

Two-Point Functions and S-Parameter in QCD-like Theories

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Abstract

We calculated the vector, axial-vector, scalar and pseudo-scalar two-point functions up to two-loop level in the low-energy effective field theory for three different QCD-like theories. In addition we also calculated the pseudo-scalar decay constant G_M . The QCD-like theories we used are those with fermions in a complex, real or pseudo-real representation with in general n flavours. These case correspond to global symmetry breaking pattern of $SU(n)_L \times SU(n)_R \rightarrow SU(n)_V$, $SU(2n) \rightarrow SO(2n)$ or $SU(2n) \rightarrow Sp(2n)$. We also estimated the S parameter for those different theories.

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1 Introduction

The different global symmetry breaking patterns of QCD-like theories with a vector-like gauge group have been summarized in [1, 2, 3] around 30 years ago. The global symmetry and its spontaneous breaking depend on whether the fermions live in a complex, real and pseudo-real representation of the gauge group. For n identical fermions this corresponds to the symmetry breaking pattern $SU(n)_L \times SU(n)_R \rightarrow SU(n)_V$, $SU(2n) \rightarrow SO(2n)$ and $SU(2n) \rightarrow Sp(2n)$ respectively. These theories can be used to characterize some of technicolor models with vector-like gauge bosons. QCD-like theories are also important in the theory of finite baryon density. Here the real and pseudo-real case allow to investigate the mechanism of diquark condensate and finite density without the sign problem. A main nonperturbative tool in studying strongly interacting theories is lattice gauge theory. Numerical calculations are performed at finite fermion mass and need in general to be extrapolated to the zero mass limit. In the case of QCD Chiral Perturbation Theory (ChPT) is used to help with this extrapolation. Our work has the intention of providing similar formulas

for the QCD-like theories using the effective field theory (EFT) appropriate for the alternative global symmetry patterns.

These EFT have been used at lowest order (LO) [4] with earlier work to be found in [5, 6, 7] and some studies at next-to-leading order (NLO) have also appeared [8, 9, 10]. The former two are the usual QCD case with n flavours. In our earlier papers [11, 12] we have systematically studied the effective field theory of these three different QCD-like theories to next-to-next-to-leading order (NNLO). We managed to write the EFT of these cases in an extremely similar form. We calculated the quark-antiquark condensates, the mass and decay constant of the pseudo-Goldstone bosons [11], and meson-meson scattering [12]. In this paper we extend the analysis to two-point correlation functions. We obtain expressions for the vector, axial-vector, scalar and pseudo-scalar two-point functions as well as the pion pseudo-scalar coupling G_M to NNLO¹ or order p^6 .

In our earlier work [11, 12], we called the three different cases QCD or complex, adjoint or real and two-colour or pseudo-real. In this paper we use only the latter, more general, terminology.

One motivation for this set of work was the study of strongly interacting Higgs sectors, reviews are [13, 14]. For any model beyond the Standard Model, passing the test of oblique corrections, or precision LEP observables, is crucial [15, 16]. Over the years, the impact of the oblique corrections in those models have been studied quite intensively but in strongly interacting cases mainly an analogy with QCD has been invoked. Lattice gauge theory methods allow to study strongly interacting models from first principles. The contributions from these theories to the S -parameter can be calculated using the two-point functions studied here and our formulas are useful to perform the extrapolation to the massless case. This was in fact the major motivation for the present work but we included the other two-point functions for completeness.

The paper is organized as follows. In Section 2 we give a brief introduction to EFT for the three different cases. Section 3 is the main part of the paper. We define the fermion currents and the two-point functions in Section 3.1. In Sections 3.2 to 3.5, we present the calculation of vector, axial-vector, scalar, pseudo-scalar two-point functions. In Section 4, we discuss the oblique corrections and the S -parameter. Section 5 summarizes our main results and we present the definition.

2 Effective Field Theory

In this section we briefly review the EFT of QCD-like theories, the details can be found in the earlier paper [11]. The basic methods are those of Chiral Perturbation Theory [17, 18]. The counting of orders is in all cases the same as in ChPT, we count momenta as order p and the fermion mass m as order p^2 .

2.1 Complex representation: QCD and CHPT

The case of n fermions in a complex representation is essentially like QCD. The Lagrangian with external left and right vector, scalar and pseudoscalar external

¹We use LO, NLO and NNLO as synonyms for order p^2 , order p^4 and order p^6 calculations even if the order p^2 vanishes.

sources, l_μ, r_μ, s and p , is

$$\begin{aligned} \mathcal{L} = & \bar{q}_{Li} i\gamma^\mu D_\mu q_{Li} + \bar{q}_{Ri} i\gamma^\mu D_\mu q_{Ri} + \bar{q}_{Li} \gamma^\mu l_{\mu ij} q_{Lj} + \bar{q}_{Ri} \gamma^\mu r_{\mu ij} q_{Rj} \\ & - \bar{q}_{Ri} \mathcal{M}_{ij} q_{Lj} - \bar{q}_{Li} \mathcal{M}_{ij}^\dagger q_{Rj} \quad i, j = 1, 2, \dots, n. \end{aligned} \quad (1)$$

The covariant derivative is given by $D_\mu q = \partial_\mu q - iG_\mu q$, and the mass matrix $\mathcal{M} = s - ip$. The sums shown are over the flavour index. The sums over gauge group indices are implicit.

The Lagrangian (1) has a symmetry $SU(n)_L \times SU(n)_R$ which is made local by the external sources [8, 18]. The quark-anti-quark condensate $\langle \bar{q}q \rangle$ breaks $SU(n)_L \times SU(n)_R$ spontaneously to the diagonal subgroup $SU(n)_V$. According to the Nambu-Goldstone theorem, $n^2 - 1$ Goldstone Bosons will thus be generated. We add a small fermion mass m explicitly by setting $s = m + s$. This mass term explicitly breaks the symmetry $SU(n)_L \times SU(n)_R$ down to $SU(n)_V$ as well and gives the Goldstone bosons a small mass.

The Goldstone boson manifold $SU(n)_L \times SU(n)_R / SU(n)_V$ can be parametrized by

$$u = \exp\left(\frac{i}{\sqrt{2}F} \pi^a T^a\right) \quad a = 1, 2, \dots, n^2 - 1. \quad (2)$$

The T^a are the generators of $SU(n)$ normalized to $\langle T^a T^b \rangle = \delta^{ab}$. The notation $\langle A \rangle$ stands for the trace over flavour indices. u transforms under $g_L \times g_R \in SU(n)_L \times SU(n)_R$ as $u \rightarrow g_R u h^\dagger = h u g_L^\dagger$ where h is the ‘‘compensator’’ and is a function of u, g_L and g_R . The methods are those of [19], but we use the notation of [20, 21]. We can construct quantities which transform under the unbroken group H as : $O \rightarrow h O h^\dagger$

$$\begin{aligned} u_\mu &= i[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - l_\mu)u^\dagger], \\ \nabla_\mu O &= \partial_\mu O + \Gamma_\mu O - O\Gamma_\mu, \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \\ f_{\pm\mu\nu} &= u l_{\mu\nu} u^\dagger \pm u^\dagger r_{\mu\nu} u. \end{aligned} \quad (3)$$

The field strengths $l_{\mu\nu}$ and $r_{\mu\nu}$ are

$$\begin{aligned} l_{\mu\nu} &= \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu], \\ r_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu]. \end{aligned} \quad (4)$$

The covariant derivative ∇_μ contains

$$\Gamma_\mu = \frac{1}{2}[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - l_\mu)u^\dagger]. \quad (5)$$

χ contains the matrix \mathcal{M} , which is the combination of scalar and pseudo-scalar sources

$$\chi = 2B_0 \mathcal{M} = 2B_0(s - ip). \quad (6)$$

Using the quantities in (3), we can find the leading order, p^2 , Lagrangian which is invariant under Lorentz and chiral symmetry:

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle. \quad (7)$$

The subscript ‘‘2’’ stands for the order of p^2 . The p^4 and p^6 Lagrangian will be explained in Section 2.3.

2.2 Real and Pseudo-Real representation

The case of n fermions in a real or pseudo-real representation of the gauge group we can treat in a similar way as the complex case. In the real case, the global symmetry breaking pattern is $SU(2n) \rightarrow SO(2n)$, and the number of generated Goldstone bosons is $2n^2 + n - 1$. In the pseudo-real case, the symmetry breaking is $SU(2n) \rightarrow Sp(2n)$, and the number of generated Goldstone is $2n^2 - n - 1$. In both cases anti-fermions are in the same representation of the gauge group and can be put together in a $2n$ vector \hat{q} , see [11] for more details.

The condensate can now be a diquark condensate as well as a quark-antiquark condensate. Our choice of vacuum corresponds to a quark-anti-quark condensate. In terms of the $2n$ fermion vector \hat{q} they can be written as

$$\text{Real : } \quad \langle \hat{q}^T C J_S \hat{q} \rangle + \text{h.c.} \quad J_S = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \quad (8)$$

$$\text{Pseudo - Real : } \quad \langle \hat{q}_\alpha \epsilon_{\alpha\beta} C J_A \hat{q}_\beta \rangle + \text{h.c.} \quad J_A = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}. \quad (9)$$

Here C is the charge conjugation operator. J_S and J_A are symmetric and anti-symmetric $2n \times 2n$ matrices, I is the $n \times n$ unit matrix. Since J_S and J_A often appear in the same way in the expressions, we use J for both cases unless a distinction is needed.

The generators, T^a , of the global symmetry group $SU(2n)$ can be separated into belonging to the broken, X^a , or unbroken part, Q^a . They satisfy the following relations with J :

$$JQ^a = -Q^{aT} J, \quad JX^a = X^{aT} J, \quad (10)$$

The Goldstone boson manifold can be parametrized with

$$U = uJu^T \rightarrow gUg^T, \quad \text{with} \quad u = \exp\left(\frac{i}{\sqrt{2}F}\pi^a X^a\right). \quad (11)$$

where $J = J_S$ and a runs from 1 to $2n^2 + n - 1$ for the real case and $J = J_A$ and a runs from 1 to $2n^2 - n - 1$ for the pseudo-real case.

In our earlier paper [11], we constructed quantities similar to those in (3-5)

$$\begin{aligned} u_\mu &= i[u^\dagger(\partial_\mu - iV_\mu)u - u(\partial_\mu + iJV_\mu^T J)u^\dagger], \\ \Gamma_\mu &= \frac{1}{2}[u^\dagger(\partial_\mu - iV_\mu)u + u(\partial_\mu + iJV_\mu^T J)u^\dagger], \\ f_{\pm\mu\nu} &= JuV_{\mu\nu}u^\dagger J \pm uV_{\mu\nu}u^\dagger, \\ \chi_\pm &= u^\dagger\chi Ju^\dagger \pm uJ\chi^\dagger u. \end{aligned} \quad (12)$$

The $2n \times 2n$ matrix V_μ includes the left and right-handed external sources

$$V_\mu = \begin{pmatrix} r_\mu & 0 \\ 0 & -l_\mu^T \end{pmatrix} \quad (13)$$

and $V_{\mu\nu}$ is the field strength

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i(V_\mu V_\nu - V_\nu V_\mu). \quad (14)$$

χ include the matrix $\hat{\mathcal{M}}$ via $\chi = 2B_0\hat{\mathcal{M}}$ [11]. Those quantities behave similarly as those (3) if we take

$$-JV_\mu^T J \rightarrow l_\mu, \quad V_\mu \rightarrow r_\mu. \quad (15)$$

With this correspondence, the Lagrangian of the real and pseudo-real case has the same form as the complex one. However one has to remember there are differences in the generators, external sources, coupling constants,... Anyway, now we can use the techniques of ChPT to perform the calculations.

2.3 High Order Lagrangians and Renormalization

Using Lorentz and chiral invariance, we can write down the p^4 EFT lagrangian [18] for all three cases using the quantities listed in (3) and (12):

$$\begin{aligned} \mathcal{L}_4 = & L_0\langle u^\mu u^\nu u_\mu u_\nu \rangle + L_1\langle u^\mu u_\mu \rangle\langle u^\nu u_\nu \rangle + L_2\langle u^\mu u^\nu \rangle\langle u_\mu u_\nu \rangle + L_3\langle u^\mu u_\mu u^\nu u_\nu \rangle \\ & + L_4\langle u^\mu u_\mu \rangle\langle \chi_+ \rangle + L_5\langle u^\mu u_\mu \chi_+ \rangle + L_6\langle \chi_+ \rangle^2 + L_7\langle \chi_- \rangle^2 + \frac{1}{2}L_8\langle \chi_+^2 + \chi_-^2 \rangle \\ & - iL_9\langle f_{+\mu\nu} u^\mu u^\nu \rangle + \frac{1}{4}L_{10}\langle f_+^2 - f_-^2 \rangle + H_1\langle l_{\mu\nu} l^{\mu\nu} + r_{\mu\nu} r^{\mu\nu} \rangle + H_2\langle \chi\chi^\dagger \rangle. \end{aligned} \quad (16)$$

To do the renormalization, we use the ChPT $\overline{\text{MS}}$ scheme with dimensional regularization [18, 8, 21]. The bare coupling constants L_i are defined as

$$L_i = (c\mu)^{d-4} [\Gamma_i\Lambda + L_i^r(\mu)], \quad (17)$$

where the dimension $d = 4 - 2\epsilon$, and

$$\Lambda = \frac{1}{16\pi^2(d-4)}, \quad (18)$$

$$\ln c = -\frac{1}{2}[\ln 4\pi + \Gamma'(1) + 1]. \quad (19)$$

The coefficients Γ_i for the complex case have been obtained in [8], for the real and pseudo-real case we have calculated them earlier in [11]. However, there are mistakes in the coefficients of L_9, L_{10} and H_1 in the Table 1 of [11]. These mistakes had no effects on our previous calculations. We therefore list all the coefficients here again in Table 1.

The p^6 Lagrangian for the complex case and general n has been obtained in [20], it contains 112+3 terms. The divergence structure of the bare coupling constants K_i in the p^6 can be written as²

$$K_i = (c\mu)^{2(d-4)} \left[K_i^r - \Gamma_i^{(2)}\Lambda^2 - \left(\frac{1}{16\pi^2}\Gamma_i^{(1)} + \Gamma_i^{(L)} \right) \Lambda \right]. \quad (20)$$

The coefficients $\Gamma_i^{(2)}$, $\Gamma_i^{(1)}$ and $\Gamma_i^{(L)}$ for the complex case have been obtained in [21].

For the real and pseudo-real case, the p^6 Lagrangian has the same form as in the complex case but some terms might be redundant. The divergence structure as given in (20) still holds but the coefficients are not known. One check on our results that remains is that all the non-local divergences cancel.

²The K_i have been made dimensionless by including a factor of $1/F^2$ explicitly in the order p^6 Lagrangian.

i	complex	real	pseudo-real
0	$n/48$	$(n+4)/48$	$(n-4)/48$
1	$1/16$	$1/32$	$1/32$
2	$1/8$	$1/16$	$1/16$
3	$n/24$	$(n-2)/24$	$(n+2)/24$
4	$1/8$	$1/16$	$1/16$
5	$n/8$	$n/8$	$n/8$
6	$(n^2+2)/(16n^2)$	$(n^2+1)/(32n^2)$	$(n^2+1)/(32n^2)$
7	0	0	0
8	$(n^2-4)/(16n)$	$(n^2+n-2)/(16n)$	$(n^2-n-2)/(16n)$
9	$n/12$	$(n+1)/12$	$(n-1)/12$
10	$-n/12$	$-(n+1)/12$	$-(n-1)/12$
1'	$-n/24$	$-(n+1)/24$	$-(n-1)/24$
2'	$(n^2-4)/(8n)$	$(n^2+n-2)/(8n)$	$(n^2-n-2)/(8n)$

Table 1: The coefficients Γ_i for the three cases that are needed to absorb the divergences at NLO. The last two lines correspond to the terms with H_1 and H_2 . This is the same as Table 1 in [11] but with the error for L_9, L_{10} and H_1 corrected.

3 Two-Point Functions

3.1 Definition

The effective action of the fermion level theory with external sources is

$$\exp\{iZ(l_\mu, r_\mu, s, p)\} = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}G_\mu \exp\left\{i \int d^4x \mathcal{L}_{QCD}(q, \bar{q}, G_\mu, l_\mu, r_\mu, s, p)\right\} \quad (21)$$

At low energies, i.e. below 1 GeV in QCD, the effective action can be obtained also from the low-energy effective theory

$$\exp\{iZ(l_\mu, r_\mu, s, p)\} = \int \mathcal{D}U \exp\left\{i \int d^4x \mathcal{L}_{eff}(U, l_\mu, r_\mu, s, p)\right\}. \quad (22)$$

With this effective action, the n-point Green functions can be easily derived by taking the functional derivative w.r.t. the external sources of $Z(J)$

$$G^{(n)}(x_1, \dots, x_n) = \frac{\delta^n}{i^n \delta j(x_1) \dots \delta j(x_n)} Z[J] \Big|_{J=0}. \quad (23)$$

Here j stands for any of the external sources l_μ, r_μ, s, p and J for the whole set of them.

The vector current v_μ and axial-vector current a_μ are included via

$$l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu. \quad (24)$$

In this paper we will calculate the two-point functions of vector, axial-vector, scalar and pseudo-scalar currents. The fermion currents in the complex case are

defined as

$$V_\mu^a(x) = \bar{q}_i T_{ij}^a \gamma_\mu q_j, \quad (25)$$

$$A_\mu^a(x) = \bar{q}_i T_{ij}^a \gamma_\mu \gamma_5 q_j, \quad (26)$$

$$S^a(x) = -\bar{q}_i T_{ij}^a q_j, \quad (27)$$

$$P^a(x) = i\bar{q}_i T_{ij}^a \gamma_5 q_j. \quad (28)$$

T^a is an $SU(n)$ generator³ or in addition for the singlet scalar and pseudo-scalar current the unit matrix which we label by T^0 . These currents also exist for the real and pseudo-real case. In this case also currents with two fermions or two anti-fermions exist. These can be combined with those above. The generators can then become $SU(2n)$ generators. All conserved generators are like the vector or scalar case while the broken generators are like the axial-vector or pseudo-scalar case. All those cases are related to the ones with the currents of (25)-(28) via transformations under the unbroken part of the symmetry group.

The definitions of the two-point functions are

$$\begin{aligned} \Pi_{V_a\mu\nu}(q) &\equiv i \int d^4x e^{iq\cdot x} \langle 0|T(V_\mu^a(x)V_\nu^a(0))^\dagger|0\rangle, \\ \Pi_{A_a\mu\nu}(q) &\equiv i \int d^4x e^{iq\cdot x} \langle 0|T(A_\mu^a(x)A_\nu^a(0))^\dagger|0\rangle, \\ \Pi_{SM_a\mu}(q) &\equiv i \int d^4x e^{iq\cdot x} \langle 0|T(V_\mu^a(x)S^a(0))^\dagger|0\rangle, \\ \Pi_{PM_a\mu}(q) &\equiv i \int d^4x e^{iq\cdot x} \langle 0|T(A_\mu^a(x)P^a(0))^\dagger|0\rangle, \\ \Pi_{S_a}(q) &\equiv i \int d^4x e^{iq\cdot x} \langle 0|T(S^a(x)S^a(0))^\dagger|0\rangle, \\ \Pi_{P_a}(q) &\equiv i \int d^4x e^{iq\cdot x} \langle 0|T(P^a(x)P^a(0))^\dagger|0\rangle. \end{aligned} \quad (29)$$

Using Lorentz invariance the two-point functions with vectors and axial-vectors can be decomposed in scalar functions

$$\Pi_{V_a\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_{V_a}^{(1)}(q^2) + q_\mu q_\nu \Pi_{V_a}^{(0)}(q^2). \quad (30)$$

where $\Pi_{V_a}^{(1)}(q^2)$ is the transverse part and $\Pi_{V_a}^{(0)}(q^2)$ is the longitudinal part or alternatively the spin one and spin 0 part. The same definition holds for the axial-vector two-point functions. The mixed functions can be decomposed as

$$\begin{aligned} \Pi_{SM_a\mu} &= q_\mu \Pi_{SM_a}, \\ \Pi_{PM_a\mu} &= iq_\mu \Pi_{PM_a}. \end{aligned} \quad (31)$$

Using the divergence of fermion currents and equal time commutation relations, we find that some two-point functions are related to each other by Ward identities. In the equal mass case considered here, they are

$$\begin{aligned} \Pi_{V_a}^{(0)} &= \Pi_{SM_a} = 0, \\ q^2 \Pi_{A_a}^{(0)} &= 2m \Pi_{PM_a}, \\ q^4 \Pi_{A_a}^{(0)} &= 4m^2 \Pi_{P_a} + 4m \langle \bar{q}q \rangle. \end{aligned} \quad (32)$$

³We have defined here the current with fermion-anti-fermion operators, hence the $SU(n)$ for n fermions. For the real and pseudo-real case, the unbroken symmetry relates them also to di-fermion or di-anti-fermion operators.

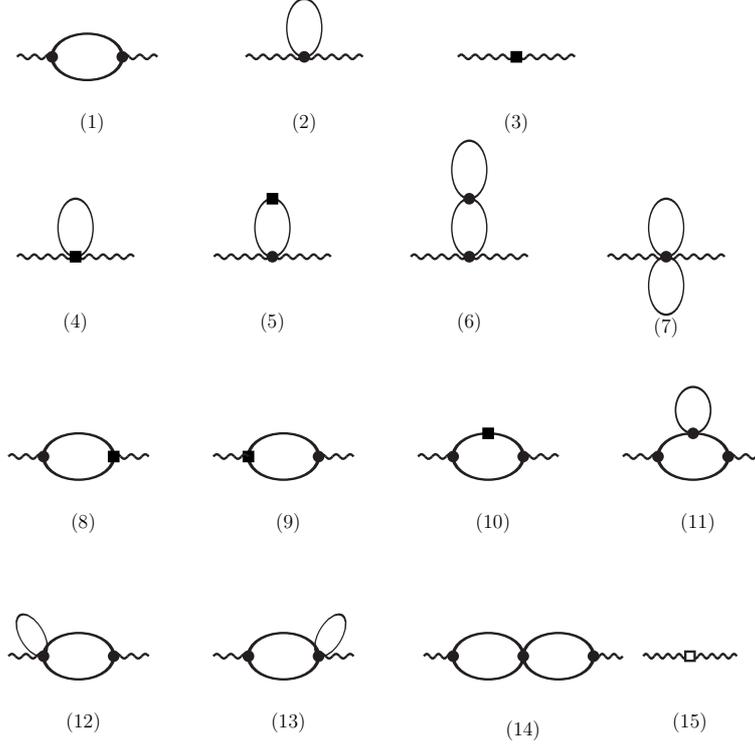


Figure 1: The diagrams for the vector two-point function. A filled circle is a vertex from \mathcal{L}_2 , a filled square a vertex from \mathcal{L}_4 , and an open square a vertex from \mathcal{L}_6 . The top line is order p^4 . The remaining ones are order p^6 .

The vacuum expectation value is the single quark-anti-quark one. We will use the last relation to double check our results of axial-vector and pseudo-scalar two-point functions.

The mixed two-point functions, Π_{SMa} and Π_{PMa} we do not discuss further since they are fully given by the Ward identities.

3.2 The Vector Two-Point Function

The vector two-point function is defined in (29). The longitudinal part vanishes for all three cases because of the Ward identities.

The Feynman diagrams for the vector two-point function are listed in Fig. 1. There is no diagram at lowest order. The diagrams at NLO are (1–3) in Fig. 1. The NNLO diagrams are (4–15). The 3-flavour QCD case is known to NNLO [22, 23].

We have rewritten the results in terms of the physical mass and decay constant. For these we use the notation M_M and F_M rather than the M_{phys} and F_{phys} used in [11, 12]. Their expression in terms of the lowest order quantities F and $M^2 = 2B_0m$ can be found in [11]. We also use the quantities

$$L = \frac{1}{16\pi^2} \log \frac{M_M^2}{\mu^2} \quad \text{and} \quad \pi_{16} = \frac{1}{16\pi^2}. \quad (33)$$

The loop integral \overline{B}_{22} is defined in Appendix A.1.

The results up to NNLO for three different cases are listed below, where the first line in each case is the NLO contributions, the rest are NNLO contributions.

Complex

$$\begin{aligned} \Pi_{VV}^{(1)} &= -\frac{n}{q^2} \left[4\overline{B}_{22}(M_M^2, M_M^2, q^2) + 2LM_M^2 \right] - 4L_{10}^r - 8H_1^r \\ &+ \frac{1}{F_M^2} \left\{ \left(\frac{4M_M^2}{q^2} Ln^2 - 16L_9^r n \right) \overline{B}_{22}(M_M^2, M_M^2, q^2) + \frac{4n^2}{q^2} [\overline{B}_{22}(M_M^2, M_M^2, q^2)]^2 \right. \\ &\quad \left. + \frac{M_M^4}{q^2} L^2 n^2 - 8q^2 K_{115}^r + 8M_M^2 (LL_{10}^r n - 4K_{81}^r - 4K_{82}^r n) \right\}, \end{aligned} \quad (34)$$

Real

$$\begin{aligned} \Pi_{VV}^{(1)} &= -\frac{1}{q^2} (n+1) \left[4\overline{B}_{22}(M_M^2, M_M^2, q^2) + 2M_M^2 L \right] - 4L_{10}^r - 8H_1^r \\ &+ \frac{1}{F_M^2} \left\{ \left[\frac{4M_M^2}{q^2} L(n+1)^2 - 16(n+1)L_9^r \right] \overline{B}_{22}(M_M^2, M_M^2, q^2) \right. \\ &\quad + \frac{4}{q^2} (n+1)^2 [\overline{B}_{22}(M_M^2, M_M^2, q^2)]^2 + \frac{M_M^4}{q^2} L^2 (n+1)^2 \\ &\quad \left. - 8q^2 K_{115}^r + 8M_M^2 [LL_{10}^r (n+1) - 4K_{81}^r - 8K_{82}^r n] \right\}, \end{aligned} \quad (35)$$

Pseudo – Real

$$\begin{aligned} \Pi_{VV}^{(1)} &= -\frac{1}{q^2} (n-1) \left[4\overline{B}_{22}(M_M^2, M_M^2, q^2) + 2M_M^2 L \right] - 4L_{10}^r - 8H_1^r \\ &+ \frac{1}{F_M^2} \left\{ \left[\frac{4M_M^2}{q^2} L(n-1)^2 - 16L_9^r (n-1) \right] \overline{B}_{22}(M_M^2, M_M^2, q^2) \right. \\ &\quad + \frac{4}{q^2} (n-1)^2 [\overline{B}_{22}(M_M^2, M_M^2, q^2)]^2 + \frac{M_M^4}{q^2} L^2 (n-1)^2 \\ &\quad \left. - 8q^2 K_{115}^r + 8M_M^2 [LL_{10}^r (n-1) - 4K_{81}^r - 8K_{82}^r n] \right\}. \end{aligned} \quad (36)$$

The complex result with $n = 3$ agrees with [22] when the masses there are set equal.

3.3 The Axial-Vector Two-Point Function

The axial-vector two-point function is defined in (29). Similar to the vector two-point function, it also can be decomposed in a transverse and longitudinal part.

$$\Pi_{AA}^{\mu\nu} = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{AA}^{(1)}(q^2) + q^\mu q^\nu \Pi_{AA}^{(0)}(q^2). \quad (37)$$

The diagrams contributing at LO are shown in (1–2) in Fig. 2. The LO results are the same for all three cases. The result is

$$\Pi_{AA}^{\mu\nu}(q^2) = 2F \left(g^{\mu\nu} - q^\mu q^\nu \frac{1}{q^2 - M^2} \right). \quad (38)$$

F and M are the LO decay constant and mass respectively. Note that in the massless limit this has only a transverse part as follows from the Ward identities.

The diagrams at NLO are (3–10) in Fig. 2 and the NNLO diagrams are (11–48) in Fig. 3.

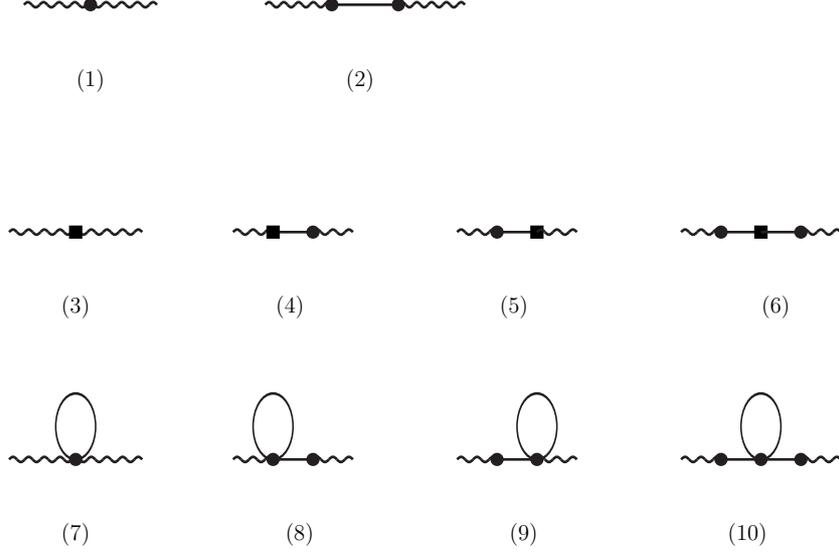


Figure 2: The axial-vector two-point function at LO and NLO. The filled circle is a vertex from \mathcal{L}_2 , The filled square is a vertex from \mathcal{L}_4 , and the open square is a vertex from \mathcal{L}_6 .

Many of the diagrams are one-particle-reducible and at first sight have double and triple poles at $q^2 = M^2$. From general properties of field theory these should be resumable in into a single pole at the physical mass, $q^2 = M_M^2$ and a nonsingular part that only has cuts. The residue at the pole is the decay constant squared. We must thus find contributions that allow for the last term in (38) the lowest order F^2, M^2 to be replaced by F_M^2, M_M^2 . It turns out to be advantageous to also do this in the first term. Most of the corrections are already included in this way.

At NLO the remaining part is only from the tree level diagram (3) in Fig. 2 and is

$$\Pi_{AA}^{(1)} = 4L_{10}^r - 8H_1^r, \quad \Pi_{AA}^{(0)} = 0. \quad (39)$$

So we can express our result up to NNLO as

$$\begin{aligned} \Pi_{AA}^{\mu\nu}(q^2) &= 2F_M^2 \left(g^{\mu\nu} - q^\mu q^\nu \frac{1}{q^2 - M_M^2} \right) + (q^\mu q^\nu - q^2 g^{\mu\nu})(4L_{10}^r - 8H_1^r) \\ &+ \frac{1}{F_M^2} \left[(q^\mu q^\nu - q^2 g^{\mu\nu}) \hat{\Pi}_{AA}^{(1)}(q^2) + q^\mu q^\nu \hat{\Pi}_{AA}^{(0)}(q^2) \right]. \end{aligned} \quad (40)$$

The $\hat{\Pi}_{AA}^{(0)}(q^2)$ and $\hat{\Pi}_{AA}^{(1)}(q^2)$ are the remainders at NNLO and have no singularity at $q^2 = M_M^2$.

The transverse part can be obtained from the part containing $g_{\mu\nu}$ as an overall factor. So the transverse part cannot come from the one-particle reducible diagrams and only gets contributions from diagrams (11–16) at NNLO. The sunset integrals H^F and H_{21}^F appearing in the results are defined in Appendix A.2.

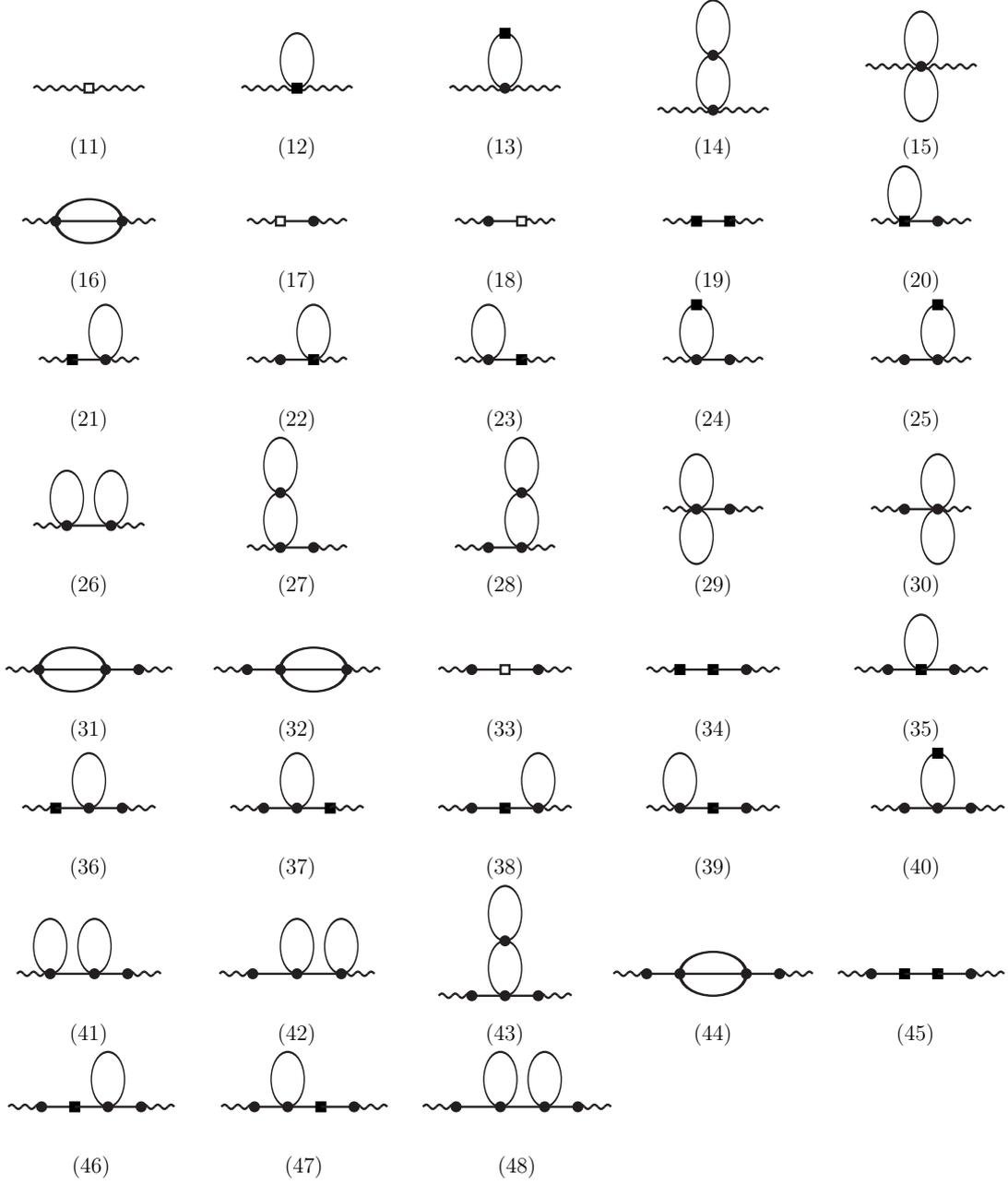


Figure 3: The axial-vector two-point function at NNLO. The filled circle is a vertex from \mathcal{L}_2 , the filled square is a vertex from \mathcal{L}_4 , and the open square is a vertex from \mathcal{L}_6

The longitudinal part gets at NNLO contributions from all diagrams shown in Fig 3. In order to rewrite the results into the single pole we need to expand the integrals around the mass. This introduces derivatives of the sunsetintegrals. These always show up in the combinations H^M and H_{21}^M defined in Appendix A.2.

Complex

$$\begin{aligned}
\hat{\Pi}_{AA}^{(1)}(q^2) &= \frac{n^2}{2} \left[\frac{M_M^2}{q^2} H^F(M_M^2, M_M^2, M_M^2, q^2) - H_{21}^F(M_M^2, M_M^2, M_M^2, q^2) \right] \\
&+ M_M^2 \left[\frac{L^2 n^2}{6} - 8nLL_{10}^r - 32(K_{102}^r + K_{103}^r n + K_{17}^r + K_{18}^r n) \right] \\
&- q^2(16K_{109}^r + 8K_{115}^r) - \frac{M_M^4}{q^2} \left(8K_{113}^r + \frac{3}{2}n^2 L^2 \right) \\
&+ \pi_{16} L \left[\frac{M_M^4}{q^2} \left(\frac{13n^2}{8} + \frac{2}{n^2} - \frac{2}{3} \right) + \frac{M_M^2}{6} n^2 \right] \\
&+ \pi_{16}^2 \left[\frac{M_M^4}{q^2} \left(\frac{7}{2n^2} - \frac{1}{8}n^2\pi^2 - \frac{17n^2}{64} - \frac{7}{6} \right) \right. \\
&\quad \left. + M_M^2 n^2 \left(\frac{\pi^2}{36} + \frac{1}{72} \right) + \frac{n^2}{96} q^2 \right], \tag{41}
\end{aligned}$$

$$\begin{aligned}
\hat{\Pi}_{AA}^{(0)}(q^2) &= \left[M_M^4 \left(\frac{4}{3} - \frac{4}{n^2} \right) - \frac{M_M^6}{2q^2} n^2 \right] H^M(M_M^2, M_M^2, M_M^2, q^2) \\
&- \frac{3M_M^4}{2} n^2 H_{21}^M(M_M^2, M_M^2, M_M^2, q^2) + \frac{8M_M^4}{q^2} K_{113}^r \\
&+ \frac{M_M^4}{q^2} \left[\pi_{16} L \left(\frac{2}{3} - \frac{n^2}{8} - \frac{2}{n^2} \right) + \pi_{16}^2 \left(\frac{7}{6} - \frac{15n^2}{64} - \frac{7}{2n^2} \right) \right], \tag{42}
\end{aligned}$$

Real

$$\begin{aligned}
\hat{\Pi}_{AA}^{(1)}(q^2) &= \frac{n}{2}(n+1) \left[\frac{M_M^2}{q^2} H^F(M_M^2, M_M^2, M_M^2, q^2) - H_{21}^F(M_M^2, M_M^2, M_M^2, q^2) \right] \\
&+ M_M^2 \left[\frac{L^2}{6} n(n+1) - 8nLL_{10}^r - 32(K_{102}^r + 2nK_{103}^r + K_{17}^r + 2nK_{18}^r) \right] \\
&- q^2(16K_{109}^r + 8K_{115}^r) - \frac{M_M^4}{q^2} \left[8K_{113}^r + \frac{3}{2}n(n+1)L^2 \right] \\
&+ \pi_{16} L \left[\frac{M_M^4}{q^2} \left(\frac{13n^2}{8} + \frac{1}{2n^2} + \frac{43n}{24} - \frac{1}{2n} - \frac{1}{6} \right) + \frac{M_M^2}{6} n(n+1) \right] \\
&+ \pi_{16}^2 \left[\frac{M_M^4}{q^2} \left(-\frac{1}{8}n^2\pi^2 - \frac{n\pi^2}{8} - \frac{17n^2}{64} + \frac{7}{8n^2} + \frac{5n}{192} - \frac{7}{8n} - \frac{7}{24} \right) \right. \\
&\quad \left. + M_M^2 n(n+1) \left(\frac{\pi^2}{36} + \frac{1}{72} \right) + \frac{n}{96} (n+1)q^2 \right], \tag{43}
\end{aligned}$$

$$\begin{aligned}
\hat{\Pi}_{AA}^{(0)}(q^2) &= \left[-\frac{M_M^6}{q^2} \frac{n}{2}(n+1) + M_M^4 (n-1) \left(\frac{1}{n^2} - \frac{1}{3} \right) \right] H^M(M_M^2, M_M^2, M_M^2, q^2) \\
&- \frac{3}{2} M_M^4 n(n+1) H_{21}^M(M_M^2, M_M^2, M_M^2, q^2) + \frac{8M_M^4}{q^2} K_{113}^r
\end{aligned}$$

$$\begin{aligned}
& +\pi_{16}\frac{M_M^4}{q^2}L\left(-\frac{n^2}{8}-\frac{1}{2n^2}-\frac{7n}{24}+\frac{1}{2n}+\frac{1}{6}\right) \\
& +\pi_{16}^2\frac{M_M^4}{q^2}\left(-\frac{15n^2}{64}-\frac{7}{8n^2}-\frac{101n}{192}+\frac{7}{8n}+\frac{7}{24}\right), \tag{44}
\end{aligned}$$

Pseudo – Real

$$\begin{aligned}
\hat{\Pi}_{AA}^{(1)}(q^2) &= \frac{n}{2}(n-1)\left[\frac{M_M^2}{q^2}H^F(M_M^2, M_M^2, M_M^2, q^2) - H_{21}^F(M_M^2, M_M^2, M_M^2, q^2)\right] \\
& -32M_M^2(K_{102}^r + 2nK_{103}^r + K_{17}^r + 2nK_{18}^r) - 8q^2(2K_{109}^r + K_{115}^r) \\
& -\frac{8M_M^4}{q^2}K_{113}^r - \frac{M_M^4}{q^2}\frac{3n}{2}(n-1)L^2 + \frac{M_M^2}{6}n(n-1)L^2 - 8M_M^2nLL_{10}^r \\
& +\pi_{16}\frac{LM_M^4}{q^2}\left(\frac{13n^2}{8}+\frac{1}{2n^2}-\frac{43n}{24}+\frac{1}{2n}-\frac{1}{6}\right) + \pi_{16}LM_M^2\left(\frac{n^2}{6}-\frac{n}{6}\right) \\
& +\pi_{16}^2\left[\frac{M_M^4}{q^2}\left(-\frac{1}{8}n^2\pi^2-\frac{17n^2}{64}+\frac{7}{8n^2}+\frac{n\pi^2}{8}-\frac{5n}{192}+\frac{7}{8n}-\frac{7}{24}\right)\right. \\
& \quad \left.+M_M^2n(n-1)\left(\frac{\pi^2}{36}+\frac{1}{72}\right)+q^2\frac{n}{96}(n-1)\right], \tag{45}
\end{aligned}$$

$$\begin{aligned}
\hat{\Pi}_{AA}^{(0)}(q^2) &= \left[-\frac{M_M^6}{q^2}\frac{n}{2}(n-1) - M_M^4(n+1)\left(\frac{1}{n^2}-\frac{1}{3}\right)\right]H^M(M_M^2, M_M^2, M_M^2, q^2) \\
& -\frac{3}{2}M_M^4n(n-1)H_{21}^M(M_M^2, M_M^2, M_M^2, q^2) + \frac{8M_M^4}{q^2}K_{113}^r \\
& +\pi_{16}\frac{M_M^4}{q^2}L\left(-\frac{n^2}{8}-\frac{1}{2n^2}+\frac{7n}{24}-\frac{1}{2n}+\frac{1}{6}\right) \\
& +\pi_{16}^2\frac{M_M^4}{q^2}\left(-\frac{15n^2}{64}-\frac{7}{8n^2}+\frac{101n}{192}-\frac{7}{8n}+\frac{7}{24}\right). \tag{46}
\end{aligned}$$

The axial two-point function is known in 3-flavour ChPT [22, 24]. We have checked that our result agrees with the one in [22] in the limit of equal masses.

3.4 The Scalar Two-Point Functions

The scalar two-point function is defined in (29), which contains the unbroken generator case ($T^a = Q^a$) and the singlet case ($a = 0$).

The Feynman diagrams for both cases are the same as those for the vector two-point function shown in Figure 1 except that diagrams (2) and (5–7) are absent. Diagrams (1) and (3) are at NLO, and the diagrams (4) and (8–11) are at NNLO.

3.4.1 Q^a case

The scalar two-point functions are similar to the vector two-point functions, the LO results are zero for all the three cases since the vertex at LO is absent. We have rewritten again everything in terms of the physical mass and decay constant, M_M^2 and F_M . The results for the three cases are given below. The first line is the NLO contribution and the remainder is the NNLO contribution.

Complex

$$\begin{aligned}
\Pi_{SS} = & B_0^2 \left\{ 8H_2^r + 16L_8^r + \frac{1}{n} (n^2 - 4) \overline{B}(m^2, q^2) \right\} \\
& + \frac{B_0^2}{F_M^2} \left\{ q^2 (8K_{113}^r + 32K_{47}^r) + M_M^2 \left(192K_{25}^r + 64K_{26}^r n \right) \right. \\
& \quad \left. + M_M^2 L \left[\left(\frac{64}{n} - 32n \right) L_8^r - 64L_7^r + (n^2 - 4) \frac{16}{n} L_5^r \right] \right. \\
& \quad \left. + \overline{B}(m^2, q^2) (n^2 - 4) \left[\frac{8q^2}{n} L_5^r + M_M^2 \left(\frac{2}{n^2} L - 16L_4^r - \frac{32}{n} L_5^r - 32L_6^r + \frac{64}{n} L_8^r \right) \right] \right. \\
& \quad \left. + \overline{B}(m^2, q^2)^2 (n^2 - 4) \left(\frac{q^2}{4} - \frac{2M_M^2}{n^2} \right) \right\}, \tag{47}
\end{aligned}$$

Real

$$\begin{aligned}
\Pi_{SS} = & B_0^2 \left\{ 8H_2^r + 16L_8^r + \frac{1}{n} (n-1)(n+2) \overline{B}(m^2, q^2) \right\} \\
& + \frac{B_0^2}{F_M^2} \left\{ q^2 (8K_{113}^r + 32K_{47}^r) + M_M^2 \left(192K_{25}^r + 128nK_{26}^r \right) \right. \\
& \quad \left. + M_M^2 L \left[\left(\frac{32}{n} - 32 - 32n \right) L_8^r - 64L_7^r + (n-1)(n+2) \frac{16}{n} L_5^r \right] \right. \\
& \quad \left. + \overline{B}(m^2, q^2) (n-1)(n+2) \left[\frac{8q^2}{n} L_5^r + M_M^2 \left(\left(-\frac{1}{n} + \frac{1}{n^2} \right) L - 32L_4^r \right. \right. \right. \\
& \quad \left. \left. \left. - \frac{32}{n} L_5^r + 64L_6^r + \frac{64}{n} L_8^r \right) \right] \right. \\
& \quad \left. + \overline{B}(m^2, q^2)^2 (n-1)(n+2) \left[\frac{q^2}{4} + M_M^2 \left(\frac{1}{2n} - \frac{1}{n^2} \right) \right] \right\}, \tag{48}
\end{aligned}$$

Pseudo – Real

$$\begin{aligned}
\Pi_{SS} = & B_0^2 \left\{ 8H_2^r + 16L_8^r + \frac{1}{n} (n+1)(n-2) \overline{B}(m^2, q^2) \right\} \\
& + \frac{B_0^2}{F_M^2} \left\{ q^2 (8K_{113}^r + 32K_{47}^r) + M_M^2 \left(192K_{25}^r + 128nK_{26}^r \right) \right. \\
& \quad \left. + M_M^2 L \left[\left(\frac{32}{n} + 32 - 32n \right) L_8^r - 64L_7^r + (n+1)(n-2) \frac{16}{n} L_5^r \right] \right. \\
& \quad \left. + \overline{B}(m^2, q^2) (n+1)(n-2) \left[\frac{8q^2}{n} L_5^r + M_M^2 \left(\left(\frac{1}{n} + \frac{1}{n^2} \right) L - 32L_4^r \right. \right. \right. \\
& \quad \left. \left. \left. - \frac{32}{n} L_5^r + 64L_6^r + \frac{64}{n} L_8^r \right) \right] \right. \\
& \quad \left. + \overline{B}(m^2, q^2)^2 (n+1)(n-2) \left[\frac{q^2}{4} + M_M^2 \left(-\frac{1}{2n} - \frac{1}{n^2} \right) \right] \right\}. \tag{49}
\end{aligned}$$

The definition of the one-loop function $\overline{B}(m^2, q^2)$ can be found in Appendix A.1.

3.4.2 Singlet case

We have also calculated the singlet case. This is the quark-antiquark combination that shows up in the mass term.

We write the expression up to NNLO as:

Complex

$$\begin{aligned}
\Pi_{SS}^0 &= B_0^2 \left\{ 8nH_2^r + 32n^2L_6^r + 16nL_8^r + 2(n^2 - 1)\overline{B}(m^2, q^2) \right\} \\
&+ \frac{B_0^2}{F_M^2} \left\{ 8q^2 \left(nK_{113}^r + 4nK_{47}^r + 4n^2K_{48}^r \right) + 192M_M^2 \left(nK_{25}^r + n^2K_{26}^r + n^3K_{27}^r \right) \right. \\
&\quad + M_M^2 L \left(n^2 - 1 \right) \left(32nL_4^r + 32L_5^r - 64nL_6^r - 64L_8^r \right) \\
&\quad + \overline{B}(m^2, q^2) \left(n^2 - 1 \right) \left[16q^2 \left(nL_4^r + L_5^r \right) \right. \\
&\quad \quad \left. \left. + M_M^2 \left(\frac{4}{n}L + 64 \left(2L_8^r + 2nL_6^r - L_5^r - nL_4^r \right) \right) \right] \right. \\
&\quad \left. + \overline{B}(m^2, q^2)^2 \left(n^2 - 1 \right) \left(nq^2 - \frac{2M_M^2}{n} \right) \right\}, \tag{50}
\end{aligned}$$

Real

$$\begin{aligned}
\Pi_{SS}^0 &= B_0^2 \left\{ 16nH_2^r + 128n^2L_6^r + 32nL_8^r + 2(2n^2 + n - 1)\overline{B}(m^2, q^2) \right\} \\
&+ \frac{B_0^2}{F_M^2} \left\{ 16q^2 \left(nK_{113}^r + 4nK_{47}^r + 8n^2K_{48}^r \right) + 384M_M^2 \left(nK_{25}^r + 2n^2K_{26}^r + 4n^3K_{27}^r \right) \right. \\
&\quad + M_M^2 L \left(2n^2 + n - 1 \right) \left(64nL_4^r + 32L_5^r - 128nL_6^r - 64L_8^r \right) \\
&\quad + \overline{B}(m^2, q^2) \left(2n^2 + n - 1 \right) \left[16q^2 \left(2nL_4^r + L_5^r \right) \right. \\
&\quad \quad \left. \left. + M_M^2 \left(\left(-2 + \frac{2}{n} \right) L + 64 \left(2L_8^r + 4nL_6^r - L_5^r - 2nL_4^r \right) \right) \right] \right. \\
&\quad \left. + \overline{B}(m^2, q^2)^2 \left(2n^2 + n - 1 \right) \left[nq^2 + M_M^2 \left(1 - \frac{1}{n} \right) \right] \right\}, \tag{51}
\end{aligned}$$

Pseudo – Real

$$\begin{aligned}
\Pi_{SS}^0 &= B_0^2 \left\{ 16nH_2^r + 128n^2L_6^r + 32nL_8^r + 2(2n^2 - n - 1)\overline{B}(m^2, q^2) \right\} \\
&+ \frac{B_0^2}{F_M^2} \left\{ 16q^2 \left(nK_{113}^r + 4nK_{47}^r + 8n^2K_{48}^r \right) + 384M_M^2 \left(nK_{25}^r + 2n^2K_{26}^r + 4n^3K_{27}^r \right) \right. \\
&\quad + M_M^2 L \left(2n^2 - n - 1 \right) \left(64nL_4^r + 32L_5^r - 128nL_6^r - 64L_8^r \right) \\
&\quad + \overline{B}(m^2, q^2) \left(2n^2 - n - 1 \right) \left[16q^2 \left(2nL_4^r + L_5^r \right) \right. \\
&\quad \quad \left. \left. + M_M^2 \left(\left(2 + \frac{2}{n} \right) L + 64 \left(2L_8^r + 4nL_6^r - L_5^r - 2nL_4^r \right) \right) \right] \right.
\end{aligned}$$

$$+\overline{B}(m^2, q^2)^2 (2n^2 - n - 1) \left[nq^2 + M_M^2 \left(-1 - \frac{1}{n} \right) \right] \}. \quad (52)$$

We also written the result in term of physical M_M^2 and F_M . Notice that all loop diagrams are proportional to the number of Goldstone bosons in each case, i.e. $n^2 - 1$, $2n^2 + n - 1$, $2n^2 - n - 1$ for the complex, real and pseudo-real case respectively.

3.5 The Pseudo-Scalar Two-Point Functions

The pseudo-scalar two-point function is defined in (29). Just as in the case of the axial-vector two-point function there are one-particle-reducible diagrams. The diagrams are the same as those for the axial-vector two-point function with the axial-vector current replaced by a pseudo-scalar current. These are shown in Figure 2 and 3. There is also no vertex with two pseudo-scalar currents at LO so the equivalent of diagrams (1) and (7) in Figure 2 and (13–15) in Figure 3 vanish immediately. Just as in the scalar case, one should distinguish here between two cases: The adjoint case for the complex representation case which generalizes to the broken generators for the real and pseudo-real case, and the singlet operator with T^a in (28) the unit operator.

In Section 3.3 we could simplify the final expressions very much by writing the final expression with the single pole at the meson mass in terms of the decay constant. The same happens here if we instead rewrite the result in terms of the meson pseudo-scalar decay constant G_M . So we first need to obtain that quantity to NNLO.

3.5.1 The meson pseudo-scalar decay constant G_M

The decay constant of the pseudoscalar density to the mesons, G_M is defined⁴ similarly to F_M :

$$\langle 0 | \bar{q} i \gamma_5 T^a q | \pi^b \rangle = \frac{1}{\sqrt{2}} \delta^{ab} G_M \quad (53)$$

The calculation of G_M is very similar to F_M , the diagrams are exactly those shown in Figure 2 in [11] with one of the legs replaced by the pseudo-scalar current. There is here also a contribution from wave-function renormalization. In [11] we reported all the quantities M_M^2 , F_M and $\langle \bar{q} q \rangle$ as an expansion in the bare or lowest order quantities F and $M^2 = 2B_0 m$. We therefore do the same here. We therefore use the quantity

$$L_0 = \frac{1}{16\pi^2} \log \frac{M^2}{F^2} \quad (54)$$

instead of L as in the other sections of this paper.

This quantity has been calculated to NLO in two-flavour ChPT in [18] and was called G_π there. We have checked that our NLO result agrees with theirs.

At leading order, all the three case have same expression:

$$G_M = G_0 = 2B_0 F. \quad (55)$$

⁴The $\sqrt{2}$ is included in the definition to have the same normalization as [18].

We express the full results up to NNLO in terms of the LO meson mass M^2 and decay constant F as

$$G_M = 2B_0F \left(1 + \frac{M^2}{F^2} a_G + \frac{M^4}{F^4} b_G \right) \quad (56)$$

At NLO and NNLO, the coefficients a_G and b_G are

Complex

$$\begin{aligned} a_G &= \left(\frac{1}{n} - \frac{n}{2} \right) L_0 + 4(-nL_4^r - L_5^r + 4nL_6^r + 4L_8^r) \\ b_G &= -64(L_5^r + nL_4^r)(L_8^r + nL_6^r) + 24(L_5^r + nL_4^r)^2 \\ &\quad - 8n^2 K_{22}^r + 48n^2 K_{27}^r - 32K_{17}^r - 8K_{19}^r - 8K_{23}^r + 48K_{25}^r + 32K_{39}^r - 32nK_{18}^r \\ &\quad - 8nK_{20}^r - 8nK_{21}^r + 48nK_{26}^r + 32nK_{40}^r \\ &\quad + L_0 \left[- (32 - 22n^2) \left(L_4^r + \frac{1}{n} L_5^r \right) + (4 - 8n^2)(L_1^r + 4L_6^r) \right. \\ &\quad \left. + \left(\frac{80}{n} - 48n \right) L_8^r + \left(\frac{12}{n} - 10n \right) L_3^r - (8 + 2n^2) L_2^r + \left(\frac{12}{n} - 4n \right) L_0^r \right] \\ &\quad + \pi_{16} \left[(2 - n^2) \left(\frac{8}{n} L_8^r + 8L_6^r - \frac{4}{n} L_5^r - 4L_4^r - \frac{1}{n} L_3^r \right) \right. \\ &\quad \left. + n^2 L_2^r + 2L_1^r + 2 \left(n - \frac{1}{n} \right) L_0^r \right] \\ &\quad + \pi_{16}^2 \left(\frac{113n^2}{256} - \frac{13}{24} + \frac{13}{8n^2} \right) - \pi_{16} L_0 \left(\frac{55n^2}{96} - 1 + \frac{7}{2n^2} \right) \\ &\quad + L_0^2 \left(\frac{3n^2}{16} - \frac{3}{2} + \frac{9}{2n^2} \right), \end{aligned} \quad (57)$$

Real

$$\begin{aligned} a_G &= - \left(\frac{n}{2} + \frac{1}{2} - \frac{1}{2n} \right) L_0 + (-8nL_4^r - 4L_5^r + 32nL_6^r + 16L_8^r) \\ b_G &= -64(L_5^r + 2nL_4^r)(L_8^r + 2nL_6^r) + 24(L_5^r + 2nL_4^r)^2 \\ &\quad - 32K_{22}^r n^2 + 192K_{27}^r n^2 - 32K_{17}^r - 8K_{19}^r - 8K_{23}^r + 48K_{25}^r + 32K_{39}^r \\ &\quad - 64K_{18}^r n - 16K_{20}^r n - 16K_{21}^r n + 96K_{26}^r n + 64K_{40}^r n \\ &\quad + L_0 \left[(-16 + 16n + 22n^2) \left(2L_4^r + \frac{1}{n} L_5^r \right) + (4 - 8n - 16n^2) L_1^r \right. \\ &\quad \left. + 16(1 - 3n - 4n^2) L_6^r - \left(40 - \frac{40}{n} + 48n \right) L_8^r - \left(6 - \frac{6}{n} + 10n \right) L_3^r \right. \\ &\quad \left. - (8 + 2n + 4n^2) L_2^r - \left(6 - \frac{6}{n} + 4n \right) L_0^r \right] \\ &\quad + \pi_{16} \left[(1 - n - n^2) \left(\frac{8}{n} L_8^r + 16L_6^r - \frac{4}{n} L_5^r - 8L_4^r - \frac{1}{n} L_3^r \right) \right. \\ &\quad \left. + (n + 2n^2) L_2^r + 2L_1^r + \left(1 - \frac{1}{n} + 2n \right) L_0^r \right] \\ &\quad + \pi_{16}^2 \left(\frac{113n^2}{256} + \frac{443n}{768} - \frac{13}{96} - \frac{13}{32n} + \frac{13}{32n^2} \right) \end{aligned}$$

$$\begin{aligned}
& -\pi_{16}L_0 \left(\frac{55n^2}{96} + \frac{67n}{96} - \frac{3}{8} - \frac{5}{8n} + \frac{7}{8n^2} \right) \\
& + L_0^2 \left(\frac{3n^2}{16} + \frac{13n}{16} - \frac{1}{8} - \frac{3}{2n} + \frac{9}{8n^2} \right), \tag{58}
\end{aligned}$$

Pseudo – Real

$$\begin{aligned}
a_G &= -L_0 \left(\frac{n}{2} - \frac{1}{2} - \frac{1}{2n} \right) + (-8nL_4^r - 4L_5^r + 32nL_6^r + 16L_8^r) \\
b_G &= -64(L_5^r + 2nL_4^r)(L_8^r + 2nL_6^r) + 24(L_5^r + 2nL_4^r)^2 \\
& - 32K_{22}^r n^2 + 192K_{27}^r n^2 - 32K_{17}^r - 8K_{19}^r - 8K_{23}^r + 48K_{25}^r \\
& + 32K_{39}^r - 64K_{18}^r n - 16K_{20}^r n - 16K_{21}^r n + 96K_{26}^r n + 64K_{40}^r n \\
& + L_0 \left[(-16 - 16n + 22n^2) \left(2L_4^r + \frac{1}{n}L_5^r \right) + (4 + 8n - 16n^2)L_1^r \right. \\
& \quad + 16(1 + 3n - 4n^2)L_6^r + \left(40 + \frac{40}{n} - 48n \right) L_8^r + \left(6 + \frac{6}{n} - 10n \right) L_3^r \\
& \quad \left. - (8 - 2n + 4n^2) L_2^r + \left(6 + \frac{6}{n} - 4n \right) L_0^r \right] \\
& + \pi_{16} \left[(1 + n - n^2) \left(\frac{8}{n}L_8^r + 16L_6^r - \frac{4}{n}L_5^r - 8L_4^r - \frac{1}{n}L_3^r \right) \right. \\
& \quad \left. + (-n + 2n^2)L_2^r + 2L_1^r + \left(-1 - \frac{1}{n} + 2n \right) L_0^r \right] \\
& + \pi_{16}^2 \left(\frac{113n^2}{256} - \frac{443n}{768} - \frac{13}{96} + \frac{13}{32n} + \frac{13}{32n^2} \right) \\
& - \pi_{16}L_0 \left(\frac{55n^2}{96} - \frac{67n}{96} - \frac{3}{8} + \frac{5}{8n} + \frac{7}{8n^2} \right) \\
& + L_0^2 \left(\frac{3n^2}{16} - \frac{13n}{16} - \frac{1}{8} + \frac{3}{2n} + \frac{9}{8n^2} \right). \tag{59}
\end{aligned}$$

3.5.2 X^a case

The pseudo-scale two point functions are similar to the axial-vector ones in the diagrams as described above. The LO result is the same for all the three cases:

$$\Pi_{PP}^a = -\frac{1}{2} \frac{G_0^2}{q^2 - M^2}. \tag{60}$$

The superscript “ a ” indicates the case with T^a in (28) an $SU(n)$ generator. For the real and pseudo-real case this is related by the conserved part of the symmetry group also to a number of diquark currents.

Subtracting the pole contribution in terms of the physical mass and decay constants, M_M^2 , F_M and G_M , absorbs the major part of the higher order corrections. The final results are thus much simpler when written in this way. The remaining part at NLO is

$$\Pi_{PP}^a = B_0^2(8H_2^r - 16L_8^r). \tag{61}$$

Thus we can define the full NNLO results as

$$\Pi_{PP}^a = -\frac{1}{2} \frac{G_M^2}{q^2 - M_M^2} + B_0^2(8H_2^r - 16L_8^r) + \frac{B_0^2}{F_M^2} \hat{\Pi}_{PP}^a, \quad (62)$$

where the $\hat{\Pi}_{PP}$ is the remainder at NNLO. Its expression for the three different cases is:

Complex

$$\begin{aligned} \hat{\Pi}_{PP}^a &= -\frac{3}{2} n^2 q^4 H_{21}^M(M_M^2, M_M^2, M_M^2, q^2) \\ &+ \left[\left(\frac{4}{3} - \frac{4}{n^2} \right) q^4 - \frac{n^2}{2} M_M^2 q^2 \right] H^M(M_M^2, M_M^2, M_M^2, q^2) \\ &+ 8q^2 K_{113}^r + 64M_M^2(K_{17}^r + nK_{18}^r - K_{39}^r - nK_{40}^r) \\ &+ L^2 M_M^2 \left(-\frac{n^2}{2} - \frac{6}{n^2} + 2 \right) + LM_M^2 \left(64L_6^r + 32nL_8^r - \frac{64L_8^r}{n} \right) \\ &+ \pi_{16} L \left[M_M^2 \left(-\frac{8}{3} - \frac{n^2}{12} + \frac{8}{n^2} \right) + \left(\frac{2}{3} - \frac{n^2}{8} - \frac{2}{n^2} \right) q^2 \right] \\ &+ \pi_{16}^2 \left[M_M^2 \left(\frac{5}{3} - \frac{85n^2}{96} - \frac{5}{n^2} \right) + q^2 \left(\frac{7}{6} - \frac{15n^2}{64} - \frac{7}{2n^2} \right) \right], \end{aligned} \quad (63)$$

Real

$$\begin{aligned} \hat{\Pi}_{PP}^a &= -\frac{3}{2} q^4 n(n+1) H_{21}^M(M_M^2, M_M^2, M_M^2, q^2) \\ &+ \left[\left(-\frac{1}{n^2} - \frac{n}{3} + \frac{1}{n} + \frac{1}{3} \right) q^4 - \frac{1}{2} M_M^2 q^2 n(n+1) \right] H^M(M_M^2, M_M^2, M_M^2, q^2) \\ &+ 8q^2 K_{113}^r + 64M_M^2(K_{17}^r + 2nK_{18}^r - K_{39}^r - 2nK_{40}^r) \\ &+ M_M^2 L^2 \left(-\frac{n^2}{2} - \frac{3}{2n^2} - n + \frac{3}{2n} + \frac{1}{2} \right) + 32M_M^2 L \left(2L_6^r + L_8^r n - \frac{L_8^r}{n} + L_8^r \right) \\ &+ \pi_{16} L \left[M_M^2 \left(-\frac{n^2}{12} + \frac{2}{n^2} + \frac{7n}{12} - \frac{2}{n} - \frac{2}{3} \right) + q^2 \left(-\frac{n^2}{8} - \frac{1}{2n^2} - \frac{7n}{24} + \frac{1}{2n} + \frac{1}{6} \right) \right] \\ &+ \pi_{16}^2 \left[M_M^2 \left(-\frac{85n^2}{96} - \frac{5}{4n^2} - \frac{125n}{96} + \frac{5}{4n} + \frac{5}{12} \right) \right. \\ &\quad \left. + q^2 \left(-\frac{15n^2}{64} - \frac{7}{8n^2} - \frac{101n}{192} + \frac{7}{8n} + \frac{7}{24} \right) \right], \end{aligned} \quad (64)$$

Pseudo - Real

$$\begin{aligned} \hat{\Pi}_{PP}^a &= \frac{3}{2} q^4 n(1-n) H_{21}^M(M_M^2, M_M^2, M_M^2, q^2) \\ &+ \left[\left(-\frac{1}{n^2} + \frac{n}{3} - \frac{1}{n} + \frac{1}{3} \right) q^4 + \frac{1}{2} M_M^2 q^2 n(1-n) \right] H^M(M_M^2, M_M^2, M_M^2, q^2) \\ &+ 8q^2 K_{113}^r + 64M_M^2(K_{17}^r + 2nK_{18}^r - K_{39}^r - 2nK_{40}^r) \\ &+ M_M^2 L^2 \left(-\frac{n^2}{2} - \frac{3}{2n^2} + n - \frac{3}{2n} + \frac{1}{2} \right) + M_M^2 L \left[64L_6^r + 32 \left(n - \frac{1}{n} - 1 \right) L_8^r \right] \\ &+ \pi_{16} L \left[M_M^2 \left(-\frac{n^2}{12} + \frac{2}{n^2} - \frac{7n}{12} + \frac{2}{n} - \frac{2}{3} \right) + q^2 \left(-\frac{n^2}{8} - \frac{1}{2n^2} + \frac{7n}{24} - \frac{1}{2n} + \frac{1}{6} \right) \right] \end{aligned}$$

$$\begin{aligned}
& +\pi_{16}^2 \left[M_M^2 \left(-\frac{85n^2}{96} - \frac{5}{4n^2} + \frac{125n}{96} - \frac{5}{4n} + \frac{5}{12} \right) \right. \\
& \quad \left. + \left(-\frac{15n^2}{64} - \frac{7}{8n^2} + \frac{101n}{192} - \frac{7}{8n} + \frac{7}{24} \right) q^2 \right]. \tag{65}
\end{aligned}$$

The loop integrals H^M and H_{21}^M are defined in Appendix A.2.

3.5.3 Singlet case

In the singlet case with $a = 0$, there is no contribution with poles. Only the one-particle-irreducible diagrams contribute. As a consequence, there is no order p^2 contribution and at order p^4 there is only a tree level contribution from the equivalent of diagram (3) in Figure 2. At order p^6 or NNLO only the one-particle-irreducible diagrams contribute and since there is no order p^2 vertex with two pseudo-scalar currents only diagram (11–12) and (16) in Figure 3 contribute.

Since there is no single pole contribution, there is also no need here to expand in the integrals around the meson mass. The integral H^F is defined in Appendix A.2.

The singlet pseudo-scalar two-point function we write as

$$\Pi_{PP}^0 = B_0^2 \bar{\Pi}_{PP}^0 + \frac{B_0^2}{F_M^2} \hat{\Pi}_{PP}^0. \tag{66}$$

The results for the three cases are

Complex :

$$\begin{aligned}
\bar{\Pi}_{PP}^0 &= 8nH_2^r - 16nL_8^r - 32n^2L_7^r, \\
\hat{\Pi}_{PP}^0 &= -\frac{2}{3n} (n^2 - 1) (n^2 - 4) H^F(M_M^2, M_M^2, M_M^2, q^2) \\
& \quad + q^2 (8K_{113}^r n - 32K_{46}^r n^2) - 64M_M^2 (K_{39}^r n + K_{40}^r n^2 + K_{41}^r n^2 + K_{42}^r n^3) \\
& \quad + L^2 M_M^2 \frac{1}{n} (n^2 - 1) (n^2 - 4) + 64LM_M^2 (n^2 - 1)(nL_7^r + L_8^r) \\
& \quad + M_M^2 \pi_{16}^2 \frac{1}{n} (n^2 - 1) (n^2 - 4) \left(\frac{\pi^2}{6} + 1 \right), \tag{67}
\end{aligned}$$

Real :

$$\begin{aligned}
\bar{\Pi}_{PP}^0 &= 16nH_2^r - 128n^2L_7^r - 32nL_8^r, \\
\hat{\Pi}_{PP}^0 &= -\frac{2}{3n} (2n^2 + n - 1) (n^2 + n - 2) H^F(M_M^2, M_M^2, M_M^2, q^2) \\
& \quad + q^2 (16K_{113}^r n - 128K_{46}^r n^2) - 128M_M^2 (nK_{39}^r + 2n^2K_{40}^r + 2n^2K_{41}^r + 4n^3K_{42}^r) \\
& \quad + L^2 M_M^2 \frac{1}{n} (2n^2 + n - 1) (n^2 + n - 2) + 64M_M^2 L (2n^2 + n - 1)(2nL_7^r + L_8^r) \\
& \quad + M_M^2 \pi_{16}^2 \frac{1}{n} (2n^2 + n - 1) (n^2 + n - 2) \left(\frac{\pi^2}{6} + 1 \right), \tag{68}
\end{aligned}$$

Pseudo – Real :

$$\begin{aligned}
\bar{\Pi}_{PP}^0 &= 16nH_2^r - 128n^2L_7^r - 32nL_8^r, \\
\hat{\Pi}_{PP}^0 &= -\frac{2}{3n} (2n^2 - n - 1) (n^2 - n - 2) H^F(M_M^2, M_M^2, M_M^2, q^2)
\end{aligned}$$

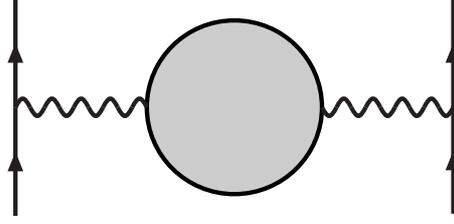


Figure 4: The one-loop oblique correction to LEP process $e^+ + e^- \rightarrow q + \bar{q}$.

$$\begin{aligned}
& +q^2 \left(16K_{113}^r n - 128K_{46}^r n^2 \right) - 128M_M^2 \left(nK_{39}^r + 2n^2K_{40}^r + 2n^2K_{41}^r + 4n^3K_{42}^r \right) \\
& + L^2 M_M^2 \frac{1}{n} \left(2n^2 - n - 1 \right) \left(n^2 - n - 2 \right) + 64M_M^2 L \left(2n^2 - n - 1 \right) \left(2nL_7^r + L_8^r \right) \\
& + M_M^2 \pi^2 \frac{1}{n} \left(2n^2 - n - 1 \right) \left(n^2 - n - 2 \right) \left(\frac{\pi^2}{6} + 1 \right). \tag{69}
\end{aligned}$$

Notice that just as for the scalar singlet two-point function, all loop contributions are proportional to the number of Goldstone bosons.

3.6 Large n

As one can see from all the explicit formulas, many of the expressions become equal for the different cases in the large n limit .

4 The Oblique Corrections and S-parameter

The physical process at the CERN LEP collider is $e^+ + e^- \rightarrow q + \bar{q}$. There are three types of one loop correction to this process: vacuum polarization corrections, vertex corrections, and box corrections. The vacuum polarization contribution is independent of the external fermions and it dominates the contributions from physics beyond SM. For the light fermions, the other two corrections are suppressed by factor of m_f^2/m_Z^2 . That's why the vacuum polarization corrections are called ‘‘oblique corrections,’’ and the vertex and box corrections are called ‘‘nonoblique corrections.’’

The oblique polarization only affect the gauge bosons propagators and their mixing. The vacuum polarization amplitude can be defined as

$$g^{\mu\nu}\Pi_{XY} + (q^\mu q^\nu \text{ terms}) = i \int d^4x e^{iq \cdot x} \langle 0 | T(J_X^\mu(x) J_Y^\nu(0)) | 0 \rangle. \tag{70}$$

The influence of new physics to the oblique corrections can be summarized to three parameters: S , T and U . One can find their definition in Ref. [15]. These parameters are chosen to be zero at a reference point in the SM. In the past 20 years, they have been studied intensively in many models beyond the Standard Model physics.

For a beyond the Standard Model with strong dynamics at the TeV scale, there will in general be many resonances and other nonperturbative effects. At low momenta one can use the EFT as described above for these cases. In this paper, we will estimate the S parameter contribution from pseudo-Goldstone Boson sector within

the EFT. The parameter T and U vanish because of the exact flavor symmetry, i.e. we work in the equal mass case.

The S parameter can be written as⁵ [15]

$$S = -2\pi [\Pi'_{VV}(0) - \Pi'_{AA}(0)] = 2\pi \frac{d}{dq^2} (q^2 \Pi_{VV}^{(1)} - q^2 \Pi_{AA}^{(1)})_{q^2=0}. \quad (71)$$

$\Pi'_{VV}(0)$ and $\Pi'_{AA}(0)$ are the derivatives of the vector and axial-vector two-point functions at $q^2 = 0$. One should keep in mind that S is defined to be zero at a particular place in the standard model, as discussed at the end of section V in [15]. Our formulas are the equivalent of (5.12) in that reference.

The result can be written as

$$S = \bar{S} + \frac{\pi M_M^2}{F_M^2} \hat{S}, \quad (72)$$

with

Complex :

$$\begin{aligned} \bar{S} &= -16\pi L_{10}^r - \frac{2n\pi}{3} (L + \pi_{16}), \\ \hat{S} &= 64 (K_{102}^r - K_{81}^r + K_{17}^r + nK_{103}^r - nfK_{82}^r + nK_{18}^r) + \frac{n^2}{3} L^2 \\ &\quad + 16n (L_9^r + 2L_{10}^r) L - \pi_{16} \frac{11n^2}{9} L + \pi_{16}^2 n^2 \left(\frac{85}{108} - \frac{5}{27} \tilde{\psi} \right) \end{aligned} \quad (73)$$

Real :

$$\begin{aligned} \bar{S} &= -16\pi L_{10}^r - \frac{2(n+1)\pi}{3} (L + \pi_{16}), \\ \hat{S} &= 64 (K_{102}^r - K_{81}^r + K_{17}^r + 2nK_{103}^r - 2nfK_{82}^r + 2nK_{18}^r) + \frac{n(n+1)}{3} L^2 \\ &\quad + 16 [(n+1)L_9^r + (2n+1)L_{10}^r] L - \pi_{16} \frac{11n(n+1)}{9} L \\ &\quad + \pi_{16}^2 n(n+1) \left(\frac{85}{108} - \frac{5}{27} \tilde{\psi} \right), \end{aligned} \quad (74)$$

Pseudo - real :

$$\begin{aligned} \bar{S} &= -16\pi L_{10}^r - \frac{2(n-1)\pi}{3} (L + \pi_{16}), \\ \hat{S} &= 64 (K_{102}^r - K_{81}^r + K_{17}^r + 2nK_{103}^r - 2nfK_{82}^r + 2nK_{18}^r) + \frac{n(n-1)}{3} L^2 \\ &\quad + 16 [(n-1)L_9^r + (2n-1)L_{10}^r] L - \pi_{16} \frac{11n(n-1)}{9} L \\ &\quad + \pi_{16}^2 n(n-1) \left(\frac{85}{108} - \frac{5}{27} \tilde{\psi} \right). \end{aligned} \quad (75)$$

The quantity $\tilde{\psi}$ is

$$\tilde{\psi} = 6\sqrt{3}\text{Cl}_2 \left(\frac{2\pi}{3} \right) = 7.0317217160684. \quad (76)$$

⁵Our two point functions are normalized differently from those in [15].

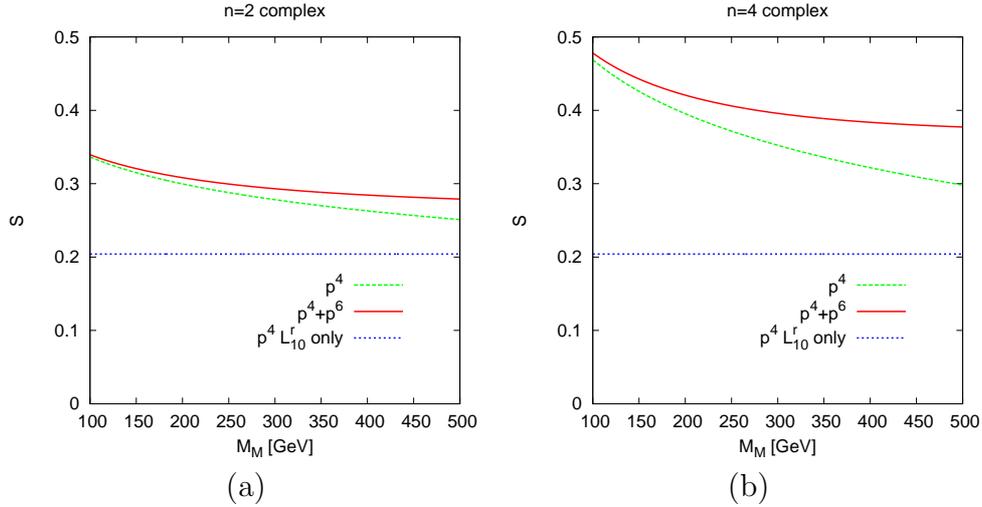


Figure 5: The S -parameter for the values of L_9^r and L_{10}^r given in the text for the complex case. (a) $n = 2$ (b) $n = 4$.

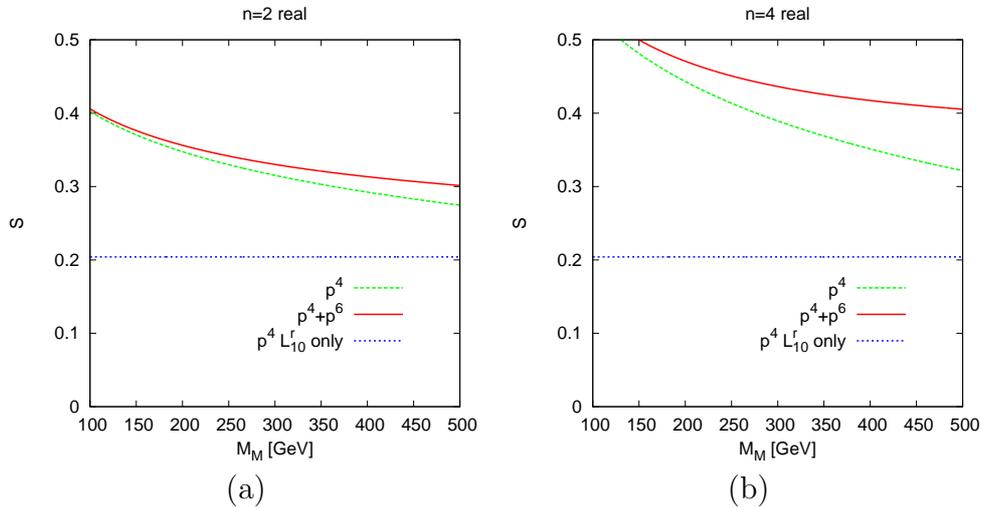


Figure 6: The S -parameter for the values of L_9^r and L_{10}^r given in the text for the real case. (a) $n = 2$ (b) $n = 4$.

The real purpose of (73)-(75) is to be able to study the S -parameter in more general theories than just scaling up from QCD. However to provide some feeling about numerical results we choose parameters as if they are scaled up from QCD/ChPT. We change $F_\pi = 0.0922$ MeV to $F_M = 243$ GeV and the subtraction scale from 0.77 GeV to 2 TeV. We set the $K_i^r = 0$ and keep $L_9^r = 0.00593$ and $L_{10}^r = -0.00406$ at their values from ChPT [25, 26].

In Figures 5, 6 and 7 we have shown the results for our three cases complex, real and pseudo-real for $n = 2$ and $n = 4$. Shown are the full p^4 and p^6 contributions as well as the p^4 part proportional to L_{10}^r only. The latter is what is the usual contribution to S corrected for the pieces that go into the reference point at p^4 . We cannot do the same for the full result since that depends on how one treats the extra pseudo-Goldstone bosons that occur in the other models.

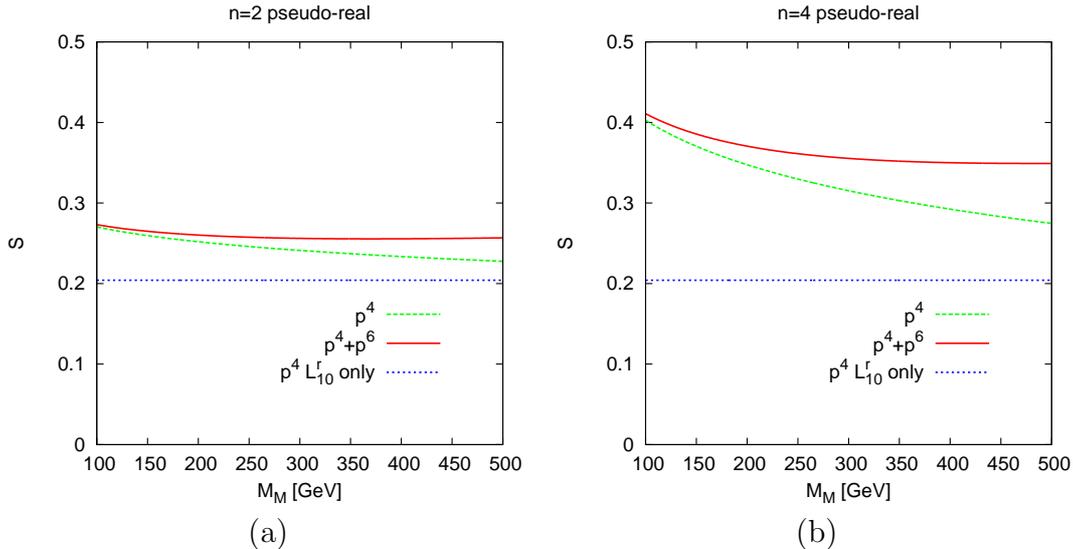


Figure 7: The S -parameter for the values of L_9^r and L_{10}^r given in the text for the pseudo-real case. (a) $n = 2$ (b) $n = 4$.

5 Conclusion

In this paper, we have calculated the two-point correlation functions of vector, axial-vector, scalar and pseudo-scalar currents for QCD-like theories.

In the beginning of the paper, we gave a very brief overview of the QCD-like theories and their EFT treatment as developed earlier.

We then gave the analytic results of those two-point functions up to NNLO. The results are significantly shortened by using the physical meson mass M_M^2 and decay constants F_M and G_M when rewriting the pole contributions.

The main use of these formulas is expected to be in extrapolations to zero fermion mass of technicolour related lattice calculations. We have therefore also included precisely the combination needed for the S -parameter.

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A Loop integrals

We use dimensional regularization and \overline{MS} scheme to evaluate the loop integrals, $d = 4 - 2\epsilon$.

A.1 One-loop integrals

The loop integral with one propagator is

$$\begin{aligned}
A(m^2) &= \frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m^2} \\
&= \frac{m^2}{16\pi^2} \left\{ \lambda_0 - \ln(m^2) + \epsilon \left[\frac{C^2}{2} + \frac{1}{2} + \frac{\pi^2}{12} + \frac{1}{2} \ln^2(m^2) - C \ln(m^2) \right] \right\} \\
&\quad + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{77}$$

Here

$$C = \ln(4\pi) + 1 - \gamma \quad \lambda_0 = \frac{1}{\epsilon} + C$$

The extra +1 in C is the ChPT version of \overline{MS} .

The loop integrals with two propagators are

$$\begin{aligned}
B(m_1^2, m_2^2, p^2) &= \frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m_1^2)((q-p)^2 - m_2^2)}, \\
B^\mu(m_1^2, m_2^2, p) &= \frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu}{(q^2 - m_1^2)((q-p)^2 - m_2^2)} \\
&= p^\mu B_1(m_1^2, m_2^2, p^2), \\
B^{\mu\nu}(m_1^2, m_2^2, p) &= \frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 - m_1^2)((q-p)^2 - m_2^2)} \\
&= p^\mu p^\nu B_{21}(m_1^2, m_2^2, p^2) + g^{\mu\nu} B_{22}(m_1^2, m_2^2, p^2).
\end{aligned} \tag{78}$$

The two last integrals can be reduced to simpler integrals A and B via

$$\begin{aligned}
B_1(m^2, m^2, p^2) &= \frac{1}{2} B(m_1^2, m_2^2, p^2), \\
B_{22}(m^2, m^2, p^2) &= \frac{1}{2(d-1)} \left[A(m^2) + \left(2m^2 - \frac{1}{2} p^2 \right) B(m^2, m^2, p^2) \right], \\
B_{21}(m^2, m^2, p^2) &= \frac{1}{p^2} \left[A(m^2) + m^2 B(m^2, m^2, p^2) - dB_{22}(m^2, m^2, p^2) \right].
\end{aligned} \tag{79}$$

We quote here only the equal mass case results relevant for this paper. The explicit expression for B is

$$\begin{aligned}
B(m^2, m^2, p^2) &= \frac{1}{16\pi^2} \lambda_0 + \overline{B}(m^2, p^2) + \mathcal{O}(\epsilon), \\
\overline{B}(m^2, p^2) &= \frac{1}{16\pi^2} \left(-1 - m^2 \log \frac{m^2}{\mu^2} \right) + \bar{J}(m^2, p^2), \\
\bar{J}(m^2, p^2) &= -\frac{1}{16\pi^2} \int_0^1 dx \ln \left(\frac{m^2 - x(1-x)p^2}{m^2} \right),
\end{aligned} \tag{80}$$

The function $\bar{J}(m^2, p^2)$ is

$$\bar{J}(m^2, p^2) = \begin{cases} 2 + \sigma \ln \left(\frac{\sigma-1}{\sigma+1} \right), & p^2 < 0, \\ 2 - 2\sqrt{\frac{4}{x} - 1} \cdot \operatorname{arccot} \left(\sqrt{\frac{4}{x} - 1} \right), & 0 \leq p^2 < 4m^2, \\ 2 + \sigma \ln \left(\frac{1-\sigma}{1+\sigma} \right) + i\pi\sigma, & p^2 > 4m^2, \end{cases}$$

$$\sigma(x) = \sqrt{1 - \frac{4}{x}}, \quad x = \frac{m^2}{p^2} \notin [0, 4]. \quad (81)$$

Taking derivatives w.r.t. p^2 at $p^2 = 0$ is most easily done in the form with the Feynman parameter integration explicit.

A.2 Sunset integrals

The sunset integrals are done with the methods of [22, 28]. They are defined as

$$\langle\langle X \rangle\rangle = \frac{1}{i^2} \int \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \frac{X}{(q^2 - m_1^2)(r^2 - m_2^2)[(q+r-p)^2 - m_3^2]}, \quad (82)$$

The various sunset integrals with Lorenz indices are

$$\begin{aligned} H(m_1^2, m_2^2, m_3^2; p^2) &= \langle\langle 1 \rangle\rangle, \\ H^\mu(m_1^2, m_2^2, m_3^2; p^2) &= \langle\langle q^\mu \rangle\rangle = p^\mu H_1(m_1^2, m_2^2, m_3^2; p^2), \\ H^{\mu\nu}(m_1^2, m_2^2, m_3^2; p^2) &= \langle\langle q^\mu q^\nu \rangle\rangle \\ &= p^\mu p^\nu H_{21}(m_1^2, m_2^2, m_3^2; p^2) + g^{\mu\nu} H_{22}(m_1^2, m_2^2, m_3^2; p^2). \end{aligned} \quad (83)$$

and

$$\begin{aligned} \langle\langle r^\mu \rangle\rangle &= p^\mu H_1(m_2^2, m_1^2, m_3^2; p^2), \\ \langle\langle r^\mu r^\nu \rangle\rangle &= p^\mu p^\nu H_{21}(m_2^2, m_1^2, m_3^2; p^2) + g^{\mu\nu} H_{22}(m_2^2, m_1^2, m_3^2; p^2), \\ \langle\langle q^\mu r^\nu \rangle\rangle &= \langle\langle r^\mu q^\nu \rangle\rangle, \\ \langle\langle q^\mu r^\nu \rangle\rangle &= p^\mu p^\nu H_{23}(m_1^2, m_2^2, m_3^2; p^2) + g^{\mu\nu} H_{24}(m_1^2, m_2^2, m_3^2; p^2), \end{aligned} \quad (84)$$

The function H is fully symmetric in m_1^2, m_2^2 and m_3^2 , while H_1, H_{21} and H_{22} are symmetric under the interchange of m_2^2 and m_3^2 . The relation between the above 3 functions

$$\begin{aligned} p^2 H_{21}(m_1^2, m_2^2, m_3^2; p^2) + d H_{22}(m_1^2, m_2^2, m_3^2; p^2) = \\ m_1^2 H(m_1^2, m_2^2, m_3^2; p^2) + A(m_2^2) A(m_3^2), \end{aligned} \quad (85)$$

allows to express H_{22} in terms of H_{21} .

Similar to the integral B and B_1 , there is also a relation between H and H_1 which in the equal mass case becomes

$$H_1(m^2, m^2, m^2; p^2) = \frac{1}{3} H(m^2, m^2, m^2; p^2). \quad (86)$$

The other functions, H_{23} and H_{24} , can be written in term of H, H_1 and H_{21} by using relations derived from redefining the momenta and masses in its definition [22].

The full sunset integral expressions and the definition for finite part $H_i^F = \{H^F, H_1^F, H_{21}^F\}$ can be found in the appendix of [22]. In our case we take $m_1 = m_2 = m_3 = m$.

In order to eliminate the extra poles in the expressions, sometimes we need to expand the $H_i^F(m^2, m^2, m^2; q^2)$ around the pseudoscalar mass m^2 , and we define

$$H_i^M(m^2, m^2, m^2; q^2) = \frac{1}{(q^2 - m^2)^2} \left[H_i^F(m^2, m^2, m^2; q^2) - H_i^F(m^2, m^2, m^2; m^2) - (q^2 - m^2) H_i^{F'}(m^2, m^2, m^2; m^2) \right], \quad (87)$$

where

$$H_i^{F'}(m^2, m^2, m^2; m^2) = \left. \frac{\partial H_i^F(m^2, m^2, m^2; q^2)}{\partial q^2} \right|_{q^2=m^2}. \quad (88)$$

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