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Hard Pion Chiral Perturbation Theory for $B \to \pi$ and $D \to \pi$ Formfactors

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Abstract

We use one-loop Heavy Meson Chiral Perturbation Theory (HMCHPT) as well as a relativistic formulation to calculate the chiral logarithms $m_\pi^2 \log\left(m_\pi^2/\mu^2\right)$ contributing to the formfactors of the semileptonic $B \to \pi$ decays at momentum transfer q^2 away from $q^2_{\rm max} = (m_B - m_\pi)^2$. We give arguments why this chiral behavior is reliable even in the energy regime with hard or fast pions. These results can be used to extrapolate the formfactors calculated on the lattice to lower light meson masses.

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1 Introduction

The study of the formfactors of semileptonic $B \to \pi$ and $D \to \pi$ decays has become a task of primary importance for the determination of the KM matrix elements $|V_{ub}|$ and V_{cd} respectively. Unfortunately this is a rather difficult mission. The kinematically accessible region is large, the physical pictures emerging at the two extremities of the q^2 range are quite different, requiring different approximation methods and this makes calculations directly from QCD hard.

Nevertheless a lot of effort has been put into studying this decay, both in experiment and in theory. The q^2 spectrum has been measured by different collaborations (CLEO [1, 2], Belle [3], Babar [4]) The QCD based theoretical calculations are either on QCD light-cone sum rules (LCSR), which provide reliable determinations at small q^2 [5], or on lattice simulations [6]. In particular lattice QCD allows to solve the non perturbative QCD effects numerically, but it is at present limited in the light quark masses that can be reached. Thus a final extrapolation in the light quark masses is needed.

At low energies Chiral Perturbation Theory (ChPT) [7, 8] provides a way to do this extrapolation on a theoretically sound basis. For processes with all pions soft this works fine and was extended to include heavy mesons in [9, 10]. The B and B^* mesons were there included using a heavy quark like formalism known as Heavy Meson ChPT (HMChPT). This has been used to extrapolate the behaviour of the form factors near the endpoint, $q^2_{\max} = (m_B - m_\pi)^2$, where the pions are soft [11, 12]. But a description of the light quark mass dependence of the form factors in the entire range of energy is still missing. Therefore we lack extrapolation formulas in the region away from maximum momentum transfer.

A similar problem exists in the case of $K_{\ell 3}$ decay. Here two-flavour ($SU(2)$) ChPT provides a well defined scheme for the calculation of the formfactors near $q^2_{\max} = (m_K - m^2_\pi)$. At other values of q^2 including $q^2 \simeq 0$, the (two-flavour) power counting scheme breaks down due to the presence of a large momentum pion in the final state. However, the authors of [13] argued that also in this latter case the coefficient of the chiral logarithm $m^2_\pi \log m^2_\pi$ is calculable and thus it can be used for the extrapolations on the lattice at q^2 away from $q^2_{\max}$. In [14] the argument was clarified and extended to the case of the $K \to \pi\pi$ decays. It was also argued there that this was a much more general circumstance.

The aim of this paper is to perform the same calculations for heavy meson semileptonic decays. These results can then be used to perform the extrapolation to light quark masses of lattice results also for values of q^2 away from the end-point $q^2_{\max}$. The arguments as presented in [14] show that also in this case the coefficient of the logarithm should be calculable as discussed in Sect. 5. The discussion implies that both the HMChPT formalism or a relativistic one can be used. We have performed the calculations in both formalisms as a consistency check and have also reproduced the known results for the masses, decay constants and formfactors at $q^2_{\max}$ in both.

The paper is structured as follow. After a short description of HMChPT in Sect. 2, we introduce in Sect. 3 the relativistic Lagrangian that we used as a consistency check, since the off-shell behaviour in both formalisms is rather different. In Sect. 4 we define the formfactors involved, and how to include the weak current in the Lagrangians of both formalisms. Sect. 5 gives the arguments why this procedure should produce the correct nonanalytic behaviour in the light quark masses where they are different from [13, 14]. Finally the results for the coefficients are shown in Sect. 6 where we provide also some checks of the validity of our assumptions. The appendix gives some results for the needed expansions of the loop integrals.

Throughout the paper we focus on $B \to \pi \ell \nu_\ell$ decay, but the same procedure and calculations go through also in the D semileptonic decays. All formulas are applicable to both cases. We are extending this work to the three flavour case as well as to other vector formfactors like $B \to D$ [15].

2 Heavy Meson Chiral Perturbation Theory

In this section we review the main features of HMChPT [9, 10], see also the lectures by Wise [16] and the book [17]. Chiral Lagrangians can be used to describe the interactions of light mesons, as pions and kaons, with hadrons containing a heavy quark. HMChPT makes use of spontaneously broken $SU(N_f) \times SU(N_f)$ chiral symmetry on the light quarks, and spin-flavour symmetry on the heavy quarks. This formulation lets us study chiral symmetry breaking effects in a chiral-loop expansion by simultaneously performing an expansion in powers of the inverse of the heavy meson mass.

In this paper we deal only with two-flavour ChPT [7] but the theory can be easily extended in the case of three flavours [8], thus including kaons in the description. The notation is the same as in [18]. The lowest order Lagrangian describing the strong interactions of the light mesons is

$$\mathcal{L}^{(2)}_{\pi\pi} = \frac{F^2}{4} \left(\langle u_\mu u^\mu \rangle + \langle \chi_+ \rangle \right) , \tag{1}$$

with

$$\begin{aligned} u_\mu &= i\{u^\dagger(\partial_\mu - i r_\mu)u - u(\partial_\mu - i l_\mu)u^\dagger\} \, , \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u \, , \\ u &= \exp\left(\frac{i}{\sqrt{2}F}\phi\right) \, , \\ \chi &= 2B(s + ip), \\ \phi &= \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix} \, . \end{aligned} \tag{2}$$

The fields s, p, $l_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$ are the standard external scalar, pseudoscalar, left- and right- handed vector fields introduced by Gasser and Leutwyler [7, 8].

The field u and u_μ transform under a chiral transformation $g_L \times g_R \in SU(2)_L \times SU(2)_R$ as

$$u \longrightarrow g_R u h^\dagger = h u g_L^\dagger, \qquad u_\mu \longrightarrow h u_\mu h^\dagger. \tag{3}$$

In (3) h depends on u, g_L and g_R and is the so called compensator field. The notation $\langle X \rangle$ stands for trace over up and down quark indices and all matrices are 2×2 matrices.

We now begin with a brief synopsis of the formalism of HMChPT for the two-flavour case. The three flavour case was the original formulation [9, 10]. In the limit $m_b \to \infty$, the pseudoscalar B and the vector $B^\star$ mesons are degenerate. In the following we neglect the mass splitting $\Delta = m_{B^\star} - m_B$. To implement the heavy quark symmetries it is convenient to assemble them into a single field

$$H^a(v) \;=\; \frac{1+\not{v}}{2}\left[B^{*a}_\mu(v)\gamma^\mu - B^a(v)\gamma_5\right], \tag{4}$$

where v is the fixed four-velocity of the heavy meson, a is a flavour index corresponding to the light antiquark in the B meson. $B^1 = B^+$, $B^2 = B^0$ and similarly for the vector mesons B^*_μ. In (4) the operator $(1+\not{v})/2$ projects out the particle component of the heavy meson only. The conjugate field is defined as $\overline{H}_a(v) = \gamma_0 H^\dagger_a(v)\gamma_0$. We assume the field $H^a(v)$ to transform under the chiral transformation $g_L \times g_R \in SU(2)_L \times SU(2)_R$ as

$$H_a(v) \longrightarrow h_{ab} H_b(v)\,, \tag{5}$$

so we introduce the covariant derivative as

$$D^\mu_{ab} H_b(v) = \delta_{ab}\partial^\mu H_b(v) + \Gamma^\mu_{ab} H_b(v), \tag{6}$$

where $\Gamma^\mu_{ab} = \frac{1}{2}\left[u^\dagger\left(\partial_\mu - ir_\mu\right)u + u\left(\partial_\mu - il_\mu\right)u^\dagger\right]_{ab}$, and the indices a, b run over the light quark flavours. Finally, the Lagrangian for the heavy-light mesons in the static heavy quark limit reads

$$\mathcal{L}_{\rm heavy} = -i\,{\rm Tr}\left[\overline{H}_a iv\cdot D_{ab} H_b\right] + g\,{\rm Tr}\left[\overline{H}_a u^\mu_{ab} H_b \gamma_\mu\gamma_5\right], \tag{7}$$

where g is the coupling of the heavy meson doublet to the Goldstone boson and the traces, Tr, are over spin indices, the γ-matrix indices. The Lagrangian (7) satisfies chiral symmetry and heavy quark spin flavour symmetry.

As a final remark of this section we stress that, in general, the use of HMChPT is only valid as long as the interacting pion is soft, i.e. if it has momentum much smaller then the scale of spontaneous chiral symmetry breaking ($\Lambda_{\rm ChSB} \simeq 1$ GeV). In fact, only in this regime the usual ChPT is well defined. For the semileptonic decays of heavy mesons this range of energy covers just a small fraction of the Dalitz plot. In Sect. 5 we will give an argument why the predictions on the coefficients of the logarithms appearing in the final amplitudes are reliable even outside the range of applicability of HMChPT.

3 Relativistic Theory

When $q^2 \neq q_{\rm max}$ it is possible that in the loops appear very off-shell B and B^* mesons. This in principle changes the non analyticities in the light masses of the loop functions and thus it might affect the coefficients of $m_\pi^2 \log m_\pi^2/\mu^2$. It could be that different treatments of the off-shell behaviour gave rise to different nonanalyticities. Sect. 5 argues that this should not be the case. In order to test this, we are not only calculating using HMChPT but also in a relativistic formulation. We also add some redundant higher order terms as an additional check.

For this scope, we construct a relativistic Lagrangian that respects the spin-flavour symmetries of HMChPT. It is built up starting from B^a and B_μ^{*a} fields, but now in the relativistic form, and we treat them as column-vectors in the light-flavour index a.

$$\begin{aligned}\mathcal{L}_{\rm kin} &= \nabla^\mu B^\dagger \nabla_\mu B - m_B B^\dagger B - \frac{1}{2} B^{\star\dagger}_{\mu\nu} B^{\star\mu\nu} + m_B B^{\star\dagger}_\mu B^{\star\mu}, \qquad (8)\\ \mathcal{L}_{\rm int} &= g M_0 \left(B^\dagger u^\mu B^\star_\mu + B^{\star\dagger}_\mu u^\mu B \right)\\ &+ \frac{g}{2} \epsilon^{\mu\nu\alpha\beta} \left(-B^{\star\dagger}_\mu u_\alpha \nabla_\mu B^\star_\beta + \nabla_\mu B^{\star\dagger}_\nu u_\alpha B^\star_\beta \right), \qquad (9)\end{aligned}$$

with $B^\star_{\mu\nu} = \nabla_\mu B^\star_\nu - \nabla_\nu B^\star_\mu$, and $\nabla_\mu = \partial_\mu + \Gamma_\mu$. The constant g of (9) is the same in (7), M_0 is the mass of the B meson in the chiral limit. In (8) and (9) we have suppressed flavour indices a, b for simplicity. The fields B and B^* transform under chiral transformations as $B \to hB$. The two terms of $\mathcal{L}_{\rm int}$ in (9) contain the vertices $BB^\star\pi$ and $B^\star B^\star \pi$. No interaction of the kind $BB\pi$ appears because it is forbidden by parity conservation.

From $\mathcal{L}_{\rm kin}$ in (8) we find the propagators of the B and $B^\star$ meson respectively:

$$\frac{i}{p^2 - m_B^2}, \qquad \frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_B^2}\right)}{p^2 - m_B^2}. \qquad (10)$$

This is to be contrasted with the propagator $1/v \cdot p$ in the HMChPT formalism showing the different off-shell behaviour.

4 $B \to \pi$ formfactors: formalism

In this section we review the semileptonic decay formalism. The hadronic current for pseudoscalar to pseudoscalar semileptonic decays ($P_i(\bar{q}_i, q) \to P_f(\bar{q_f}, q)\ell^+\nu_\ell$) has the structure

$$\begin{aligned}\langle P_f(p_f) | \overline{q}_i \gamma_\mu q_f | P_i(p_i) \rangle &= (p_i + p_f)_\mu f_+(q^2) + (p_i - p_f)_\mu f_-(q^2) \qquad (11)\\ &= \left[(p_i + p_f)_\mu - q_\mu \frac{(m_i^2 - m_\pi^2)}{q^2} \right] f_+(q^2) + q_\mu \frac{(m_i^2 - m_f^2)}{q^2} f_0(q^2),\end{aligned}$$

where q^μ is the momentum transfer $q^\mu = p_i^\mu - p_f^\mu$. In our case P_f is a pion, P_i is a B meson and $q_i = b$. For example, to find the $B^0 \to \pi^+$ formfactors we need then to evaluate the hadron matrix elements of the quark bilinear $\overline{b}\gamma_\mu q$, where $q = u$.

Heavy quark and chiral symmetry transformation properties of chiral currents dictate that the matching of QCD bilinears onto operators of HMChPT take the form [11, 16],

$$\begin{aligned}\bar{b}\gamma^\mu (1-\gamma_5) q_a &\to i c_L {\rm Tr}\left[\gamma^\mu (1-\gamma_5) u^\dagger_{ab} H_b(v) \right],\\ \bar{b}\gamma^\mu (1+\gamma_5) q_a &\to i c_R {\rm Tr}\left[\gamma^\mu (1+\gamma_5) u_{ab} H_b(v) \right]. \qquad (12)\end{aligned}$$

The constants c_L and c_R have to be equal because of parity invariance, therefore we can conclude

$$\bar{b}\gamma^\mu q_a \propto \mathrm{Tr}\left[\gamma^\mu \left(u^\dagger_{ab} + u_{ab}\right) H_b(v)\right] + \mathrm{Tr}\left[\gamma_5\gamma^\mu \left(u^\dagger_{ab} - u_{ab}\right) H_b(v)\right]. \quad (13)$$

If no hard pions appear in the final state we can use the definition of decay constant

$$\left\langle 0 \left| \bar{b}\gamma_\mu\gamma_5 q \right| B(p_B)\right\rangle = iF_B p^\mu_B \quad (14)$$

and state $c_L = c_R = \frac{1}{2}F_B\sqrt{m_B}$. Of course this latter result does not hold for momenta away from q^2_{max} in which case $c_L = c_R$ is just an effective coupling, as explained in Sect. 5.

In HMChPT it is convenient to use definitions in which the formfactors are independent of the heavy meson mass

$$\left\langle \pi(p_\pi) \left| \bar{b}\gamma_\mu q \right| B(v)\right\rangle_{\mathrm{HMChPT}} = \left[p_{\pi\mu} - (v\cdot p_\pi)\, v_\mu\right] f_p(v\cdot p_\pi) + v_\mu f_v(v\cdot p_\pi). \quad (15)$$

In (15) $v\cdot p_\pi$ is the energy of the pion in the heavy meson rest frame

$$v\cdot p_\pi = \frac{m^2_B + m^2_\pi - q^2}{2m_B}. \quad (16)$$

The formfactors defined in (11) and in (15) are related by matching the relativistic and the HMChPT hadronic current:

$$f_0(q^2) = \frac{1}{\sqrt{m_B}} f_v(v\cdot p_\pi), \qquad f_+(q^2) = \frac{\sqrt{m_B}}{2} f_p(v\cdot p_\pi). \quad (17)$$

The $\sqrt{m_B}$ factors in (17) are due to the different normalizations for states used in the two formalisms. In principle the relations in (17) are valid only when $q^2 \approx q^2_{\mathrm{max}}$, i.e. when HMChPT is applicable. On the other hand, for the arguments shown in Sect. 5 below, the chiral structure of the formfactors in QCD and in HMChPT is the same also for q^2 away from q^2_{max}. Therefore (17) holds, at least as far as regard the tree level term and the leading logarithms.

A matching similar to (12) has to be done also for the relativistic theory described in Sect. 3. We identify four possible operators[1]

$$J^L_\mu = \frac{1}{2}E_1 t u^\dagger \nabla_\mu B + \frac{i}{2}E_2 t u^\dagger u_\mu B + \frac{i}{2}E_3 t u^\dagger B^\star_\mu + \frac{1}{2}E_4 t u^\dagger \left(\nabla_\nu u_\mu\right) B^{\star\nu}, \quad (18)$$

where E_1,..., E_4, are effective couplings. t is a constant spurion vector transforming as $t \to t g^\dagger_L$, so that J^L_μ is invariant under $SU(2)_L$ transformations. The heavy quark symmetry implies $m_B E_1 = E_3$. Analogously we can introduce a J^R_μ current and thus an axial-vector $J^5_\mu = J^R_\mu - J^L_\mu$ and a vector $J^V_\mu = J^R_\mu + J^L_\mu$ current. They are used respectively to evaluate the amplitudes of $B \to \ell\nu_\ell$ and the $B \to \pi\ell\nu_\ell$ formfactors as defined in (11). We leave the

[1] The last one is higher order but we included it since it has a different type of contraction of the Lorentz indices and as an explicit check on the arguments of Sect. 5.

discussion for the latter in Sect. 6, while we quote here the results of the $B \to$ vacuum matrix element at one loop

$$F_B = E_1 \left[1 + \frac{1}{F^2} \left(\frac{3}{8} + \frac{9}{8} g^2 \right) \overline{A}(m_\pi^2) \right] \tag{19}$$

$\overline{A}(m_\pi^2)$ is defined in (25) in the appendix. Here we only quoted the nonanalytic dependence on the light quark masses for the one-loop part. We compare (19) with the results obtained with HMChPT [19]. We see that E_1 plays the role of $F_{H_u,d}$ in [19] and that the relativistic theory predicts the same coefficient of the chiral logarithm in $\overline{A}(m_\pi^2)$.

5 Hard Pion Chiral Perturbation Theory

In order to study the chiral behavior of the formfactors at q^2 away from $q^2_{\max}$ we can not neglect operators with an arbitrary numbers of derivatives on the external pion since now its momentum is large. Therefore we must take into account that the usual power counting of ChPT does not work and we can not be sure a priori that a loop calculation would make sense.

A similar problem arises in the case of $K_{\ell 3}$ decay. The authors of [13] dealt with it using $SU(2)$ ChPT to study the amplitudes whether the outgoing pion was soft or not. They argued they could calculate the corrections of the type $m_\pi^2 \log m_\pi^2$ even in the range of energy where the usual ChPT does not work. Their argument is based on the fact that only the soft internal pions are responsible for the chiral logarithms. These ideas were generalized by the authors of [14] who made clear that those arguments basically corresponds to use an effective Lagrangian to describe the hard part of a general loop calculation in a chiral invariant way. The situation is shown schematically in Fig. 1. The underlying argument is the same as the analysis for infra-red divergences. Since the soft lines do not see the hard or short-distance structure of the diagram, we can separate them from the rest of the process. We should thus be able to describe the hard part of any diagram by an effective Lagrangian. This effective Lagrangian should include the most general terms allowed consistent with all the symmetries and have coefficients that depend on the hard kinematical quantities and can even be complex. An two-loop example will be given in [15]. We expect that a proof along the lines of SCET [20] should be possible. Once it is accepted that one can do this, a second step is to prove that the effective Lagrangian one uses is sufficient to describe the neighbourhood of the hard process and calculate chiral logarithms.

The latter was done in [13] for the case of $K_{\ell 3}$ decays by showing that the matrix elements of operators with higher derivatives was proportional to the lowest order matrix-element up to terms of order m_π^2. In particular, the part including the coefficient of the chiral logarithms $m_\pi^2 \log m_\pi^2$ has the same coefficient relative to the tree level matrix-element as the lowest order operator. The same was proven for $K \to \pi\pi$ in [14].

As a matter of fact, the semileptonic K decay has the same structure *heavy* $\to$ *light* as the B one when M_K is treated as large compared to m_π as in [13]. The main differences

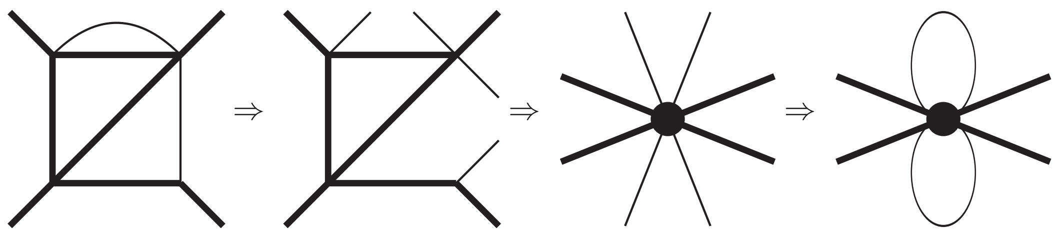

Figure 1: An example of the argument used. The thick lines contain a large momentum, the thin lines a soft momentum. Left: a general Feynman diagram with hard and soft lines. Middle-left: we cut the soft lines to remove the soft singularity. Middle-right: The contracted version where the hard part is assumed to be correctly described by a "vertex" of an effective Lagrangian. Right: the contracted version as a loop diagram. This is expected to reproduce the chiral logarithm of the left diagram. Figure from [14].

between the two processes are the energies involved and that for the B meson the corresponding vectorial particle $B^\star$ is close to its mass-shell. So in order to have a sufficiently complete effective Lagrangian in the neighbourhood we need to include the B^* as was done in the previous sections. However, we expect the kind of arguments presented in [13, 14] to work here as well.

We note that the effective Lagrangian needs to be complete enough in the neighbourhood of the underlying process. That also implies that if we have two different formalisms, both sufficiently complete, the logarithms should be the same. In particular, the HMChPT and the relativistic formalism should give the same results.

We are only concerned with terms of order 1, m_π and $m_\pi^2 \log m_\pi^2$, i.e. we are not trying to calculate terms of order $O(m_\pi^2)$ without logarithms. Here we restrict our discussion to the case of $SU(2)$ ChPT, and far from $q^2_{\rm max}$. It is clear that adding (soft) kaon loops does not change the validity of the arguments. First we analyze the case of HMChPT and thus we need to look at the matrix elements $\langle \pi(p_\pi)\,|O|\,B(v)\rangle$ where O can be any of the operators in (13) with possibly more derivatives. We want to show that matrix elements of operators with higher number of derivatives are all proportional to the lowest order ones up to terms of order m_π^2 (and without logarithms) which are of higher order. We need to look at the cases where extra covariant derivatives D_μ are added in the operators. We can distinguish different possibilities depending on which particle the derivative hits.

- The case where it hits an internal soft pion line, leads to $\int d^dp\, p_\mu/(p^2 - m_\pi^2)$ which is always suppressed by three powers of m_π.

- If the derivative hits an internal line which is not soft in a loop, it is part of the loop diagram that is described by our effective Lagrangian and is thus indirectly included via the coefficients. A simple example is when a pair of derivatives hits a B-meson.

Then we get terms like $\int d^dp\, p_B^2/(v\cdot p_\pi-\Delta)\ldots \simeq m_B^2 \int d^dp\, 1/(v\cdot p_\pi-\Delta)\ldots$, i.e. something that is proportional to the lowest order result and that can be included modifying accordingly the effective coupling. It corresponds to change the hard structure of the loop diagram, what can be described by a proper replacement of the effective coupling, as shown in Fig. 1.

- All the extra derivatives should thus act on external lines or tree-level internal lines, i.e. those not in a loop. All these can thus be transformed into masses of external particles or other kinematical quantities, as here q^2. None of these has terms of order m_π or $m_\pi^2 \log m_\pi^2$. The kinematical quantities we keep fixed and masses have corrections at most of order m_π^2 compared to the order 1 terms and the order 1 part can be absorbed into the coefficient of the lowest order term.

- Note that if the extra derivatives are contracted with a v_μ rather than another derivative this can also be put into the value of the coefficients.

Also in the case of the relativistic theory described in Sect. 3, we need to worry if more chiral logarithms arise including operators like the ones in (18) but with extra derivatives. All the above arguments also work except for derivatives that are contracted with B^*_μ. In this case the extra derivatives becomes contracted with the momenta in the B^* propagator or via $g_{\mu\nu}$ to the external current. After that the above arguments again apply. Terms involving a contraction with B^* can always be reduced to the simplest one which we included in (18), the E_4 term, and so we have also an explicit test of the last argument.

Near $q^2_{\rm max}$ the above arguments fail since kinematical quantities can contain terms of order m_π. However, here all pion lines are soft and we are in the regime of validity of standard HMChPT.

The conclusion from this section is that the coefficient of the chiral logarithm $m_\pi^2 \log m_\pi^2$ is calculable at all values of q^2.

6 The Coefficients of the Chiral Logarithms

In this section we show the results for the semileptonic decay $B \to \pi \ell \nu_\ell$ amplitudes. Hereafter, we quote only the relevant terms, i.e. the leading ones and the chiral logarithms. The tree-level diagrams contributing to the amplitude are shown in Fig. 2. The results for the formfactors at tree level for HMChPT are [11, 12]

$$f_v^{\rm Tree}(v\cdot p_\pi) = \frac{\alpha}{F}, \qquad f_p^{\rm Tree}(v\cdot p_\pi) = \frac{\alpha}{F}\frac{g}{v\cdot p_\pi+\Delta}, \tag{20}$$

where α is a constant that takes the value $\sqrt{m_B/2}F_B$ at $q^2_{\rm max}$. We also have $c_L = c_R = \alpha/\sqrt{2}$. For the relativistic theory of Sect. 3 we obtain

$$f_0^{\rm Tree}(q^2) = \frac{E_1}{F}\frac{1}{4}, \qquad f_+^{\rm Tree}(q^2) = \frac{1}{4}\frac{E_3}{F}\frac{m_B}{q^2-m_B^2}g. \tag{21}$$

(1) (2)

Figure 2: The tree-level diagrams contributing to the amplitude. A double line correspond to a B, a zigzag line to a $B^\star$, a single line to a pion. A black circle represents the insertion of a $B \to \pi$ vector current.

Near $q^2_{max} = (m_B - m_\pi)^2$ the results are obviously the same, but the propagators in the second equations of (20) and (21) become respectively $1/m_\pi$ and $1/(2m_\pi m_B)$. The different factor of 2 is due to the different normalization of states used in HMChPT and in the relativistic formulation just as in (17). Note that the relation of the coupling constant to F_B is only valid for $q^2_{max} = (m_B - m_\pi)^2$. The coupling constant are different at the different values of q^2 and can even be complex.

To proceed with the calculation at one-loop we need the wavefunction renormalization Z_π and Z_B. They are the same for HMChPT and the relativistic theory and read:

$$Z_\pi = 1 + \frac{4}{3}\overline{A}(m_\pi^2), \qquad Z_B = 1 + \frac{9}{2}g^2\overline{A}(m_\pi^2). \tag{22}$$

The one-loop diagrams are shown in Fig. 3. In Tab. 1 we present the results at q^2 away from $q^2_{\rm max}$. To find the results in HMCHPT we expanded the one-loop calculation of [12] at $v \cdot p_\pi \to m_B$, $m_\pi^2 \to 0$. In the relativistic theory we first calculated the formfactors and then we expanded the loop integrals for $m_\pi^2 \ll m_B^2, (m_B^2 - q^2)$. These latter expansions are shown in App. A. We check that the coefficients of the leading logarithms coincide in the two theories. Summing up all the results in Tab. 1 and including $(1/2)Z_\pi$ and $(1/2)Z_B$ times tree level, we find

$$\begin{aligned} f_{v/p}(v \cdot p_\pi) &= f_{v/p}^{\rm Tree}(v \cdot p_\pi)\left[1 + \left(\frac{3}{4} + \frac{9}{4}g^2\right)\overline{A}(m_\pi^2)\right], \\ f_{0/+}(q^2) &= f_{0/+}^{\rm Tree}(q^2)\left[1 + \left(\frac{3}{4} + \frac{9}{4}g^2\right)\overline{A}(m_\pi^2)\right], \end{aligned} \tag{23}$$

i.e. as expected the same coefficients in the two theories. The correction is also the same for the scalar formfactor f_0 and for f_+. In Tab. 2 we quote also the results in the limit $q^2 = q^2_{\rm max} = (m_B - m_\pi)^2$ where the two theories must give the same outcome, being one the relativistic limit of the other. So this is another check of the validity of our relativistic theory. Summing up all the results as explained above we find at $q^2_{\rm max}$

$$f_v(v \cdot p_\pi) = f_v^{\rm Tree}(v \cdot p_\pi)\left[1 + \left(\frac{11}{4} + \frac{9}{4}g^2\right)\overline{A}(m_\pi^2)\right],$$

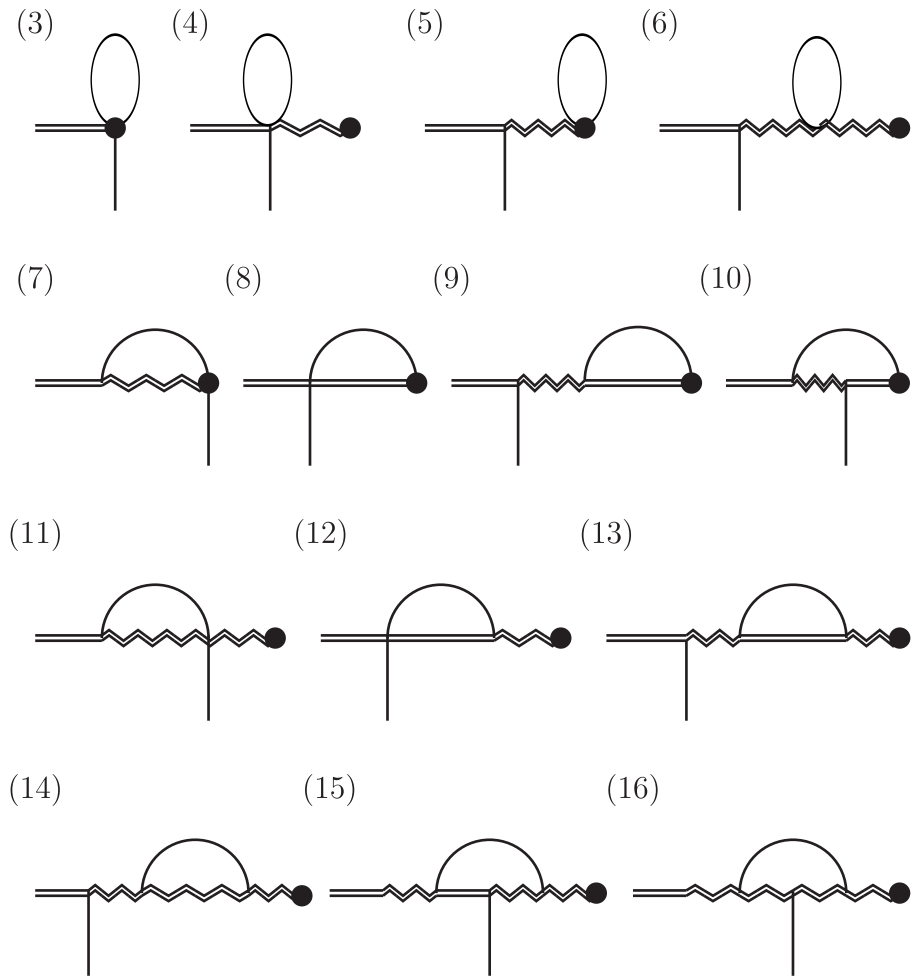

Figure 3: The one-loop diagrams contributing to the amplitude.Vertices and lines as in Fig. 2

Diagram	f_v HMChPT	f_0 Rel. Th.
(3)	$\frac{5}{24}\frac{1}{F^3}\alpha$	$\frac{5}{96}\frac{1}{F^3}E_1$
(8)	$\frac{1}{2}\frac{1}{F^3}\alpha$	$\frac{1}{8}\frac{1}{F^3}E_1$
	f_p HMChPT	f_+ Rel. Th.
(4)	$\frac{2}{6}\frac{g}{F^3}\frac{\alpha}{v\cdot p_\pi+\Delta}$	$\frac{1}{12}\frac{g}{F^3}\frac{m_B}{q^2-m_B^2}E_3$
(5)	$\frac{3}{8}\frac{g}{F^3}\frac{\alpha}{v\cdot p_\pi+\Delta}$	$\frac{3}{32}\frac{g}{F^3}\frac{m_B}{q^2-m_B^2}E_3$

Table 1: The coefficients of the chiral logarithms, $\overline{A}(m_\pi^2)$, at q^2 away from $(m_B - m_\pi)^2$ from the different diagrams in Fig. 3. The diagrams not listed in the table do not contribute with logarithms. The two Lagrangians give the same coefficients diagram per diagram provided the tree level coefficients are correctly identified.

Diagram	f_v HMChPT	f_0 Rel. Th.
(3)	$\frac{5}{24}\frac{1}{F^3}\alpha$	$\frac{5}{96}\frac{1}{F^3}E_1$
(8)	$\frac{3}{2}\frac{1}{F^3}\alpha$	$\frac{3}{8}\frac{1}{F^3}E_1$
	f_p HMChPT	f_+ Rel. Th.
(4)	$\frac{2}{6}\frac{\alpha}{F^3}\frac{g}{m_\pi}$	$\frac{1}{12}\frac{E_3}{F^3}\frac{g}{2m_\pi}$
(5)	$\frac{3}{8}\frac{\alpha}{F^3}\frac{g}{m_\pi}$	$\frac{3}{32}\frac{E_3}{F^3}\frac{g}{2m_\pi}$
(13)	$\frac{3}{4}\frac{\alpha}{F^3}\frac{g^3}{m_\pi}$	$\frac{1}{16}\frac{E_3}{F^3}\frac{g^3}{2m_\pi}$
(14)	$\frac{3}{2}\frac{\alpha}{F^3}\frac{g^3}{m_\pi}$	$\frac{1}{8}\frac{E_3}{F^3}\frac{g^3}{2m_\pi}$
(15)	$\frac{1}{12}\frac{\alpha}{F^3}\frac{g^3}{m_\pi}$	$\frac{1}{48}\frac{E_3}{F^3}\frac{g^3}{2m_\pi}$
(16)	$-\frac{1}{6}\frac{\alpha}{F^3}\frac{g^3}{m_\pi}$	$-\frac{1}{24}\frac{E_3}{F^3}\frac{g^3}{2m_\pi}$

Table 2: The coefficients of the chiral logarithms, $\overline{A}(m_\pi^2)$, at q^2_{max} from the different diagrams in Fig. 3. The diagrams not listed in the table do not contribute to the chiral logarithms. The two Lagrangians give the same coefficients provided the tree level coefficients are correctly identified.

$$
\begin{aligned}
f_0(q^2) &= f_0^{\text{Tree}}(q^2)\left[1+\left(\frac{11}{4}+\frac{9}{4}g^2\right)\overline{A}(m_\pi^2)\right], \\
f_p(v\cdot p_\pi) &= f_p^{\text{Tree}}(v\cdot p_\pi)\left[1+\left(\frac{3}{4}+\frac{43}{4}g^2\right)\overline{A}(m_\pi^2)\right], \\
f_+(q^2) &= f_+^{\text{Tree}}(q^2)\left[1+\left(\frac{3}{4}+\frac{43}{4}g^2\right)\overline{A}(m_\pi^2)\right],
\end{aligned} \tag{24}
$$

i.e. agreement between the two theories.

As a final check, we notice that the results obtained including only those diagrams where no $B^\star$ appears (i.e. (1),(3) and (8) in Fig. 2 and 3) coincide with the ones in [13] for the $K \to \pi$ amplitudes. The chiral corrections must agree in these two cases since, as we remarked above, the only difference between the two processes is the presence of the vectorial $B^\star$ particle.

7 Conclusions

In this paper we have calculated the pionic logarithms in the semileptonic $B \leftarrow \pi$ and $D \leftarrow \pi$ transitions. We have reproduced the known results near the endpoint $q^2 = (M_B - m_\pi)^2$, Eq. (24) and obtained the chiral logarithm also away from the endpoint in Eq. 23 and it was the same for both formfactors.

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A Loop integrals expansions

We collect the relevant expansions of the one-loop integrals needed to evaluate the diagrams in Fig. 3 in the framework of the relativistic theory of Sect. 3. In the calculation we need the one-, two- and three-point functions defined as ($d = 4 - 2\epsilon$)

$$
A(m_1^2) = \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m_1^2} \tag{25}
$$

$$
B(m_1^2, m_2^2, p^2) = \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_1^2)((p-k)^2-m_2^2)} \tag{26}
$$

$$C(m_1^2,m_2^2,m_3^2,p_1^2,p_2^2) = \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_1^2)((k-p_1)^2-m_2^2)((k-p_1-p_2)^2-m_3^2)} \quad . \tag{27}$$

Actually two- and three-point functions with extra powers of momenta in the numerator contribute too, but we do not intend to give their definitions here. They can be found in [22] in precisely the form used here. We only stress that all these functions can be rewritten in terms of (25), (26) and (27) [23]. The finite parts of $A(m_1^2)$ and $B(m_1^2,m_2^2,q^2)$ are [24]

$$\overline{A}(m_1^2) = -\frac{m_1^2}{16\pi^2}\log\left(\frac{m_1^2}{\mu^2}\right), \tag{28}$$

$$\bar{B}(m_1^2,m_2^2,q^2) = \frac{1}{16\pi^2}\left[-1-\int_0^1 dx\log\left(\frac{m_1x+m_2(1-x)-x(1-x)q^2}{\mu^2}\right)\right]. \tag{29}$$

In the calculation of the amplitude the three-point function $C(m_1^2,m_2^2,m_3^2,p_1^2,p_2^2)$ always depends on the masses as (m^2,M^2,M^2,m^2,q^2) with $m=m_\pi$, and $M=m_B$. It can be rewritten using Feynman parameters x, y

$$C(m^2,M^2,M^2,m^2,q^2) = -\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\,\big[m^2(1-x-2y+y^2) +M^2(x+y)^2+(q^2-M^2-m^2)(-y+y(x+y))\big]^{-1}. \tag{30}$$

In order to find the appropriate chiral logarithms we expanded (29) and (30) for small m^2/M^2 . We quote only the terms of the expansions containing the chiral logarithms $\log(m^2/\mu^2)$

$$\bar{B}(m^2,M^2,q^2) = -\frac{1}{M^2-q^2}\overline{A}(m^2), \qquad q^2\ll q^2_{\max} \tag{31}$$

$$C(m^2,M^2,M^2,m^2,q^2) = -\frac{1}{(M^2-q^2)^2}\overline{A}(m^2), \qquad q^2\ll q^2_{\max} \tag{32}$$

$$\bar{B}(m^2,M^2,M^2) = \frac{1}{2M^2}\overline{A}(m^2), \tag{33}$$

$$\bar{B}(m^2,M^2,m^2) = 0, \tag{34}$$

$$\bar{B}(m^2,M^2,(M-m)^2) = -\frac{1}{mM}\overline{A}(m^2)-\frac{1}{M^2}\overline{A}(m^2). \tag{35}$$

The expansions of the three-point functions at $q^2_{\max}$ are a bit more involved. The reason is that the reduction formulas present a singularity at $q^2_{\max}=(M-m)^2$ for $m^2=0$. Thus we expand each of them directly, from the Feynman parameter integral, without rewriting them in terms of (25), (26) and (27). To do this one n rewrites the integral in (30) using $z=x+y$ as

$$C(m^2,M^2,M^2,m^2,(M-m)^2) = -\frac{1}{16\pi^2}\int_0^1 dz\int_0^z dy\,\times \frac{1}{[M^2z^2+m^2+2mMy+(m^2(-z-y+y^2)-2mMyz)]}. \tag{36}$$

The part in the denominator in brackets is always suppressed by at least m/M compared to the first three terms for all values of z and y and we can thus expand in it. The remaining integrals can be done with elementary means. The result of the expansion is, quoting only up to the order needed for this work,

$$C(m^2, M^2, M^2, m^2, (M-m)^2) = -\frac{1}{2}\left(\frac{1}{m^2M^2}\overline{A}(m^2) + \frac{1}{mM^3}\overline{A}(m^2) + \frac{1}{M^4}\overline{A}(m^2)\right), \tag{37}$$

$$\overline{C}_{11}(m^2, M^2, M^2, m^2, (M-m)^2) = -\frac{1}{2}\frac{1}{mM^3}\overline{A}(m^2) + \frac{7}{12}\frac{1}{M^4}\overline{A}(m^2), \tag{38}$$

$$\overline{C}_{12}(m^2, M^2, M^2, m^2, (M-m)^2) = \frac{1}{3}\frac{1}{mM^3}\overline{A}(m^2) + \frac{7}{12}\frac{1}{M^4}\overline{A}(m^2), \tag{39}$$

$$\overline{C}_{21}(m^2, M^2, M^2, m^2, (M-m)^2) = -\frac{1}{6}\frac{1}{M^4}\overline{A}(m^2), \tag{40}$$

$$\overline{C}_{22}(m^2, M^2, M^2, m^2, (M-m)^2) = -\frac{7}{30}\frac{1}{M^4}\overline{A}(m^2), \tag{41}$$

$$\overline{C}_{23}(m^2, M^2, M^2, m^2, (M-m)^2) = -\frac{1}{4}\frac{1}{M^4}\overline{A}(m^2), \tag{42}$$

$$\overline{C}_{24}(m^2, M^2, M^2, m^2, (M-m)^2) = -\frac{1}{12}\frac{1}{M^2}\overline{A}(m^2). \tag{43}$$

The other three-point functions do not give any leading logarithm.

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