate the integrals with $X \in \{s_\mu, r_\mu s_\nu, s_\mu s_\nu\}$, which can be done by starting from Eq. (5.9) and working through the steps in a similar way to what was done for the $\langle\langle 1\rangle\rangle_r$ case. Again however there is a faster approach which is to make use of the similarities of the expressions in Eq. (5.9) and rewrite the unknown integrals in terms of the previously calculated ones. From those equations it is apparent that the only difference between the integrands of $\langle\langle 1\rangle\rangle_r$ and $\langle\langle s_\mu\rangle\rangle_r$ is a factor $-x(r-p)_\mu/2$, and by the linearity of the integral we must then have that

$$\langle\langle s_\mu\rangle\rangle_r = -\langle\langle \frac{x}{2}r_\mu\rangle\rangle_r - \langle\langle \frac{x}{2}p_\mu\rangle\rangle_r. \tag{5.32}$$

Making use of the results in Eqs. (5.26) and (5.29) some simple algebra then gives

$$\begin{aligned}\langle\langle s_\mu\rangle\rangle_r = &-\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdyd\lambda\frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2}\,e^{-Y_3}\times \\ &z\left(A+\frac{3\delta}{\lambda}-\frac{B}{\lambda}-\frac{C}{\lambda^2}\right)\left(\frac{xz}{2\sigma}p_\mu-\frac{iz}{4\sigma\lambda}l_{r\mu}\right).\end{aligned} \tag{5.33}$$

The integral $\langle\langle r_\mu s_\nu\rangle\rangle_r$ can then be obtained from $\langle\langle s_\mu\rangle\rangle_r$ by applying the differential operator Δ_μ and using the relations in Eq. (5.28), which gives

$$\begin{aligned}\langle\langle r_\mu s_\nu\rangle\rangle_r = &-\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdyd\lambda\frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2}\,e^{-Y_3}z\times \\ &\left[\left(A+\frac{3\delta}{\lambda}-\frac{B}{\lambda}-\frac{C}{\lambda^2}\right)\left(-\frac{z}{4\sigma\lambda}\delta_{\mu\nu}-\frac{iz}{4\sigma\lambda^2}p_\mu l_{r\nu}\right)\right. \\ &+\frac{xz\rho}{2\sigma}\left(\tau A+\frac{\delta(3\tau-x)}{\lambda}-\frac{\tau B}{\lambda}-\frac{\tau C}{\lambda^2}\right)p_\mu p_\nu \\ &\left.+\left(A+\frac{4\delta}{\lambda}-\frac{B}{\lambda}-\frac{C}{\lambda^2}\right)\left(\frac{z\rho}{8\sigma\lambda^2}l_{r\mu}l_{r\nu}+\frac{ixz\rho}{4\sigma\lambda}\{p,l_r\}_{\mu\nu}\right)\right]\end{aligned} \tag{5.34}$$

What remains now is to find an expression for the integral $\langle\langle s_\mu s_\nu\rangle\rangle_r$ which is done by starting from Eq. (5.9) and performing a partial integration with respect to x. The terms then remaining are given by

$$\begin{aligned}\langle\langle s_\mu s_\nu\rangle\rangle_r = &-\frac{1}{16\pi^2}\int\frac{d^dr}{(2\pi)^d}\frac{e^{il_r\cdot r}}{(r^2+m_1^2)^{n_1}}\int_0^1 dx\,x\times \\ &\left\{\frac{\delta_{\mu\nu}}{2}\left[\left(\frac{1}{2}-\frac{x}{4}\right)m_2^2+\frac{x}{4}m_3^2+\left(\frac{x}{4}-\frac{x^2}{6}\right)(r-p)^2\right]\right. \\ &\left.+\frac{x^2}{3}(r-p)_\mu(r-p)_\nu\right\}\frac{m_3^2-m_2^2+(1-2x)(r_\mu-p_\mu)^2}{\bar{m}^2}\end{aligned} \tag{5.35}$$

and comparing this to Eq. (5.19), we see that the above can be expressed in terms of the previous integrals according to

$$\begin{aligned}\langle\langle s_\mu s_\nu\rangle\rangle_r = &\langle\langle\frac{x^2}{3}(r-p)_\mu(r-p)_\nu\rangle\rangle_r-\langle\langle\left(\frac{x}{4}-\frac{x^2}{6}\right)(r-p)^2\rangle\rangle_r\,\delta_{\mu\nu} \\ &-\langle\langle\left(\frac{1}{2}-\frac{x}{4}\right)m_2^2+\frac{x}{4}m_3^2\rangle\rangle_r\,\delta_{\mu\nu}.\end{aligned} \tag{5.36}$$

Collecting all these terms to find an explicit expression we find that the results gets much easier if we introduce the parameters

$$\alpha_1 = m_2^2\left(\frac{1}{2}-\frac{\tau}{4y}\right)z+m_3^2\frac{\tau z}{4y}$$

$$
\begin{aligned}
\alpha_2 &= \frac{\tau z}{4y} - \frac{\tau^2 z}{6y^2} \\
\alpha_3 &= \frac{\tau^2 z}{6y^2} - 2\alpha_2 \\
\alpha_4 &= x^2\rho^2 p^2 \alpha_2 + \alpha_1 .
\end{aligned} \tag{5.37}
$$

Using these expressions and the results found for the other integrals above we see that

$$
\begin{aligned}
\langle\langle s_\mu s_\nu \rangle\rangle_r = &-\frac{1}{\Gamma(n_1)(16\pi^2)^2} \int dxdyd\lambda \frac{x^{n_1-1}\lambda^{n_1-2}}{\sigma^2} e^{-Y_3} \times \\
&\left\{ \frac{z^3}{3\sigma^2} \left(A + \frac{4\delta}{\lambda} - \frac{B}{\lambda} - \frac{C}{\lambda^2} \right) \left(x^2 p_\mu p_\nu - \frac{ix}{2\lambda}\{p, l_r\}_{\mu\nu} - \frac{l_{r\mu} l_{r\nu}}{4\lambda^2} \right) \right. \\
&\left[-\alpha_4 A + \frac{1}{\lambda}\left(\rho A(\alpha_3 + i\alpha_2 xp\cdot l_r) + 2\delta\alpha_1 - (4\delta + B)\alpha_4 \right) \right. \\
&\frac{1}{\lambda^2}\left(A\rho^2\alpha_2 \frac{l_r^2}{4} + \rho\alpha_3(3\delta + B) + ix\rho^2\alpha_2(4\delta + B)p\cdot l_r - C\alpha_4 \right) \\
&\frac{1}{\lambda^3}\left(\rho^2\alpha_2 \frac{l_r^2}{4}(4\delta + B) + C\rho\alpha_3 + ixC\rho^2\alpha_2 p\cdot l_r \right) \\
&\left. \left. + C\rho^2\alpha_2 \frac{l_r^2}{4\lambda^4} \right] \delta_{\mu\nu} \right\} .
\end{aligned} \tag{5.38}
$$

All the integrals from Eq. (5.9) have now been expressed in a way that separates the contributions from different tensor structures, and each such contribution has been ordered according to falling powers of λ.

5.2.3 Expressions in terms of modified Bessel functions

In this section the integrals $\langle\langle X \rangle\rangle_r$ with one discrete loop momentum will be given explicit expression in terms of modified Bessel functions of the second kind. This is done by carrying out the integration with respect to λ of the expressions presented in the previous section. It should however be kept in mind that what is presented in the following is only the convergent parts of the integrals, since the divergent parts have already been written out in Eqs. (5.15) and (5.16); there is also a finite part that was left after the partial integration of Eq. (5.9) that will not be included. The notational convention and values used for the arguments of the Bessel functions are

$$
\begin{aligned}
Y &= \frac{\rho}{4} l_r^2 \\
Z &= xm_1^2 + ym_2^2 + zm_3^2 + \frac{xyz}{\sigma} p^2 \\
\nu &= n_1 - 1
\end{aligned} \tag{5.39}
$$

In the following equations the arguments of the Bessel functions will be suppressed for greater clarity, and we obtain the expressions

$$\langle\langle 1\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdy\frac{x^{n_1-1}}{\sigma^2}\, e^{\frac{iyz}{\sigma}p\cdot l_r} z \times \tag{5.40}$$
$$[A\mathcal{K}_\nu + (2\delta + B)\mathcal{K}_{\nu-1} - C\mathcal{K}_{\nu-2}]$$

$$\langle\langle r_\mu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdy\frac{x^{n_1-1}}{\sigma^2}\, e^{\frac{iyz}{\sigma}p\cdot l_r} z\rho \times \tag{5.41}$$
$$[(\tau A\mathcal{K}_\nu + (\delta(2\tau - x) - \tau B)\mathcal{K}_{\nu-1} - \tau C\mathcal{K}_{\nu-2})\, p_\mu$$
$$+ \frac{i}{2}\left(A\mathcal{K}_{\nu-1} + (3\delta - B)\mathcal{K}_{\nu-2} - C\mathcal{K}_{\nu-3}\right) l_{r\mu}]$$

$$\langle\langle r_\mu r_\nu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdy\frac{x^{n_1-1}}{\sigma^2}\, e^{\frac{iyz}{\sigma}p\cdot l_r} z\rho \times \tag{5.42}$$
$$\left[\frac{1}{2}\left(A\mathcal{K}_{\nu-1} + (3\delta - B)\mathcal{K}_{\nu-2} - C\mathcal{K}_{\nu-3}\right)\delta_{\mu\nu}\right.$$
$$+\rho\tau\left(\tau A\mathcal{K}_\nu + (2\delta(\tau - x) - \tau B)\mathcal{K}_{\nu-1} - \tau C\mathcal{K}_{\nu-1}\right) p_\mu p_\nu$$
$$+\frac{i\rho}{2}\left(\tau A\mathcal{K}_{\nu-1} + (\delta(3\tau - x) - \tau B)\mathcal{K}_{\nu-2} - \tau C\mathcal{K}_{\nu-3}\right)\{p, l_r\}_{\mu\nu}$$
$$\left.-\frac{\rho}{4}\left(A\mathcal{K}_{\nu-2} + (4\delta - B)\mathcal{K}_{\nu-3} - C\mathcal{K}_{\nu-4}\right) l_{r\mu} l_{r\nu}\right]$$

$$\langle\langle s_\mu\rangle\rangle_r = -\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdy\frac{x^{n_1-1}}{\sigma^2}\, e^{\frac{iyz}{\sigma}p\cdot l_r} z \times \tag{5.43}$$
$$\left[\frac{xz}{2\sigma}\left(A\mathcal{K}_\nu + (3\delta - B)\mathcal{K}_{\nu-1} - C\mathcal{K}_{\nu-2}\right) p_\mu\right.$$
$$\left.-\frac{iz}{4\sigma}\left(A\mathcal{K}_{\nu-1} + (3\delta - B)\mathcal{K}_{\nu-2} - C\mathcal{K}_{\nu-3}\right) l_{r\mu}\right]$$

$$\langle\langle r_\mu s_\nu\rangle\rangle_r = -\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdy\frac{x^{n_1-1}}{\sigma^2}\, e^{\frac{iyz}{\sigma}p\cdot l_r} z \times \tag{5.44}$$
$$\left[-\frac{z}{4\sigma}\left(A\mathcal{K}_{\nu-1} + (3\delta - B)\mathcal{K}_{\nu-2} - C\mathcal{K}_{\nu-3}\right)\delta_{\mu\nu}\right.$$
$$-\ \frac{iz}{4\sigma}\left(A\mathcal{K}_{\nu-2} + (3\delta - B)\mathcal{K}_{\nu-3} - C\mathcal{K}_{\nu-4}\right) p_\mu l_{r\nu}$$
$$+\ \frac{xz\rho}{2\sigma}\left(\tau A\mathcal{K}_\nu + (\delta(3\tau - x) - \tau B)\mathcal{K}_{\nu-1} - \tau C\mathcal{K}_{\nu-2}\right) p_\mu p_\nu$$
$$+\ \frac{ixz\rho}{4\sigma}\left(A\mathcal{K}_{\nu-1} + (4\delta - B)\mathcal{K}_{\nu-2} - C\mathcal{K}_{\nu-3}\right)\{p, l_r\}_{\mu\nu}$$
$$\left.+\frac{z\rho}{8\sigma}\left(A\mathcal{K}_{\nu-2} + (4\delta - B)\mathcal{K}_{\nu-3} - C\mathcal{K}_{\nu-4}\right) l_{r\mu} l_{r\nu}\right]$$

$$\langle\langle s_\mu s_\nu\rangle\rangle_r = -\frac{1}{\Gamma(n_1)(16\pi^2)^2}\int dxdy\frac{x^{n_1-1}}{\sigma^2}\, e^{\frac{iyz}{\sigma}p\cdot l_r} \times \tag{5.45}$$
$$\left\{\frac{x^2 z^3}{3\sigma^2}\left(A\mathcal{K}_\nu + (4\delta - B)\mathcal{K}_{\nu-1} - C\mathcal{K}_{\nu-2}\right) p_\mu p_\nu\right.$$

$$
\begin{aligned}
&-\frac{ixz^3}{6\sigma^2}\left(A\mathcal{K}_{\nu-1}+(4\delta-B)\mathcal{K}_{\nu-2}-C\mathcal{K}_{\nu-3}\right)\{p,l_r\}_{\mu\nu}\\
&-\frac{z^3}{12\sigma^2}\left(A\mathcal{K}_{\nu-2}+(4\delta-B)\mathcal{K}_{\nu-3}-C\mathcal{K}_{\nu-4}\right)l_{r\mu}l_{r\nu}\\
&+\Bigg[-\alpha_4 A\mathcal{K}_\nu+\Big(\rho A(\alpha_3+i\alpha_2 xp\cdot l_r)+2\delta\alpha_1-(4\delta+B)\alpha_4\Big)\mathcal{K}_{\nu-1}\\
&\left(A\rho^2\alpha_2\frac{l_r^2}{4}+\rho\alpha_3(3\delta+B)+ix\rho^2(4\delta+B)p\cdot l_r-C\alpha_4\right)\mathcal{K}_{\nu-2}\\
&\left(\rho^2\alpha_2\frac{l_r^2}{4}(4\delta+B)+C\rho\alpha_3+ixC\rho^2\alpha_2 p\cdot l_r\right)\mathcal{K}_{\nu-3}\\
&+C\rho^2\alpha_2\frac{l_r^2}{4}\mathcal{K}_{\nu-4}\Bigg]\delta_{\mu\nu}\Bigg\}
\end{aligned}
$$

5.2.4 Expressions in terms of theta functions

Instead of evaluating the integral over λ in the expressions for $\langle\langle X\rangle\rangle_r$ we can choose to perform the sum over $l_{r\mu}$ analytically. As in Chapter 3 there is no simple analytic expression valid in a general frame of reference, and in order to write down the equations we first have to choose a direction of the momentum. In what follows we will present the integrals in the center-of-mass frame using a representation in terms of Jacobi's third theta function and its first and second derivative. These can be written in the form

$$
\begin{aligned}
\theta_{30}(1/\lambda) &= \sum_{k=-\infty}^{\infty} e^{-k^2/\lambda} \\
\theta_{31}(1/\lambda) &= \sum_{k=-\infty}^{\infty} k^2\, e^{-k^2/\lambda} \\
\theta_{32}(1/\lambda) &= \sum_{k=-\infty}^{\infty} k^4\, e^{-k^2/\lambda}
\end{aligned}
\tag{5.46}
$$

where the first index represents the name of the function and the second one the order of the derivative. To simplify the notation it is also convenient to introduce

$$
\begin{aligned}
\hat{\lambda} &= \frac{4\bar{\lambda}}{L^2\rho} \\
Y &= \frac{L^2\rho}{4}\hat{\lambda}(m_1^2x+m_2^2y+m_3^2z+\frac{p^2xyz}{\sigma})
\end{aligned}
\tag{5.47}
$$

so that the argument $\hat{\lambda}$ of the theta functions is dimensionless. The terms of the integrals are organized according to their Lorentz structure and their order of dependence on $\hat{\lambda}$. For the finite parts of the integrals (disregarding the part left after partial integration in Eq. (5.9)) we then have

$$\langle\langle 1\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi)^2}\int dxdyd\hat{\lambda}\,\frac{x^{n_1-1}\hat{\lambda}^{n_1-2}}{\sigma^2}e^{-Y}\left(\frac{L^2\rho}{4}\right)^{n_1-1}\times \quad (5.48)$$
$$z\left[A(\theta_{30}^3-1)+\frac{8\delta}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right]$$

$$\langle\langle r_\mu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi)^2}\int dxdyd\hat{\lambda}\,\frac{x^{n_1-1}\hat{\lambda}^{n_1-2}}{\sigma^2}e^{-Y}\left(\frac{L^2\rho}{4}\right)^{n_1-1}\times \quad (5.49)$$
$$z\left[\tau A(\theta_{30}^3-1)+\frac{4\delta(2\tau-x)}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}\right]p_\mu$$

$$\langle\langle s_\mu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi)^2}\int dxdyd\hat{\lambda}\,\frac{x^{n_1-1}\hat{\lambda}^{n_1-2}}{\sigma^2}e^{-Y}\left(\frac{L^2\rho}{4}\right)^{n_1-1}\times \quad (5.50)$$
$$\frac{xz^2}{2\sigma}\left[A(\theta_{30}^3-1)+\frac{12\delta}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right]p_\mu$$

$$\langle\langle r_\mu r_\nu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi)^2}\int dxdyd\hat{\lambda}\,\frac{x^{n_1-1}\hat{\lambda}^{n_1-2}}{\sigma^2}e^{-Y}\left(\frac{L^2\rho}{4}\right)^{n_1-1}\times \quad (5.51)$$
$$z\Bigg[\frac{2}{L^2\hat{\lambda}}\left(A(\theta_{30}^3-1)+\frac{12\delta}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right)\delta_{\mu\nu}$$
$$+\rho^2\left(\tau^2A(\theta_{30}^3-1)+\frac{8\delta\tau(\tau-x)}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4\delta\tau^2}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right)p_\mu p_\nu$$
$$-\frac{4}{L^2\hat{\lambda}^2}\left(A\theta_{31}^3+\frac{16\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{32}^3\hat{\lambda}^{-2}\right)\tilde{t}_{\mu\nu}\Bigg]$$

$$\langle\langle r_\mu s_\nu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi)^2}\int dxdyd\hat{\lambda}\,\frac{x^{n_1-1}\hat{\lambda}^{n_1-2}}{\sigma^2}e^{-Y}\left(\frac{L^2\rho}{4}\right)^{n_1-1}\times \quad (5.52)$$
$$z\Bigg[\frac{\rho z}{2\sigma}\left(x\tau A(\theta_{30}^3-1)+\frac{4x\delta(3\tau-x)}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4x\delta\tau}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right)p_\mu p_\nu$$
$$+\frac{4z}{L^2\rho\sigma\hat{\lambda}^2}\left(A\theta_{31}^3+\frac{16\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{32}^3\hat{\lambda}^{-2}\right)\tilde{t}_{\mu\nu}$$
$$-\frac{z}{L^2\rho\sigma\hat{\lambda}}\left(A(\theta_{30}^3-1)+\frac{12\delta}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right)\delta_{\mu\nu}\Bigg]$$

$$\langle\langle s_\mu s_\nu\rangle\rangle_r = \frac{1}{\Gamma(n_1)(16\pi)^2}\int dxdyd\hat{\lambda}\,\frac{x^{n_1-1}\hat{\lambda}^{n_1-2}}{\sigma^2}e^{-Y}\left(\frac{L^2\rho}{4}\right)^{n_1-1}\times \quad (5.53)$$
$$z\Bigg[\frac{x^2z^2}{3\sigma^2}\left(A(\theta_{30}^3-1)+\frac{16\delta}{L^2\rho}(\theta_{30}^3-1)\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-2}\right)p_\mu p_\nu$$
$$-\frac{4z^2}{3(L\rho\sigma\hat{\lambda})^2}\left(A\theta_{31}^3+\frac{16\delta}{L^2\rho}\theta_{31}^3\hat{\lambda}^{-1}-\frac{4\delta}{L^2\rho}\theta_{32}^3\hat{\lambda}^{-2}\right)\tilde{t}_{\mu\nu}$$
$$+\Bigg\{-\alpha_4A(\theta_{30}^3-1)+\frac{4}{L^2\rho}(A\rho\alpha_3+2\delta(\alpha_1-2\alpha_4))(\theta_{30}^3-1)\hat{\lambda}^{-1}$$

$$+\left(\frac{4}{L^2\rho}\right)^2\left(3\delta\rho\alpha_3(\theta_{30}^3-1)-\frac{\delta\rho\alpha_4L^2}{4}\theta_{31}^3+A\frac{\rho^2\alpha_2L^2}{4}\theta_{31}^3\right)\hat{\lambda}^{-2}$$
$$+\left(\frac{4}{L^2\rho}\right)^2(4\delta\rho\alpha_2-\delta\rho\alpha_3)\theta_{31}^3\hat{\lambda}^{-3}-\left(\frac{4}{L^2\rho}\right)^2\rho\delta\alpha_2\theta_{32}^3\hat{\lambda}^{-4}\Big\}\delta_{\mu\nu}\Big]$$

The expressions in a general frame are similar except we instead have to use the theta functions

$$\theta_{30}(1/\lambda,z)=\sum_{k=-\infty}^{\infty}e^{-k^2/\lambda+2ikz} \tag{5.54}$$
$$\theta_{31}(1/\lambda,z)=\sum_{k=-\infty}^{\infty}k^2\,e^{-k^2/\lambda+2ikz}$$
$$\theta_{32}(1/\lambda,z)=\sum_{k=-\infty}^{\infty}k^4\,e^{-k^2/\lambda+2ikz}$$

in the directions with non-zero momentum, and it is necessary to compute each tensor component of the integrals separately.

5.3 Sunset integrals with two discrete loop momenta

As noted in the beginning of this chapter, to find an expression for the finite volume corrections to the sunset integrals we need to compute the parts $\langle\langle X\rangle\rangle_r$ and $\langle\langle X\rangle\rangle_{rs}$ containing one and two quantized loop momenta respectively. The first one was considered in full generality in Sec. 5.2 where it was shown that for some values of n_1, n_2 and n_3 it has a non-local divergence. In contrast the second integral is always finite, since only the well-behaved parts of it remains after the division made in Eq (5.3). The purpose of this section is to carry out, for general masses and powers of the propagators, a calculation of the integral

$$\langle\langle X\rangle\rangle_{rs}=\int\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\frac{X\,e^{il_r\cdot r}e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}} \tag{5.55}$$

where $X\in\{1,r_\mu,s_\mu,r_\mu s_\nu,r_\mu r_\nu\}$. To do this we start with bringing up the propagators using Schwinger's parametrization

$$\frac{1}{a^n}=\frac{1}{\Gamma(n)}\int_0^\infty d\lambda\,\lambda^{n-1}e^{-a\lambda} \tag{5.56}$$

and then complete the squares with respect to r and s in the argument of the resulting exponential. Shifting the integration variables according to

$$\tilde{r}_\mu=\sqrt{\lambda_1+\lambda_3}\left(r_\mu-\frac{p_\mu\lambda_3-s_\mu\lambda_3}{\lambda_1+\lambda_3}+\frac{il_r}{2(\lambda_1+\lambda_3)}\right) \tag{5.57}$$

$$\tilde{s}_\mu = \sqrt{\frac{\tilde{\lambda}}{\lambda_1+\lambda_3}}\left(s_\mu + \frac{p_\mu\lambda_3^2 - p_\mu\lambda_3(\lambda_1+\lambda_3)}{\tilde{\lambda}} + \frac{il_r\lambda_3 - il_s(\lambda_1+\lambda_3)}{2\tilde{\lambda}}\right)$$

in order to put the integral in a Gaussian form we find the following general expression

$$\langle\langle X\rangle\rangle_{rs} = \int d\lambda_1 d\lambda_2 d\lambda_3 \frac{\lambda_1^{n_1-1}\lambda_2^{n_2-1}\lambda_3^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\tilde{\lambda}^2} e^{-X_2-Y_2-Z_2} \times \int \frac{d^4\tilde{r}}{(2\pi)^4}\frac{d^4\tilde{s}}{(2\pi)^4}\, X\, e^{-\tilde{r}^2} e^{-\tilde{s}^2}, \tag{5.58}$$

where $\tilde{\lambda} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ has been introduced to simplify the notation and

$$\begin{aligned} X_2 &= \frac{i\lambda_1\lambda_3}{\tilde{\lambda}}\, p\cdot l_s - \frac{i\lambda_2\lambda_3}{\tilde{\lambda}}\, p\cdot l_r \\ Y_2 &= \frac{\lambda_1}{4\tilde{\lambda}} l_s^2 + \frac{\lambda_2}{4\tilde{\lambda}} l_r^2 + \frac{\lambda_3}{4\tilde{\lambda}} l_t^2 \\ Z_2 &= \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 + \frac{\lambda_1\lambda_2\lambda_3}{\tilde{\lambda}} p^2. \end{aligned} \tag{5.59}$$

In the case where $X = 1$ the integrals over $\tilde{r}$ and $\tilde{s}$ can be performed directly using the formula of Eq. (3.6) giving

$$\langle\langle 1\rangle\rangle_{rs} = \int d\lambda_1 d\lambda_2 d\lambda_3 \frac{\lambda_1^{n_1-1}\lambda_2^{n_2-1}\lambda_3^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi\tilde{\lambda})^2} e^{-X_2-Y_2-Z_2}. \tag{5.60}$$

For the numeric evaluation it is convenient to introduce dimensionless parameters, which can be done by letting $\lambda_1 = x\bar{\lambda}$, $\lambda_2 = y\bar{\lambda}$ and $\lambda_3 = z\bar{\lambda}$ where $z = 1 - x - y$. In these new variables we obtain the representation

$$\langle\langle 1\rangle\rangle_{rs} = \int dx dy d\bar{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\bar{\lambda}^{\nu_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\, e^{-X_3-Y_3-Z_3} \tag{5.61}$$

where now

$$\begin{aligned} X_3 &= \frac{iyz}{\sigma} p\cdot l_r - \frac{ixz}{\sigma} p\cdot l_s \\ Y_3 &= \frac{1}{\bar{\lambda}}\left(\frac{x}{4\sigma} l_s^2 + \frac{y}{4\sigma} l_r^2 + \frac{z}{4\sigma} l_t^2\right) \\ Z_3 &= \bar{\lambda}\left(x m_1^2 + y m_2^2 + z m_3^2 + \frac{xyz}{\sigma} p^2\right) \\ \nu_3 &= n_1 + n_2 + n_3 - 4. \end{aligned} \tag{5.62}$$

For all other values of X it is necessary to invert the expressions for $\tilde{r}$ and $\tilde{s}$ in order to find r and s. Inserting the result into $\langle\langle r_\mu\rangle\rangle_{rs}$ and $\langle\langle s_\mu\rangle\rangle_{rs}$ and making use of the fact that integrals over odd powers of $\tilde{r}_\mu$ or $\tilde{s}_\mu$ vanish we find

$$
\begin{aligned}
\langle\langle r_\mu\rangle\rangle_{rs} &= \int dxdyd\bar{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\bar{\lambda}^{\nu_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\, e^{-X_3-Y_3-Z_3} \times \\
&\qquad \left[\frac{yz}{\sigma}p_\mu + \frac{iy}{2\sigma}l_{r\mu}\bar{\lambda}^{-1} + \frac{iz}{2\sigma}l_{t\mu}\bar{\lambda}^{-1}\right] \\
\langle\langle s_\mu\rangle\rangle_{rs} &= \int dxdyd\bar{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\bar{\lambda}^{\nu_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\, e^{-X_3-Y_3-Z_3} \times \\
&\qquad \left[\frac{xz}{\sigma}p_\mu + \frac{ix}{2\sigma}l_{s\mu}\bar{\lambda}^{-1} - \frac{iz}{2\sigma}l_{t\mu}\bar{\lambda}^{-1}\right].
\end{aligned}
\tag{5.63}
$$

There is nothing new introduce in the calculations of the remaining integrals, and ordering the results according to their tensor structure and in falling powers of $\bar{\lambda}$ we find

$$
\begin{aligned}
\langle\langle r_\mu r_\nu\rangle\rangle_{rs} =& \int dxdyd\bar{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\bar{\lambda}^{\nu_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\, e^{-X_3-Y_3-Z_3} \times \\
& \left[\frac{y^2z^2}{\sigma^2}p_\mu p_\nu + \left(\frac{\delta_{\mu\nu}}{2}\frac{y+z}{\sigma} + \frac{iy^2z}{2\sigma^2}\{p,l_r\}_{\mu\nu} + \frac{iyz^2}{2\sigma^2}\{p,l_t\}_{\mu\nu}\right)\bar{\lambda}^{-1}\right. \\
& \left. - \left(\frac{y^2}{4\sigma^2}l_{r\mu}l_{r\nu} + \frac{z^2}{4\sigma^2}l_{t\mu}l_{n\nu} + \frac{yz}{4\sigma^2}\{l_r,l_t\}_{\mu\nu}\right)\bar{\lambda}^{-2}\right] \\
\langle\langle s_\mu s_\nu\rangle\rangle_{rs} =& \int dxdyd\bar{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\bar{\lambda}^{\nu_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\, e^{-X_3-Y_3-Z_3} \times \\
& \left[\frac{x^2z^2}{\sigma^2}p_\mu p_\nu + \left(\frac{\delta_{\mu\nu}}{2}\frac{x+z}{\sigma} + \frac{ix^2z}{2\sigma^2}\{p,l_s\}_{\mu\nu} - \frac{ixz^2}{2\sigma^2}\{p,l_t\}_{\mu\nu}\right)\bar{\lambda}^{-1}\right. \\
& \left. - \left(\frac{x^2}{4\sigma^2}l_{s\mu}l_{s\nu} + \frac{z^2}{4\sigma^2}l_{t\mu}l_{n\nu} - \frac{xz}{4\sigma^2}\{l_s,l_t\}_{\mu\nu}\right)\bar{\lambda}^{-2}\right] \\
\langle\langle r_\mu s_\nu\rangle\rangle_{rs} =& \int dxdyd\bar{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\bar{\lambda}^{\nu_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\, e^{-X_3-Y_3-Z_3} \left[\frac{xyz^2}{\sigma^2}p_\mu p_\nu\right. \\
& + \left(-\delta_{\mu\nu}\frac{z}{2\sigma} + \frac{ixyz}{2\sigma^2}(p_\mu l_{s\nu} + l_{r\mu}p_\nu) - \frac{iyz^2}{2\sigma^2}p_\nu l_{n\nu} + \frac{ixz^2}{2\sigma^2}l_{t\mu}p_\nu\right)\bar{\lambda}^{-1} \\
& \left. - \left(\frac{yz}{4\sigma^2}l_{r\mu}l_{n\nu} + \frac{z^2}{4\sigma^2}l_{t\mu}l_{n\nu} - \frac{xz}{4\sigma^2}l_{t\mu}l_{s\nu} - \frac{xy}{4\sigma^2}l_{r\mu}l_{s\nu}\right)\bar{\lambda}^{-2}\right]
\end{aligned}
$$

Like in the previous section it is now possible to proceed in two different directions, where one choice would be to perform the integral over $\bar{\lambda}$ to get a representation in terms of modified Bessel functions, and another to first carry out the summations to get a representation in terms of Jacobi and Riemann theta functions, and then try to evaluate the resulting integrals numerically. Both these approaches will be investigated below, starting with the first.

5.3.1 Expressions in terms of modified Bessel functions

In this section the integrals $\langle\langle X\rangle\rangle_{rs}$ are given explicitly in terms of modified Bessel functions of the second kind, which can be achieved by integrating the expressions in the previous section with respect to $\bar{\lambda}$. The notational conventions and values used for the arguments of the Bessel functions are

$$\begin{aligned} Y &= \frac{x}{4\sigma}l_s^2 + \frac{y}{4\sigma}l_r^2 + \frac{z}{4\sigma}l_t^2 \\ Z &= xm_1^2 + ym_2^2 + zm_3^2 + \frac{xyz}{\sigma}p^2 \\ \nu &= n_1 + n_2 + n_3 - 4. \end{aligned} \tag{5.64}$$

In the following equations the arguments of the Bessel functions will be suppressed for greater clarity, and we find the expressions

$$\begin{aligned} \langle\langle 1\rangle\rangle_{rs} &= \int dxdy\,\frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\,e^{iyzp\cdot l_r/\sigma - ixzp\cdot l_s/\sigma}\mathcal{K}_\nu \\ \langle\langle r_\mu\rangle\rangle_{rs} &= \int dxdy\,\frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\,e^{iyzp\cdot l_r/\sigma - ixzp\cdot l_s/\sigma}\times \\ &\quad \left[\frac{yz}{\sigma}p_\mu\mathcal{K}_\nu + \left(\frac{iy}{2\sigma}l_{r\mu} + \frac{iz}{2\sigma}l_{t\mu}\right)\mathcal{K}_{\nu-1}\right] \\ \langle\langle s_\mu\rangle\rangle_{rs} &= \int dxdy\,\frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\,e^{iyzp\cdot l_r/\sigma - ixzp\cdot l_s/\sigma}\times \\ &\quad \left[\frac{xz}{\sigma}p_\mu\mathcal{K}_\nu + \left(\frac{ix}{2\sigma}l_{s\mu} - \frac{iz}{2\sigma}l_{t\mu}\right)\mathcal{K}_{\nu-1}\right] \\ \langle\langle r_\mu r_\nu\rangle\rangle_{rs} &= \int dxdy\,\frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\,e^{iyzp\cdot l_r/\sigma - ixzp\cdot l_s/\sigma}\times \\ &\quad \Bigg[\frac{y^2z^2}{\sigma^2}p_\mu p_\nu\mathcal{K}_\nu + \Bigg(\frac{y+z}{2\sigma}\delta_{\mu\nu} \\ &\quad +\frac{iy^2z}{2\sigma^2}\{p,l_r\}_{\mu\nu} + \frac{iyz^2}{2\sigma^2}\{p,l_t\}_{\mu\nu}\Bigg)\mathcal{K}_{\nu-1} \\ &\quad -\left(\frac{y^2}{4\sigma^2}l_{r\mu}l_{r\nu} + \frac{z^2}{4\sigma^2}l_{t\mu}l_{n\nu} + \frac{yz}{4\sigma^2}\{l_r,l_t\}_{\mu\nu}\right)\mathcal{K}_{\nu-2}\Bigg] \\ \langle\langle s_\mu s_\nu\rangle\rangle_{rs} &= \int dxdy\,\frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\,e^{iyzp\cdot l_r/\sigma - ixzp\cdot l_s/\sigma}\times \\ &\quad \Bigg[\frac{x^2z^2}{\sigma^2}p_\mu p_\nu\mathcal{K}_\nu + \Bigg(\frac{x+z}{2\sigma}\delta_{\mu\nu} \\ &\quad +\frac{ix^2z}{2\sigma^2}\{p,l_s\}_{\mu\nu} - \frac{ixz^2}{2\sigma^2}\{p,l_t\}_{\mu\nu}\Bigg)\mathcal{K}_{\nu-1} \\ &\quad -\left(\frac{x^2}{4\sigma^2}l_{s\mu}l_{s\nu} + \frac{z^2}{4\sigma^2}l_{t\mu}l_{n\nu} - \frac{xz}{4\sigma^2}\{l_r,l_t\}_{\mu\nu}\right)\mathcal{K}_{\nu-2}\Bigg] \end{aligned} \tag{5.65}$$

$$
\begin{aligned}
\langle\langle r_\mu s_\nu \rangle\rangle_{rs} = \int dxdy \, & \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2} \, e^{iyzp\cdot l_r/\sigma - ixzp\cdot l_s/\sigma} \times \\
& \left[\frac{xyz^2}{\sigma^2} p_\mu p_\nu \mathcal{K}_\nu + \left(\frac{ixz^2}{2\sigma^2} l_{t\mu} p_\nu - \frac{z}{2\sigma} \delta_{\mu\nu} \right.\right. \\
& \left. + \frac{ixyz}{2\sigma^2} (p_\mu l_{s\nu} + l_{r\mu} p_\nu) - \frac{iyz^2}{2\sigma^2} p_\mu l_{n\nu} \right) \mathcal{K}_{\nu-1} \\
& \left. + \left(\frac{yz}{4\sigma^2} l_{r\mu} l_{n\nu} - \frac{xy}{4\sigma^2} l_{r\mu} l_{s\nu} + \frac{z^2}{4\sigma^2} l_{t\mu} l_{n\nu} - \frac{xz}{4\sigma^2} l_{t\mu} l_{s\nu} \right) \mathcal{K}_{\nu-2} \right]
\end{aligned}
$$

5.3.2 Expressions in terms of Riemann theta functions

In this section the integrals $\langle\langle X \rangle\rangle_{rs}$ will be expressed using Jacobi and Riemann theta functions. In order to facilitate the numerical evaluation we make the arguments of the theta functions dimensionless using the change of variables $\bar{\lambda} = L^2 \hat{\lambda}/4\sigma$, which gives

$$
\begin{aligned}
X_4 &= \frac{iyzL}{\sigma} \mathbf{p} \cdot \mathbf{n} - \frac{ixzL}{\sigma} \mathbf{p} \cdot \mathbf{m} \\
Y_4 &= \frac{1}{\hat{\lambda}} (x\mathbf{m}^2 + y\mathbf{n}^2 + z(\mathbf{m} - \mathbf{n})^2) \\
Z_4 &= \frac{L^2}{4\sigma} \hat{\lambda} \left(xm_1^2 + ym_2^2 + zm_3^2 + \frac{xyz}{\sigma} p^2 \right)
\end{aligned}
\tag{5.66}
$$

if we write $l_{r\mu} = (0, L\mathbf{n})$ and $l_{s\mu} = (0, L\mathbf{m})$. To carry out the summation we use the following expression for the Riemann theta function

$$
\theta(u|\Omega) = \sum_{\mathbf{n},\mathbf{m} \in \mathbb{Z}^3} e^{2\pi i (k^T u + \frac{1}{2} k^T \Omega k)} \tag{5.67}
$$

where $k = (\mathbf{m}, \mathbf{n})$, $u \in \mathbb{C}^2$ and $\Omega \in \mathbb{H}_2 = \{\Omega \in M(2, \mathbb{C}) : \Omega = \Omega^T, \mathrm{Im}\,\Omega > 0\}$ i.e. it is a symmetric matrix with positive definite imaginary part. Comparing with the expressions for X_4 and Y_4 above it is easy to identify

$$
\begin{aligned}
u &= \frac{zL\mathbf{p}}{2\pi\sigma} (x, -y) \\
\Omega &= \frac{i}{\pi\hat{\lambda}} \begin{pmatrix} x+z & -z \\ -z & y+z \end{pmatrix}.
\end{aligned}
\tag{5.68}
$$

To write down the integrals explicitly it is necessary to choose a direction for the momentum $\mathbf{p}$, so from now on we will assume that we are in the center-of-mass frame. Since sums over odd powers of l then vanish by symmetry, we immediately find the following result for the simplest three integrals

$$
\langle\langle 1 \rangle\rangle_{rs} = \int dxdyd\hat{\lambda} \, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\hat{\lambda}^{\nu-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2} \left(\frac{L^2}{4\sigma} \right)^\nu e^{-Z} \theta(0|\Omega)
$$

$$\langle\langle r_\mu\rangle\rangle_{rs} = \int dxdyd\hat{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\hat{\lambda}^{\nu-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\left(\frac{L^2}{4\sigma}\right)^\nu e^{-Z}\theta(0|\Omega)\frac{yz}{\sigma}p_\mu$$
$$\langle\langle s_\mu\rangle\rangle_{rs} = \int dxdyd\hat{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\hat{\lambda}^{\nu-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\left(\frac{L^2}{4\sigma}\right)^\nu e^{-Z}\theta(0|\Omega)\frac{xz}{\sigma}p_\mu$$

where $\nu = n_1 + n_2 + n_3 - 4$ and $Z = Z_4$.

For the other three integrals we need the derivatives of the theta function with respect to the components of u, the effect of which is to bring down an extra factor of $\mathbf{m}$ or $\mathbf{n}$. More precisely, taking the derivative $\partial_{u_1}\theta(u|\Omega) = \mathbf{v}\cdot\nabla_u\theta(u|\Omega)$ in the direction $\mathbf{v} = (1,0)$ brings down an $\mathbf{m}$, while doing the same in the direction $\mathbf{v} = (0,1)$ bring down an $\mathbf{n}$. To get higher powers like $\mathbf{m}^2$, $\mathbf{n}^2$ or $\mathbf{m}\cdot\mathbf{n}$ it turns out to be simpler to take the derivative with respect to the elements Ω_{11}, Ω_{22} or Ω_{12} of the matrix Ω. Using this we find in the center-of-mass frame that

$$\begin{aligned}\langle\langle r_\mu r_\nu\rangle\rangle_{rs} = &\int dxdyd\hat{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\hat{\lambda}^{\nu-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\left(\frac{L^2}{4\sigma}\right)^\nu e^{-Z}\\ &\Bigg[p_\mu p_\nu\frac{y^2z^2}{\sigma^2}\theta(0|\Omega) + \delta_{\mu\nu}\frac{2(x+y)}{L^2}\theta(0|\Omega)\hat{\lambda}^{-1}\\ &+\frac{12t_{\mu\nu}}{L^2}\hat{\lambda}^{-2}\Bigg(yz\frac{\partial}{\partial\Omega_{11}}\theta(0|\Omega) - y(y+z)\frac{\partial}{\partial\Omega_{22}}\theta(0|\Omega)\\ &-z(y+z)\left\{\frac{\partial}{\partial\Omega_{11}}\theta(0|\Omega) + \frac{\partial}{\partial\Omega_{22}}\theta(0|\Omega) - \frac{\partial}{\partial\Omega_{12}}\theta(0|\Omega)\right\}\Bigg)\Bigg]\end{aligned}$$

$$\begin{aligned}\langle\langle s_\mu s_\nu\rangle\rangle_{rs} = &\int dxdyd\hat{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\hat{\lambda}^{\nu-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\left(\frac{L^2}{4\sigma}\right)^\nu e^{-Z}\\ &\Bigg[p_\mu p_\nu\frac{y^2z^2}{\sigma^2}\theta(0|\Omega) + \delta_{\mu\nu}\frac{2(x+z)}{L^2}\theta(0|\Omega)\hat{\lambda}^{-1}\\ &+\frac{12t_{\mu\nu}}{L^2}\hat{\lambda}^{-2}\Bigg(xz\frac{\partial}{\partial\Omega_{22}}\theta(0|\Omega) - x(x+z)\frac{\partial}{\partial\Omega_{11}}\theta(0|\Omega)\\ &-z(x+z)\left\{\frac{\partial}{\partial\Omega_{11}}\theta(0|\Omega) + \frac{\partial}{\partial\Omega_{22}}\theta(0|\Omega) - \frac{\partial}{\partial\Omega_{12}}\theta(0|\Omega)\right\}\Bigg)\Bigg]\end{aligned}$$

$$\begin{aligned}\langle\langle r_\mu s_\nu\rangle\rangle_{rs} = &\int dxdyd\hat{\lambda}\, \frac{x^{n_1-1}y^{n_2-1}z^{n_3-1}\hat{\lambda}^{\nu-1}}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2\sigma)^2}\left(\frac{L^2}{4\sigma}\right)^\nu e^{-Z}\\ &\Bigg[p_\mu p_\nu\frac{xyz^2}{\sigma^2}\theta(0|\Omega) - \delta_{\mu\nu}\frac{2z}{L^2}\theta(0|\Omega)\hat{\lambda}^{-1}\\ &+\frac{12t_{\mu\nu}}{L^2}\hat{\lambda}^{-2}\Bigg((xz-\frac{\sigma}{2})\frac{\partial}{\partial\Omega_{11}}\theta(0|\Omega) + (yz-\frac{\sigma}{2})\frac{\partial}{\partial\Omega_{22}}\theta(0|\Omega)\\ &+(z^2+\frac{\sigma}{2})\left\{\frac{\partial}{\partial\Omega_{11}}\theta(0|\Omega) + \frac{\partial}{\partial\Omega_{22}}\theta(0|\Omega) - \frac{\partial}{\partial\Omega_{12}}\theta(0|\Omega)\right\}\Bigg)\Bigg]\end{aligned}$$

In the next chapter we will present numerical results both in and out of the center-of-mass frame.

Chapter 6

Sunset Integral Numerical Results

In the previous chapter it was shown that the finite volume corrections to the sunset integrals can be given formal expressions in terms of either modified Bessel functions or Riemann and Jacobi theta functions. To estimate the importance of these corrections it remains to numerically evaluate the integrals and relate them to the infinite volume contribution. How this is done in detail will be the subject of this chapter, and it depends on which representation we work with as well as the frame of reference in which the calculation is carried out.

To evaluate the integrals of Chapter 5 in the center-of-mass frame is relatively straightforward, especially in the case when they are given by Bessel functions. After applying Eq. (3.4) or (5.4) we obtain a one-dimensional sum of integrals for $\langle\langle X\rangle\rangle_r$ and a two-dimensional sum for $\langle\langle X\rangle\rangle_{rs}$, which can be computed directly using MATHEMATICA and the built-in function `BesselK`. In the first case it is no problem to include terms up to $k \simeq 200$ within a reasonable amount of computing time, while in the second we have to truncate the sum at $k_r, k_s, k_n \simeq 40$. In both cases this is enough to give a good convergence of the sums for $L \gtrsim 3$ fm, as can be seen in the figures below.

When using the theta functions it is bit intricate to make sure that the contributions from the terms with $l_r = 0$, $l_s = 0$ and $l_r = l_s$ not are included, since the built-in functions `EllipticTheta` and `SiegelTheta` (MATHEMATICA's version of the Riemann theta function) don't take this into consideration. For the integrals $\langle\langle X\rangle\rangle_r$ it is taken care of by making the replacement $\theta_{30}^3 \to \theta_{30}^3 - 1$, and noting that for θ_{31} and θ_{32} the unwanted contributions automatically vanish because of the k's in front of the exponential. In the case of the $\langle\langle X\rangle\rangle_{rs}$ integrals we have to make the replacement

$$\theta(0|\Omega) \to \theta(0|\Omega) - \theta_{30}(x+z) - \theta_{30}(y+z) - \theta_{30}(-z) + 2 \qquad (6.1)$$

to remove the extra terms, where each term is there to remove the contribution from when $l_r = 0$, $l_s = 0$ or $l_r = l_s$ respectively, and the 2 has to be added to correct for the fact that we've subtracted the term with $l_r = l_s = 0$ three times. Similar replacements can be done for the various derivatives of the Riemann function, with the difference that some terms will vanish automatically due to the extra factors of $\mathbf{m}$ and $\mathbf{n}$ that come from the differentiation.

A complication associated with the computation of the $\langle\langle X\rangle\rangle_{rs}$ integrals is that the differential operator in MATHEMATICA doesn't work together with the function `SiegelTheta`. In order to evaluate the integrals depending on these derivatives we therefore had to program them our selves, which was done using the `Compile` command and the method in [11]. To increase the speed of convergence of the sum it is for certain values of the variables x and y better to compute the function using the inverse matrix Ω^{-1}, related to the original function by the Siegel transformation

$$\theta(u|\Omega) = e^{-i\pi u^T \Omega^{-1} u} \sqrt{\det(-i\Omega)}\theta(\Omega^{-1}u| - \Omega^{-1}), \tag{6.2}$$

as described in Appendix A. Since the speed of convergence for each value of x and y depends on the magnitude of the eigenvalues of Ω, we have used the criterion $\det(\Omega) < 1$ to trigger the transformation and have through this reduced the computation time substantially. We have also adapted the maximum value of the summation variables $\mathbf{m}$ and $\mathbf{n}$ in such a way that the sum is truncated when its terms gets smaller than some pre-defined value, which is chosen in order to obtain a certain precision. The function has been tested against `SiegelTheta` in the case of no derivative, and they can be made to agree to arbitrary precision.

In the case of a moving frame where $p \cdot l_r$ or $p \cdot l_s$ is non-vanishing, it is no longer possible to make use of Eqs. (3.4) and (5.4) to reduce the number of independent summation variables. This makes it much more demanding to evaluate the integrals using Bessel functions, and instead of going to $k \simeq 200$ we have to stop at $|l_r| \simeq 10$ in the case of $\langle\langle X\rangle\rangle_r$ and at $|l_r|, |l_s|, |l_t| \simeq 5$ for $\langle\langle X\rangle\rangle_{rs}$. For large values of L the sum converges fast so the precision doesn't suffer too much, but in the case of $L \lesssim 3$ fm this effect is important.

Presented in the graphs that follow are numerical results for $\langle\langle X\rangle\rangle_r$ and $\langle\langle X\rangle\rangle_{rs}$ in the cases where $X \in \{1, r_\mu, r_\mu r_\nu\}$. Each of these integrals are given as a function of L in fm for three different values of the particle masses m_1, m_2 and m_3 and the external momentum p, with the powers of the propagators as $n_1 = n_2 = n_3 = 1$. The first row of each figure shows the integral for the case $m_1 = m_2 = m_3 = 0.1$ GeV and $p^2 = 0.015$ GeV^2 in the center-of-mass frame, and in the second row the values of the parameters are the same but the momentum is in the x-direction. The third row shows the case where $m_1 = 0.1$ GeV, $m_2 = 0.9m_1$ and $m_3 = 0.8m_1$ in the center-of-mass frame with $p^2 = 0.015$ GeV^2. In a physical calculation only momenta with the values $p = 2\pi n/L$ of the components would be allowed, but here we have used the same momentum for all values of L to simplify the presentation.

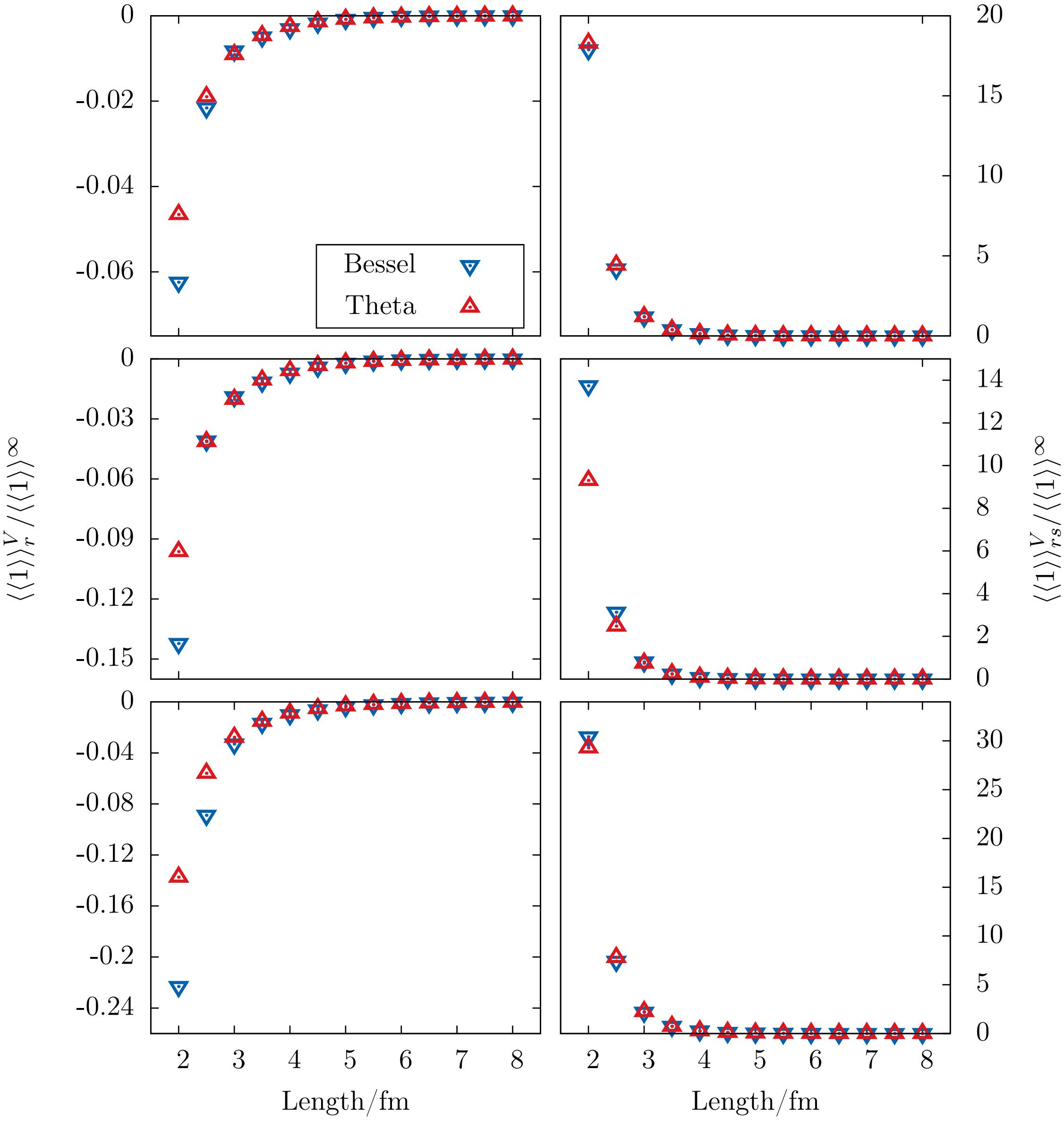

Figure 6.1: The relative value of the integrals $\langle\langle 1\rangle\rangle_r$ and $\langle\langle 1\rangle\rangle_{rs}$ as compared to the infinite volume case, for different values of the particle masses and the external momentum.

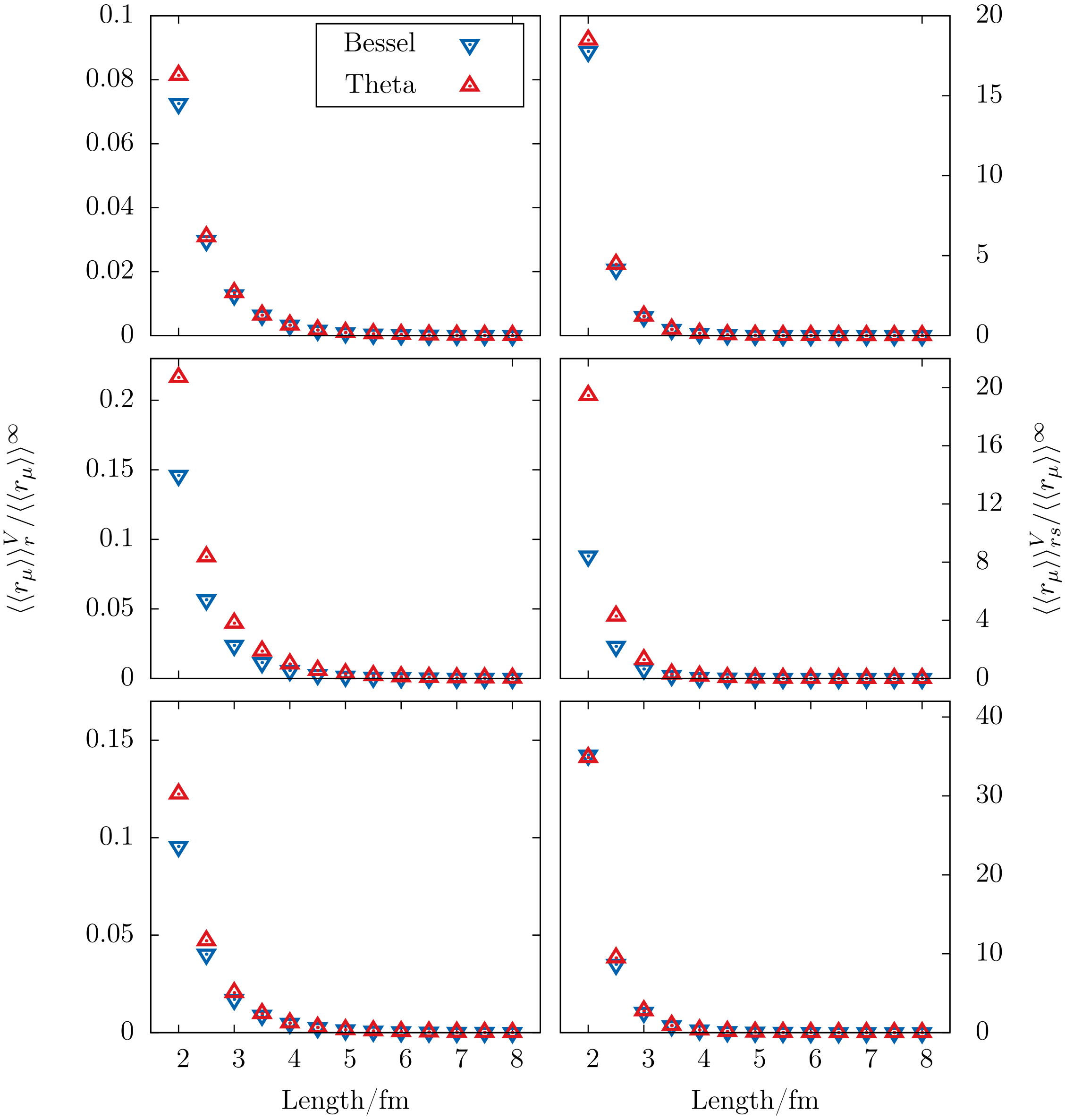

Figure 6.2: The relative value of the integrals $\langle\langle r_\mu\rangle\rangle_r$ and $\langle\langle r_\mu\rangle\rangle_{rs}$ as compared to the infinite volume case, for different values of the particle masses and the external momentum.

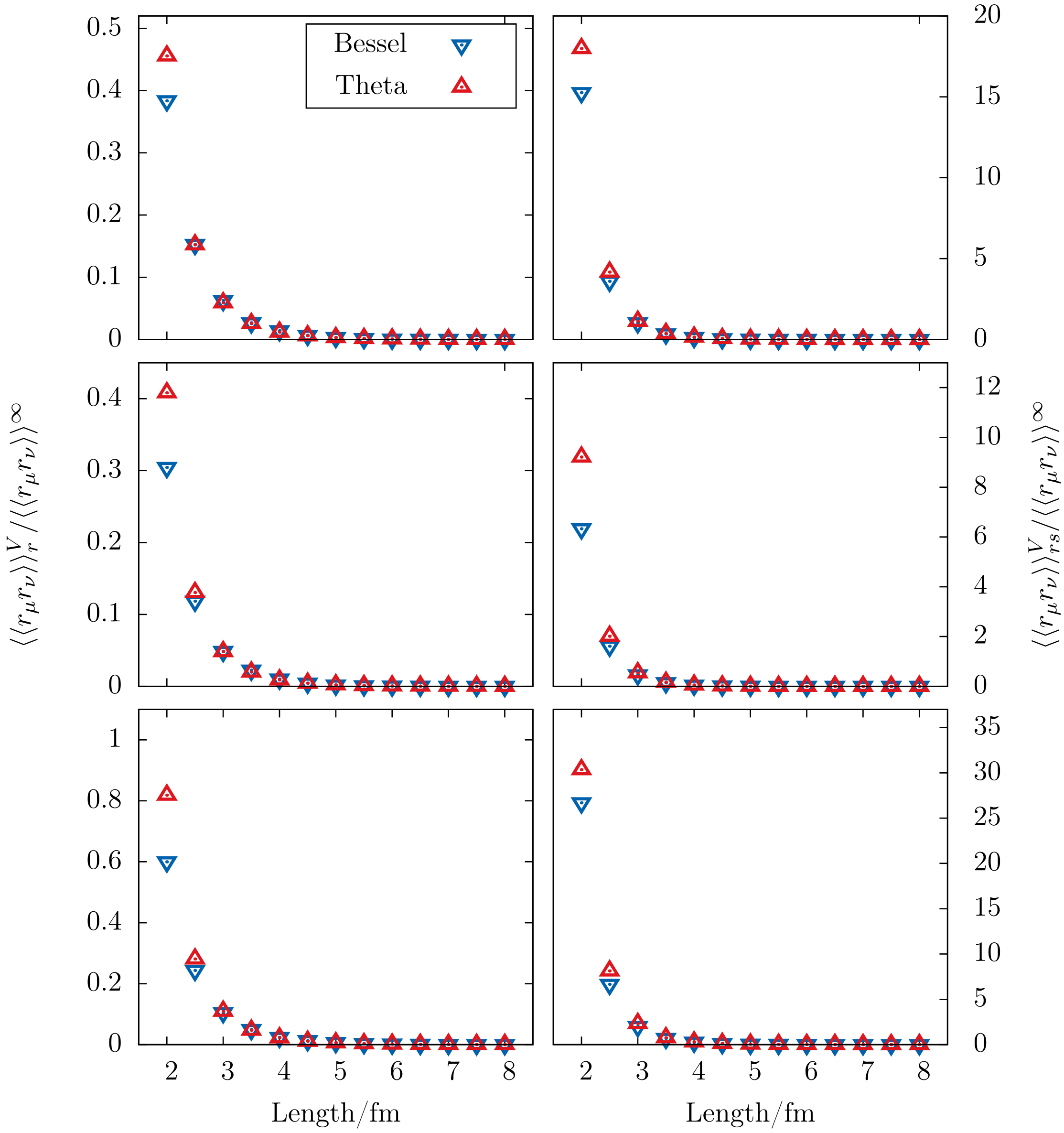

Figure 6.3: The relative value of the parts of the integrals $\langle\langle r_\mu r_\nu \rangle\rangle_r$ multiplying $p_\mu p_\nu$ and $\langle\langle r_\mu r_\nu \rangle\rangle_{rs}$ as compared to the infinite volume case, for different values of the particle masses and the external momentum.

Chapter 7

Conclusions

From the analytic and numerical results of the previous chapters we can extract some general information about the behavior of finite volume corrections to one- and two-loop integrals. When the mass of the particles are increased the values of the integrals decreases, and the same holds in most cases when we move out of the center-of-mass frame. This has important consequences for lQCD where the simulations now run with quite high unphysical masses, since if these masses are decreased to around the pion mass our results show that finite volume effects gets important. This conclusion is even more apparent when it comes to the size of the volume, since the integrals increase rapidly when the volume gets smaller.

When it comes to the computational aspect there is much that still could be added and improved. In this work we have only considered volumes where the side length is equal in all spatial directions, but this could easily be generalized to a non-symmetric case. It would put a higher demand on computation power since we would then like in the moving frame case have to sum over the components of the vectors l_r and l_s individually. There are also a lot of numerical results in moving frames that are yet to be generated, but these are included in the in the present formalism and do not pose a big challenge. To perform the tasks above it would be worthwhile to reprogram the Riemann theta function in a more efficient way, which can be done using the approach presented in [11].

Finally we have not included numerical values of the total sunset integrals in this work, since they involve the integrals $\langle\langle r_\mu \rangle\rangle_s$ and $\langle\langle r_\mu \rangle\rangle_t$ which have not yet been computed. There are no formal obstacles stopping this since these integrals can be obtained from $\langle\langle r_\mu \rangle\rangle_r$ by variable substitutions, as argued in Chapter 5. It is however important to make sure that the conventions for which terms are to be absorbed into the divergent parts are the same in the finite and infinite volume case, which is not yet the case for the programs we have used.

Appendix A

Special functions

For the numerical evaluation of the loop integrals we make use of the modified Bessel functions of the second kind $K_\nu(z)$, the third Jacobi theta functions $\theta_{3n}(\lambda, z)$ and the Riemann theta function $\theta(z|\Omega)$. It is therefore worthwhile to investigate some properties of these functions in order to better understand the results of Chapters 3 and 5.

As the definition of a two argument version of the Bessel function we use the integral representation

$$\mathcal{K}_\nu(Y, Z) = \int_0^\infty d\lambda \, \lambda^{\nu-1} e^{-Z\lambda - Y/\lambda}, \tag{A.1}$$

which is convenient since this type of integral appears frequently in the calculations above. This function is related to the usual Bessel function $K_\nu(z)$ by

$$\mathcal{K}_\nu(Y, Z) = 2\left(\frac{Y}{Z}\right)^\nu K_\nu\left(2\sqrt{YZ}\right). \tag{A.2}$$

In the limit of large z it can be shown that the Bessel function behaves as

$$K_\nu(z) \sqrt{\frac{\pi}{2z}} e^{-z} + \mathcal{O}\left(\frac{e^{-z}}{z^{3/2}}\right) \tag{A.3}$$

and so decays exponentially. This means that as L grows large the finite volume corrections will fall of exponentially, as shown for the case of theta functions in the main text.

In the expansion around the reduced dimension that shows up in the computation of the divergent parts of $\langle\langle X\rangle\rangle_r$ we have used the function

$$\tilde{\mathcal{K}}_\nu(Y, Z) = \frac{1}{2}\ln\left(\frac{Y}{Z}\right)\mathcal{K}_\nu(Y, Z) + 2\left(\frac{Y}{Z}\right)^{\nu/2}\tilde{K}_\nu(Y, Z) \tag{A.4}$$

where $\tilde{K}_\nu(Y, Z)$ is the derivative of the Bessel function with respect to the order ν. It is easy to obtain by taking the derivative of Eq. A.2 with respect to ν and using the product rule.

Turning now to some version of the third Jacobi theta function, we will use the definition

$$\theta_{3n}(\tau, z) = \sum_{k=-\infty}^{\infty} k^{2n} e^{-k^2\tau + 2ikz}, \tag{A.5}$$

where putting $n = 0$ gives back the usually encountered case. This function has a number of properties, of which the most important for our sake is the inversion relation

$$\theta_{30}(\tau, z) = \sqrt{\frac{\pi}{\tau}} e^{-z^2/\tau} \theta_{30}\left(\frac{i\pi}{\tau}, \frac{z}{i\tau}\right) \tag{A.6}$$

which allows us to evaluate the function using the argument $1/\tau$ instead of τ. This is favorable for small values of τ, since it will reduce the number of terms it takes the sum to converge considerably. Similar relations can be found for the functions θ_{3n} with higher values of n by differentiating the equation above, but they will become increasingly more complicated.

A generalized version of the functions above is the Riemann theta function which in two dimensions is defined as

$$\theta(z|\Omega) = \sum_{n \in \mathbb{Z}^2} e^{i\pi(n^T \Omega n + 2n^T z)} \tag{A.7}$$

for $z \in \mathbb{C}^2$ and $\Omega \in \mathbb{H}_2 = \{\Omega \in M(2, \mathbb{C}) : \Omega = \Omega^T, \operatorname{Im}\Omega > 0\}$, the space of symmetric matrices with positive definite imaginary part. Similar to the inversion relation for the Jacobi functions is the so-called Siegel transformation

$$\theta(z|\Omega) = e^{-i\pi z^T \Omega^{-1} z} \sqrt{\det(-i\Omega)}\, \theta(\Omega^{-1} z | - \Omega^{-1}) \tag{A.8}$$

which allows us to evaluate the Riemann function with the inverse matrix Ω^{-1} as the argument instead of Ω. The Siegel transformation is good to use when the eigenvalues of the matrix Ω becomes small, since the inverse matrix then will have large eigenvalues. In the numerical evaluation above this has been incorporated using the criterion $\det(\Omega) < 1$ to decide when to use the transformation or not.

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