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Scalar Form Factors in SU(3) Chiral Perturbation Theory

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Abstract

The scalar form factors for the pions and kaons are calculated in SU(3) Chiral Perturbation Theory at order p^6, in the isospin limit $m_u = m_d = \hat{m}$. We find in general sizable corrections of $\mathcal{O}(p^6)$. We use our results to obtain results on the $1/N_c$ suppressed low energy constants L_4^r and L_6^r as well as on two $\mathcal{O}(p^6)$ low energy constants. We present some numerical results for masses and decay constants as well.

We derive a relation for the scalar form factors analogous to Sirlin's relation for vector form factors.

1 Introduction

Chiral Perturbation Theory (ChPT) in its effective Lagrangian form was introduced by Weinberg [1] and developed into its standard form, for both two and three light flavours (referred to as $SU(2)$ and $SU(3)$) formalisms, up to order p^4, by Gasser and Leutwyler [2, 3]. The underlying assumption of ChPT is that the chiral $SU(n)_L \times SU(n)_R$ symmetry present in the QCD Lagrangian is spontaneously broken to its vector subgroup, $SU(n)_L \times SU(n)_R \to SU(n)_V$. This breaking is due to the finite quark condensates[1]

$$\langle 0|\bar{q}q|0\rangle \neq 0, \quad \text{for } q{=}u,d \text{ and } s. \tag{1}$$

The quark and gluon degrees of freedom are integrated out and replaced by pseudo-scalar fields representing the lightest pseudo-scalar mesons triplet ($SU(2)$) or octet ($SU(3)$), these are the Goldstone bosons resulting from the spontaneous chiral symmetry breaking. Introducing non-zero quark masses renders the chiral symmetry only approximate. This results in masses for the mesons.

ChPT is thus an effective theory of QCD, perturbatively expanded in powers of the relevant momentum and the quark masses, and is non-renormalizable. Loop corrections produce ultra-violet divergences which require the introduction of new counterterms and associated low energy constants (LEC) at each order in the parameters expansion. The number of new operators is, however, finite at each order, allowing ChPT to be predictive. Recent introductions to ChPT can be found in Ref. [4].

A basic set of calculations in $SU(3)$ ChPT at order p^4 was performed in [3, 5] and used to determine the LEC at order p^4, the L_i^r, to that order. L_4^r and L_6^r were assumed to be negligible there using the large N_c limit of QCD [6]. This assumption was studied at order p^4 in [7] by considering scalar form factors and a scalar two-point function.

The state of ChPT in the meson sector now is such that many calculations have been pushed to the next order, p^6 or next-to-next-to-leading (NNLO) or two-loop order. There are two main reasons to explore the two-loop region of $SU(3)$ Chiral Theory. First, the accuracy of experimental results has improved so this accuracy is now routinely needed and we need to determine all LECs to the required precision. Second, spontaneous chiral,symmetry breaking of QCD does not have to be driven by the quark condensates, Eq. (1). An alternative bookkeeping method in the Lagrangian expansion might be necessary, leading to a Generalized ChPT (GChPT), see Ref. [8] and references therein. The combination of the two-loop calculation of $\pi\pi$-scattering in $SU(2)$ ChPT [9, 10], together with the dispersive Roy equation analysis [11], lead to sharp predictions for the $\pi\pi$ scattering lengths [12] confirmed by the E865 experiment [13] leading to the conclusion that $SU(2)$ ChPT is indeed driven by the quark condensate [14]. See also [8, 15, 16] for relevant discussions. In the case of $SU(3)$ ChPT the possibility of a small quark condensate is not ruled out [7, 17, 18]. The pion scalar form factor has been evaluated to order p^6 in [19] after earlier dispersive work [20].

[1] See below for the alternative view of generalized ChPT.

The two-loop calculations in $SU(3)$ ChPT of the masses [21, 22] indicated the possibility of sizable NNLO corrections. This behaviour persisted when the calculations also included $K_{\ell 4}$ [23, 24] such that the L_i^r could be determined with p^6 accuracy under similar assumptions as the p^4 determination [3]. In [24] a search was done for possible values of L_4^r and L_6^r that would lead to acceptable convergence of all quantities considered there. In [25] effects of isospin violation in the masses were taken into account also to order p^6 as well as well as the new E865 $K_{\ell 4}$ data of [13]. In this paper we calculate the scalar form factors of pion and kaons to order p^6. This allows to relax some of the assumptions made in the previous L_i^r determinations as well as being interesting in its own right. In particular, the curvature of the scalar form factor in $K_{\ell 3}$ decays [26] can in principle be determined from the quantities considered here. Unfortunately as discussed at the end of Sect. 5.5 the numerical accuracy obtainable is rather low.

The organization of the present paper is as follows. We first present the framework of $SU(3)$ ChPT in the isospin limit and including external scalar fields in Section 2 and sketch the calculation of the scalar form factors in Section 3. We then present the analytical results of this calculation in Section 4.1 and App. B as well as a few plots of the form factors. We made use of dispersive analysis, Section 5.1, to extract experimental information before presenting numerical results for some of the LEC's of the theory, Sections 5.3 and 5.5. We also present some updated results on masses and decay constants in Sect. 5.6.

A byproduct of this calculation is a relation between the scalar form factors that has only second order corrections in quark mass differences. This relation, Eq. (18) is proven in general in App. A and is valid for all t where an expansion in mass differences is valid.

2 The ChPT Lagrangian

This section contains a very short overview of ChPT and serves also to define our conventions. More detailed introductions can be found in [4]. The Lagrangians of ChPT are ordered in the p counting, powers of momenta ($\mathcal{O}(p)$), quark masses or external scalar or pseudo scalar fields ($\mathcal{O}(p^2)$) and external vector or axial-vector fields ($\mathcal{O}(p)$). We keep to the three flavour case here.

The Lagrangian for the strong and semi leptonic mesonic sector to NNLO can be written as

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6\,, \tag{2}$$

where the subscript refers to the chiral order. The lowest order Lagrangian

$$\mathcal{L}_2 = \frac{F_0^2}{4}\langle u_\mu u^\mu + \chi_+\rangle\,. \tag{3}$$

The mesonic fields enter via

$$u = \exp\left(\frac{iM}{F_0\sqrt{2}}\right)\,, \quad M = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & \frac{-2}{\sqrt{6}}\eta \end{pmatrix} \tag{4}$$

and the quantity u_μ introduces the external vector (v_μ) and axial-vector (a_μ) currents

$$u_\mu = i(u^\dagger \partial_\mu u - \partial_\mu u u^\dagger - i u^\dagger r_\mu u + i u l_\mu u^\dagger)\,, \quad l_\mu(r_\mu) = v_\mu - (+) a_\mu\,, \tag{5}$$

while the scalar (s) and pseudo scalar (p) currents are contained in

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u\,, \quad \chi = 2B_0\,(s + ip)\,. \tag{6}$$

The p^4 or NLO Lagrangian $\mathcal{L}_4$ was introduced in Ref. [3] and reads

$$\begin{aligned}
\mathcal{L}_4 \;=\; & L_1\langle u_\mu u^\mu\rangle^2 + L_2\langle u_\mu u_\nu\rangle\langle u^\mu u^\nu\rangle \\
& + L_3\langle u^\mu u_\mu u^\nu u_\nu\rangle + L_4\langle u^\mu u_\mu\rangle\langle\chi_+\rangle \\
& + L_5\langle u^\mu u_\mu \chi_+\rangle + L_6\langle\chi_+\rangle^2 \\
& + L_7\langle\chi_-\rangle^2 + L_8\langle\chi_+\chi_-\rangle \\
& - i\,L_9\langle F^R_{\mu\nu} u u^\mu u^\nu u^\dagger + F^L_{\mu\nu} u^\dagger u^\mu u^\nu u\rangle + L_{10}\langle F^R_{\mu\nu} F^{L\mu\nu}\rangle \\
& + H_1\langle F^R_{\mu\nu} F^{\mu\nu R} + F^L_{\mu\nu} F^{\mu\nu L}\rangle + H_2\langle\chi_+^2 - \chi_-^2\rangle/4\,.
\end{aligned} \tag{7}$$

The L_9 and L_{10} terms introduce also the field strength tensor

$$F^{L(R)}_{\mu\nu} = \partial_\mu l(r)_\nu - \partial_\nu l(r)_\mu - i\,[l(r)_\mu, l(r)_\nu]\,. \tag{8}$$

The two terms proportional to H_1 and H_2 are contact terms and do not enter physical amplitudes. In the present case, we keep only the relevant scalar current interactions from which we extract and separate the quark masses contribution:

$$s = \begin{pmatrix} \hat{m} & & \\ & \hat{m} & \\ & & m_s \end{pmatrix} + \tilde{s}\,, \quad p = l_\mu = r_\mu = 0\,. \tag{9}$$

We quote the schematic form of the NNLO Lagrangian in the $SU(3)$ case

$$\mathcal{L}_6 = \sum_{i=1,94} C_i\, O_i \tag{10}$$

and refer to [27], where this was first constructed after the earlier attempt of [28], for the full expressions. The last four terms are contact terms [27].

All ultra-violet divergences produced by loop diagrams of order p^4 and p^6 cancel in the process of renormalization with the divergences extracted form the low energy constants L_i's and C_i's. We use here dimensional regularization and the standard modified minimal subtraction ($\overline{MS}$) version used in ChPT. An exhaustive description of the regularization and renormalization procedure including a description of the freedom involved can be found in Ref. [10] and [29].

The subtraction of divergences is done explicitly by

$$L_i = (c\mu)^{d-4}[\hat{\Gamma}_i \Lambda + L_i^r(\mu)] \tag{11}$$

and

$$C_i = \frac{(c\mu)^{2(d-4)}}{F^2}\left[C_i^r(\mu) - \left(\Gamma_i^{(1)} + \Gamma_i^{(L)}(\mu)\right)\Lambda - \Gamma_i^{(2)}\Lambda^2\right] \tag{12}$$

where c and Λ are defined by

$$\ln c = -\frac{1}{2}\left[\ln 4\pi + \Gamma'(1) + 1\right], \tag{13}$$

$$\Lambda = \frac{1}{16\pi^2(d-4)}. \tag{14}$$

The coefficients $\hat{\Gamma}_i$, $\Gamma_i^{(1)}$ and $\Gamma_i^{(2)}$ are constants while the $\Gamma_i^{(L)}$'s are linear combinations of the $L_i^r(\mu)$'s. Their explicit expressions can all be found in [29] where they have been calculated in general.

Scalar form factors can be calculated using functional derivatives w.r.t. the external scalar field $\tilde{s}$ [3].

3 The scalar form factors: definitions and overview of the calculation

3.1 Definitions

The scalar form factors for the pions and kaons are defined as follows:

$$\langle M_2(p)|\bar{q}_i q_j|M_1(q)\rangle = F_{ij}^{M_1 M_2}(t) \tag{15}$$

with $t = (p-q)^2$ and $i,j = u,d,s$ being indices in the flavour basis and M_1, M_2 a meson state with the indicated momentum.

In the present case of isospin symmetry $m_u = m_d = \hat{m}$, the various pion scalar form factors obey

$$\begin{aligned} F_S^\pi(t) &\equiv 2F_{uu}^{\pi^0\pi^0}(t) = 2F_{dd}^{\pi^0\pi^0}(t) = 2F_{uu}^{\pi^+\pi^+}(t) = 2F_{dd}^{\pi^+\pi^+}(t) = \\ &= -2\sqrt{2}F_{du}^{\pi^0\pi^+}(t) = 2\sqrt{2}F_{ud}^{\pi^0\pi^-}(t), \\ F_{Ss}^\pi &\equiv F_{ss}^{\pi^+\pi^+} = F_{ss}^{\pi^0\pi^0}. \end{aligned} \tag{16}$$

The kaon currents are related by the following rotations in flavour space

$$\begin{aligned} F_{Su}^K(t) &\equiv F_{uu}^{K^+K^+(t)} = F_{dd}^{K^0K^0}(t), \\ F_{Sd}^K(t) &\equiv F_{dd}^{K^+K^+}(t) = F_{uu}^{K^0K^0}(t), \\ F_{Ss}^K(t) &\equiv F_{ss}^{K^+K^+}(t) = F_{ss}^{K^0K^0}(t), \\ F_S^{K\pi}(t) &\equiv F_{su}^{K^0\pi^-}(t) = \sqrt{2}F_{su}^{K^+\pi^0}(t), \\ F_{Sq}^K(t) &\equiv F_{Su}^K(t) + F_{Sd}^K(t). \end{aligned} \tag{17}$$

The other scalar form factors can be related to these using charge conjugation and time reversal. In (16) and (17) we have also given the notation we shall use for the form factors in the remainder.

The scalar form factor $F_S^{K\pi}(t)$ is proportional to the form factor $f_0(t)$ used in $K_{\ell 3}$ decays.

The scalar form factors can be shown to obey a relation similar to the Sirlin [30] relation for the vector form factor

$$F_S^\pi(t) - 2F_{Ss}^\pi(t) - 2F_{Sd}^K(t) + 2F_{Ss}^K(t) - 4F_S^{K\pi}(t) = \mathcal{O}\left((m_s - \hat{m})^2\right) . \tag{18}$$

The proof of this relation is in App. A.

The values at zero momentum transfer are related to the derivatives of the masses w.r.t. to quark masses because of the Feynman-Hellman theorem (see e.g. [5])

$$\begin{aligned} F_S^\pi(0) &= \frac{\partial}{\partial \hat{m}} m_\pi^2 , \qquad F_{Ss}^\pi(0) = \frac{\partial}{\partial m_s} m_\pi^2 , \\ F_{Su}^K(0) &= \frac{\partial}{\partial m_u} m_K^2 , \qquad F_{Ss}^K(0) = \frac{\partial}{\partial m_s} m_K^2 , \\ F_{Sd}^K(0) &= \frac{\partial}{\partial m_d} m_K^2 . \end{aligned} \tag{19}$$

This also shows that we can expect large corrections at $t = 0$. The argument is fairly simple. If

$$m_K^2 \approx B_0 m_s + \beta (B_0 m_s)^2 + \gamma (B_0 m_s)^3 , \tag{20}$$

then

$$F_{Ss}^K(0) \approx B_0 + 2\beta B_0 m_s + 3\gamma (B_0 m_s)^3 . \tag{21}$$

So we see that in the scalar form factors the relative p^6 corrections can get enhanced by factors of order 3 compared to the masses.

3.2 Diagrams

The relevant diagrams for the present case are identical to those involved in the electromagnetic form factors [31] with the electromagnetic current replaced by the scalar one. We list the diagrams appearing at each order but refer to [24, 31] for a deeper discussion of the checks to perform and the renormalization.

3.2.1 Leading and next-to-leading orders

The diagrams are shown in Fig 1. The black square represents an external scalar insertion from $\mathcal{L}_2$ - diagram a) - and the crossed square a scalar insertion form $\mathcal{L}_4$ - diagram d). Also included, are the tadpole contributions with two possible insertions.

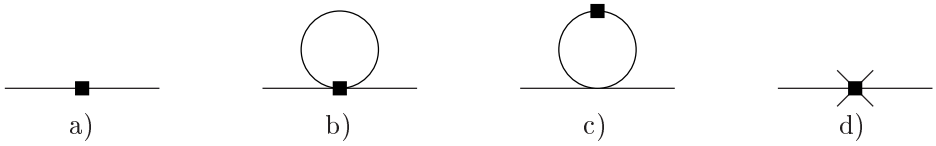

Figure 1: Order p^2 and p^4.

The lowest order contributions are

$$\begin{aligned} F_S^\pi(t) &= 2B_0\,, \quad F_{Ss}^\pi(t) = 0\,, \\ F_{Sq}^K(t) &= B_0\,, \quad F_{Ss}^K(t) = B_0 \\ F_S^{K\pi}(t) &= B_0\,. \end{aligned} \tag{22}$$

3.2.2 Next-to-next-to-leading order

There are four distinct topologies involved in order p^6 diagrams. Firstly, the two loop diagrams, built exclusively on $O(p^2)$ vertices, are represented in Fig 2. There again does a black square represent a scalar interaction.

Secondly, one loop diagrams can contribute, with one vertex built on $\mathcal{L}_4$, possibly including a scalar insertion. These are represented in Fig 3.

There are also sunset integrals and irreducible contributions to include through the non-factorizable diagrams of Fig 4. See [21] and [31] for a treatment of sunset integrals and irreducible two loops integrals.

Of course, one must finally include the $\mathcal{L}_6$ tree contribution, Fig 5.

4 Analytical and First Numerical Results

4.1 Analytical expressions - $O(p^6)$ LEC's

We present here the dependence of all the scalar form factors on the LECs of order p^6. It can be easily checked that the relation (18) is satisfied.

$$\begin{aligned} \langle\pi|\bar{u}u+\bar{d}d|\pi\rangle &= \frac{16B_0}{F_\pi^4}\Big\{ m_\pi^2\, m_K^2\,(-16C_{13}^r - 8C_{15}^r + 8C_{16}^r + 8C_{20}^r + 72C_{21}^r + 16C_{32}^r) \\ &\quad + m_\pi^2\, t\,(4C_{12}^r + 8C_{13}^r + 2C_{14}^r + 3C_{15}^r + 4C_{16}^r + 2C_{17}^r + 2C_{34}^r + 2C_{36}^r) \\ &\quad + m_\pi^4\,(-12C_{12}^r - 16C_{13}^r - 6C_{14}^r - 8C_{15}^r - 14C_{16}^r - 6C_{17}^r + 18C_{19}^r \\ &\qquad + 30C_{20}^r + 30C_{21}^r + 12C_{31}^r + 16C_{32}^r) + 2m_K^2\, t\, C_{15}^r \\ &\quad + m_K^4\,(-8C_{16}^r + 8C_{20}^r + 24C_{21}^r) - t^2\,(C_{12}^r + 2C_{13}^r)\Big\}\,. \end{aligned} \tag{23}$$

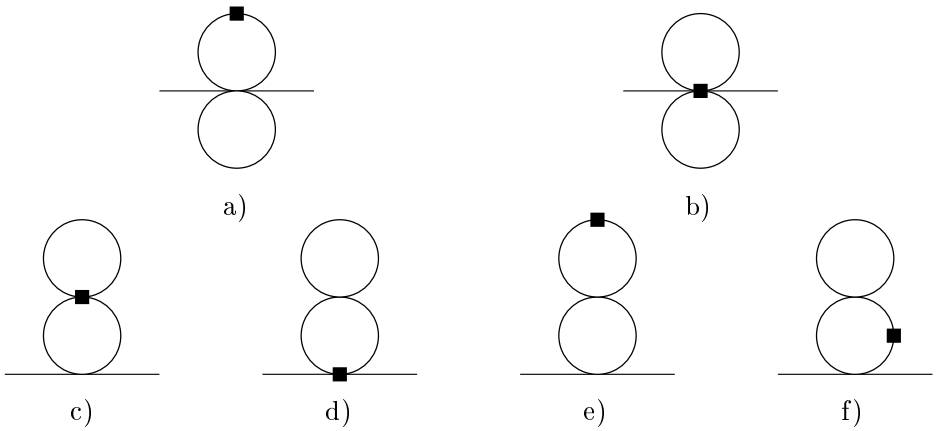

Figure 2: The two loop corrections diagrams.

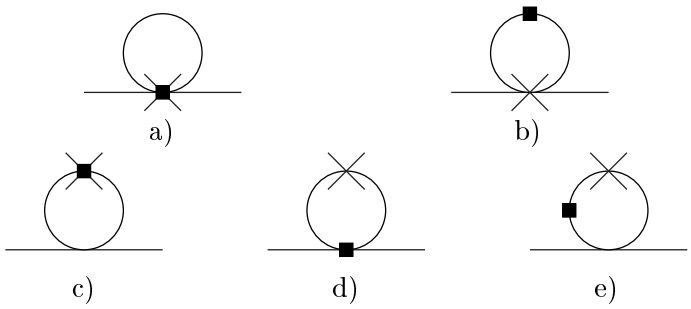

Figure 3: One loop diagrams involving an $O(p^4)$ vertex.

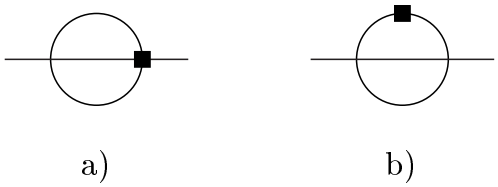

Figure 4: Irreducible two-loop diagrams.

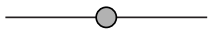

Figure 5: The $\mathcal{L}_6$ contribution.

$$\begin{aligned}\langle\pi|\bar{s}s|\pi\rangle &= \frac{8B_0}{F_\pi^4}\Big\{m_\pi^2\,m_K^2\,(-16C_{16}^r+16C_{20}^r+48C_{21}^r)+m_\pi^2\,t\,(8C_{13}^r+2C_{15}^r-4C_{16}^r+2C_{36}^r)\\ &\quad+m_\pi^4\,(-8C_{13}^r-4C_{15}^r+8C_{16}^r+24C_{21}^r+8C_{32}^r)+8\,m_K^2\,t\,C_{16}^r-2\,t^2\,C_{13}^r\Big\} \qquad (24)\end{aligned}$$

$$\begin{aligned}\langle K^+|\bar{u}u+\bar{d}d|K^+\rangle &= \frac{8B_0}{F_\pi^4}\Big\{+m_\pi^2\,m_K^2\,(-8C_{13}^r-4C_{15}^r-8C_{16}^r+16C_{20}^r+72C_{21}^r+8C_{32}^r)\\ &\quad+m_\pi^2\,t\,(2C_{14}^r+C_{15}^r+8C_{16}^r-2C_{17}^r)\\ &\quad+m_\pi^4\,(-2C_{14}^r-6C_{16}^r+2C_{17}^r+6C_{19}^r+6C_{20}^r+6C_{21}^r)\\ &\quad+m_K^2\,t\,(4C_{12}^r+16C_{13}^r+6C_{15}^r+4C_{17}^r+2C_{34}^r+4C_{36}^r)\\ &\quad+m_K^4\,(-12C_{12}^r-32C_{13}^r-4C_{14}^r-16C_{15}^r-8C_{16}^r-8C_{17}^r+12C_{19}^r\\ &\quad+40C_{20}^r+120C_{21}^r+12C_{31}^r+32C_{32}^r)-t^2\,(C_{12}^r+4C_{13}^r)\Big\} \qquad (25)\end{aligned}$$

$$\begin{aligned}\langle K^+|\bar{s}s|K^+\rangle &= \frac{8B_0}{F_\pi^4}\Big\{+m_\pi^2\,m_K^2\,(-8C_{13}^r+8C_{14}^r-4C_{15}^r+16C_{16}^r-8C_{17}^r-24C_{19}^r-8C_{20}^r\\ &\quad+48C_{21}^r+8C_{32}^r)+m_\pi^2\,t\,(-2C_{14}^r+C_{15}^r-4C_{16}^r+2C_{17}^r)\\ &\quad+m_\pi^4\,(-2C_{14}^r-6C_{16}^r+2C_{17}^r+6C_{19}^r+6C_{20}^r+6C_{21}^r)\\ &\quad+m_K^2\,t\,(4C_{12}^r+8C_{13}^r+4C_{14}^r+4C_{15}^r+8C_{16}^r+2C_{34}^r+2C_{36}^r)\\ &\quad+m_K^4\,(-12C_{12}^r-24C_{13}^r-12C_{14}^r-12C_{15}^r-24C_{16}^r+36C_{19}^r+48C_{20}^r\\ &\quad+72C_{21}^r+12C_{31}^r+24C_{32}^r)-t^2\,(C_{12}^r+2C_{13}^r)\Big\}\,. \qquad (26)\end{aligned}$$

$$\begin{aligned}\langle\pi^-|\bar{s}u|K^0\rangle &= \frac{8B_0}{F_\pi^4}\Big\{+m_\pi^2\,m_K^2\,(-2C_{12}^r-12C_{13}^r-6C_{15}^r+8C_{16}^r-4C_{17}^r+4C_{20}^r+24C_{21}^r\\ &\quad+4C_{31}^r+12C_{32}^r+2C_{34}^r)+m_\pi^2\,t\,(2C_{12}^r+C_{15}^r+2C_{17}^r+C_{34}^r)\\ &\quad+m_\pi^4\,(-5C_{12}^r-4C_{13}^r-2C_{14}^r-2C_{15}^r-6C_{16}^r-2C_{17}^r+6C_{19}^r+10C_{20}^r\\ &\quad+6C_{21}^r+4C_{31}^r+4C_{32}^r-C_{34}^r)+m_K^2\,t\,(2C_{12}^r+2C_{14}^r+2C_{15}^r+C_{34}^r)\\ &\quad+m_K^4\,(-5C_{12}^r-8C_{13}^r-4C_{14}^r-4C_{15}^r-8C_{16}^r+12C_{19}^r+16C_{20}^r+24C_{21}^r\\ &\quad+4C_{31}^r+8C_{32}^r-C_{34}^r)-t^2\,C_{12}^r\Big\} \qquad (27)\end{aligned}$$

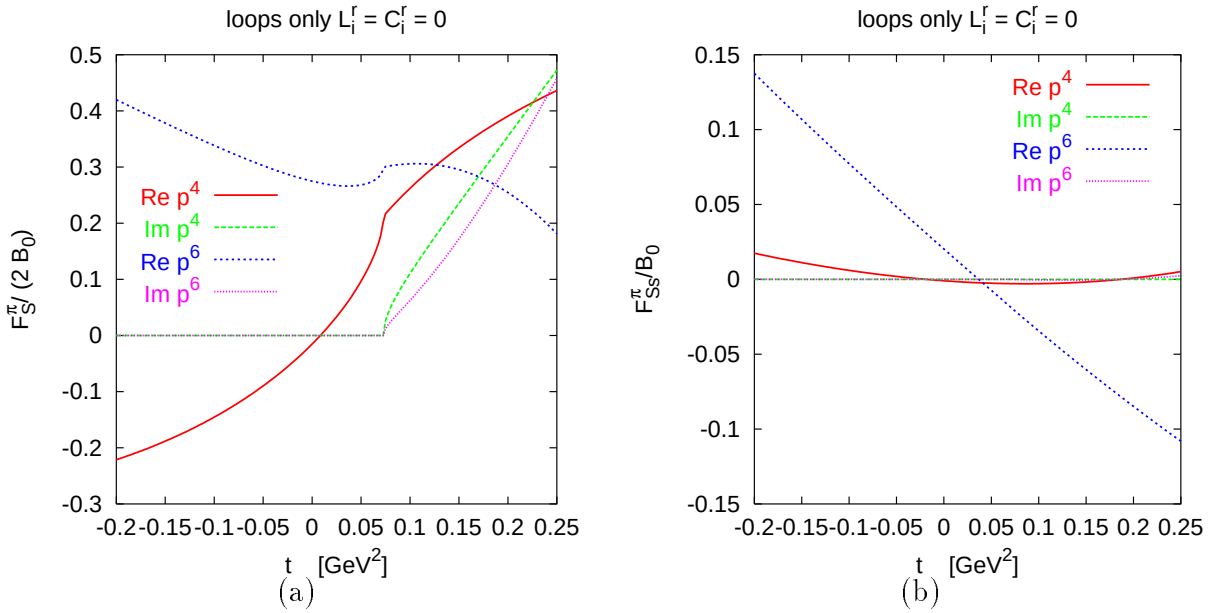

Figure 6: The various contributions to (a) $F_S^\pi(t)/(2B_0)$ and (b) $F_{Ss}^\pi(t)/B_0$ as a function of t for the case with $L_i^r = C_i^r = 0$. Plotted are the real and imaginary parts of the p^4 and the p^6 contributions separately.

4.2 Loop corrections

We present only the results for F_S^π. The expressions for the others are obtainable on request from the authors. Form factors including external etas can be calculated as well if required. The order p^4 results are in agreement with those of [5]. The form given in App. B is the one that corresponds to the p^6 expressions we use and which can also be found in App. B.

4.3 First Numerical Results

In this part we present some plots of the sizes of the various corrections for the case with $C_i^r = 0$, we have used the L_i^r values of [25], fit 10, and the neutral pion, neutral kaon and physical eta masses as well as $F_\pi = 92.4$ MeV.

In Fig. 6a we have plotted various contributions to the pion scalar form factor normalized to $2B_0$. There is a sizable two-loop contribution in this case. The pion strange scalar form factor, $F_{Ss}^\pi(t)/B_0$, is plotted in Fig. 6b. The corrections here are moderate.

In Fig. 7a we have plotted various contributions to the kaon light quark scalar form factor normalized to B_0. There are large two-loop contributions in this case. The kaon strange scalar form factor, $F_{Ss}^K(t)/B_0$, is plotted in Fig. 7b. The corrections here are also large.

We will do a more extensive analysis in the next section but in order to show the effects of the L_i^r we have plotted in Fig. 8a the effect of the L_i^r of fit 10 if [25] on $F_S^\pi(t)/(2B_0)$ compared to the case with the $L_i^r = 0$ and with the same nonzero values of the other L_i^r

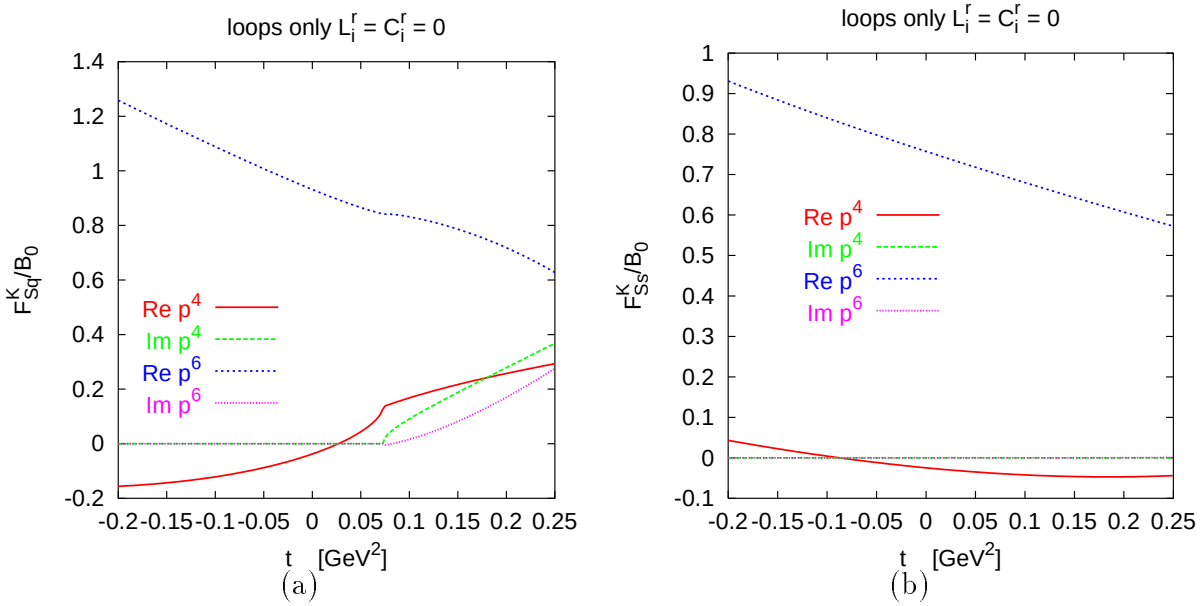

Figure 7: The various contributions to (a) $F^K_{Sq}(t)/B_0$ and (b) $F^K_{Ss}(t)/B_0$ as a function of t for the case with $L_i^r = C_i^r = 0$. Plotted are the real and imaginary parts of the p^4 and the p^6 contributions separately.

but $L_4^r = -0.0003$. Notice that the effect of such a small L_4^r is fairly large. The effect on the pion strange scalar form factor, $F^\pi_{Ss}(t)/B_0$ is also rather large. But notice that the effect of the p^6 corrections is larger than the change due to this value of L_4^r.

5 Numerical Analysis

5.1 Dispersive Inputs

There is no direct experimental measurement of the scalar form factors but information can still be extracted from elastic scattering experiments. We recall here the principal results of dispersive analyses and refer to [32] and [7] for a more thorough treatment.

5.1.1 The Muskhelishvili-Omnès (MO) problem

We consider first the one channel case, *i.e.* where only pion interactions are considered. The problem of finding the functions satisfying the pion form factor's analytical properties is referred to as the Muskhelishvili-Omnès problem. Provided some convergence conditions on the form factor and the phase shift $\delta_\pi(s)$ are satisfied, the solution is of the form:

$$F_\pi(s) = P(s)\, \exp\left[\frac{s}{\pi}\int_{s_0}^{\infty} ds' \frac{\delta_\pi(s')}{s'\,(s'-s)}\right]. \tag{28}$$

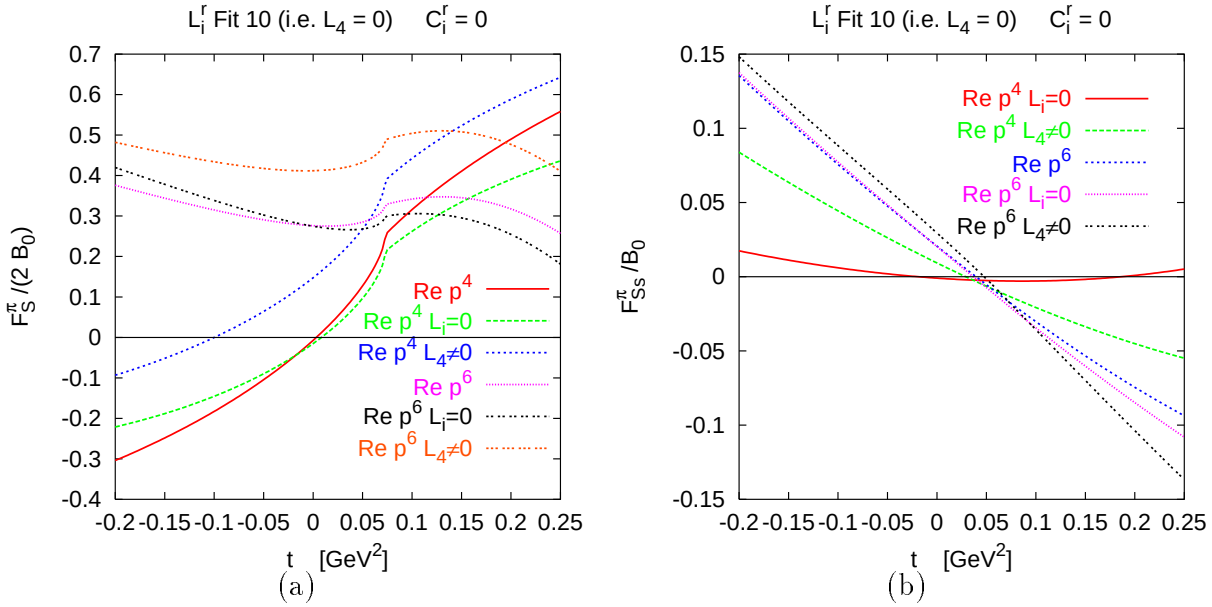

Figure 8: The effects of the L_i^r on the various contributions to (a) $F_S^\pi(t)/(2B_0)$ and (b) $F_{Ss}^\pi(t)/B_0$ as a function of t for the case with $C_i^r = 0$. Plotted are the real and imaginary parts of the p^4 and the p^6 contributions separately.

Watson's theorem then relates the phase shift to the scattering phases δ_I^J, here $\delta_\pi = \delta_0^0$. This can be extended to two or more channels. We restrict here to the two channel case, $\pi\pi$ and $K\bar{K}$ in the isospin and angular momentum zero case, $I = 0$ S-wave. The MO problem becomes a system of coupled equations of the two independent contributions

$$F_1(s) = \langle 0|\hat{X}_0|\pi\pi\rangle \qquad \text{and} \qquad F_2(s) = \frac{2}{\sqrt{3}}\langle 0|\hat{X}_0|K\bar{K}\rangle\,, \tag{29}$$

where $\hat{X}_0$ is an isospin zero operator, *e.g.* $(\bar{u}u + \bar{d}d)$ or $\bar{s}s$. The interesting feature of the MO solution is the possibility to write the general solution using two independent sets of solutions $\{F_1^{(i)}, F_2^{(i)}\}_{i=1,2}$ fulfilling the initial conditions

$$F_i^{(j)}(0) = \delta_{ij}, \tag{30}$$

and this *independently* of the exact form of the operator $\hat{X}_0$. The solution is then built on the value of the form factors at the origin:

$$\left\{ \begin{array}{rcl} F_1(s) & = & F_1(0)\,F_1^{(1)}(s) \;+\; F_2(0)\,F_1^{(2)}(s) \\ F_2(s) & = & F_1(0)\,F_2^{(1)}(s) \;+\; F_2(0)\,F_2^{(2)}(s) \end{array} \right. \tag{31}$$

5.1.2 The solutions

The method and coding of the solutions were borrowed from Moussallam [7, 33], where the solutions are obtained by solving the linear system of equations – equivalent to (28)

	$F_1(0)$	$F_1'(0)$	$1/2\,F_1''(0)$	$F_2(0)$	$F_2'(0)$	$1/2\,F_2''(0)$
$\mathrm{Au}^{(1)}$	1.	2.33	9.94	0.	0.92	5.31
$\mathrm{Au}^{(2)}$	0.	0.31	1.03	1.	0.78	0.98
$\mathrm{lm}^{(1)}$	1.	2.43	10.28	0.	0.75	3.49
$\mathrm{lm}^{(2)}$	0.	0.27	0.93	1.	0.84	1.00
$\mathrm{Abm}^{(1)}$	1.	2.45	10.61	0.	1.03	5.36
$\mathrm{Abm}^{(2)}$	0.	0.21	0.72	1.	0.81	0.98

Table 1: Results of the numerical computation of the Omnès solutions for the different T_{ij} models. The superscript refers to the respective inequivalent sets of solutions. The first two solutions are from Ref. [7], the last from [33].

for the two channel case – given by the discretized form of

$$\mathrm{Re}\,F_i(s) = PV\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\,F_i(s)}{s'-s} \tag{32}$$

and

$$\mathrm{Im}\,F_i(s) = 2i\sum_{j=1,2}\sigma_j(s)T_{ij}(s)F_j(s)^* \tag{33}$$

where

$$\sigma_j(s) = \sqrt{1-\frac{4m_j^2}{s}}\,\theta(s-4m_j^2)\,, \quad m_j = m_\pi, m_K \tag{34}$$

and T_{ij} are the T-matrix elements of the needed $\pi\pi$, KK scattering channels. Three T-matrix models of scattering are used, from Au *et al.* [34], Kamiński *et al.* [35] and Ananthanaryan et al. [33], and we refer again to [7, 33] for details and differences. The results of our calculation for the different derivatives of the canonical solutions using the programs of [7, 33] at the origin are presented in Table 1. We have only used the last solution, since they use a better physical model for the $\pi\pi$ to KK amplitude and newer $\pi\pi$ phases.

Similar work exists for the $K\pi$ form factor, $F_S^{K\pi}(t)$, in [36]. We have not obtained their results in a form we can use in the same way.

5.2 Other experimental inputs

The other experimental inputs are the same as those used in [25] corresponding to fit 10 there. This differed from the main fits reported there by using the then preliminary data of [13]. We have, similarly to the analysis performed in [24] done the same fit but with a variety of fixed values of $L_4^r(m_\rho)$ and $L_6^r(m_\rho)$ as input. In all these cases we have used the resonance saturation contributions to the $K_{\ell 4}$ constants of p^6 as derived in [24] and those

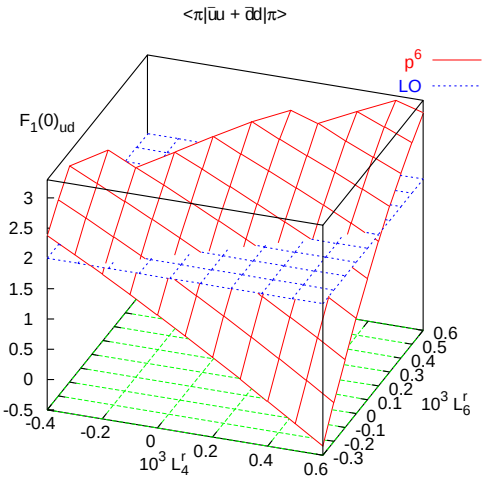

Figure 9: The result for $F_1(0)_{ud} = F_S^\pi(0)/B_0$ as a function of L_4^r and L_6^r. Also shown is the lowest order value 2.

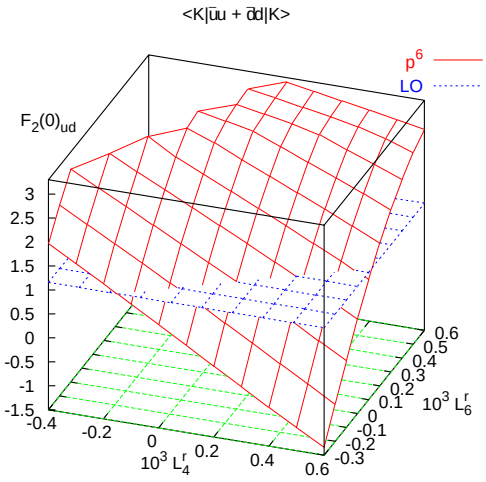

Figure 10: The result for $F_2(0)_{ud} = 2/\sqrt{3}\, F_{Sq}^K(0)/B_0$ as a function of L_4^r and L_6^r. Also shown is the lowest order value $2/\sqrt{3}$.

for the decay constants and the masses as derived in [23] but we have chosen the value of d_m there to be zero. The reason for this is that the estimate of the parameter performed there was very uncertain and the naive value obtained there led to p^6 contributions to the masses which were enormous. We have performed the fits at a grid

$$10^4\, L_4^r = -4, \ldots, 6 \quad \text{and} \quad 10^4\, L_6^r = -3, \ldots, 6\,. \tag{35}$$

For some values of L_4^r and L_6^r no decent fits could be obtained, the values fits with a value of χ^2 below 1 in the grid satisfy

$$L_6^r \lesssim 0.6 L_4^r + 0.0004\,. \tag{36}$$

In the numerical results quoted below we have used the neutral kaon and pion masses and a subtraction scale $\mu = 770$ MeV.

5.3 Variation with L_4^r and L_6^r of the scalar form factors at $t = 0$.

It should be kept in mind that the values of the other L_i^r used change as well with the values of L_4^r and L_6^r in accordance with the fit to the other experimental values.

We first show the results for $F_S^\pi(0)/B_0$ in Fig. 9. We also show the lowest order expectation of 2 for comparison. Only the points with good fits are shown.

We show similarly in Fig. 10 the value of $F_2(0)_{ud}(0) = 2/\sqrt{3}\, F_{Sq}^K/B_0$ together with the lowest order value of $2/\sqrt{3}$. Notice that in both cases we have large corrections for most values of L_4^r and L_6^r.

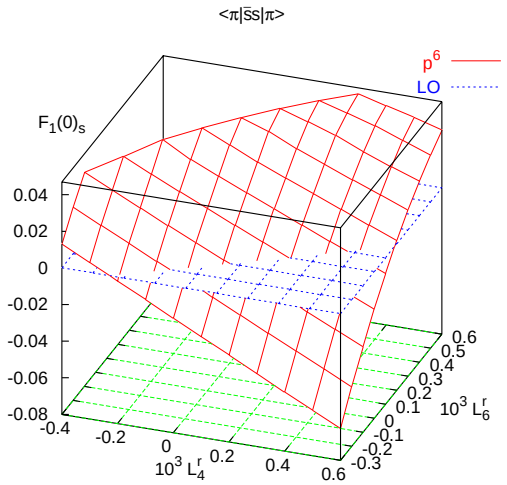

Figure 11: The result for $F_1(0)_s = F^\pi_{Ss}(0)/B_0$ as a function of L_4^r and L_6^r. Also shown is the lowest order value 0. Notice the strange quark content of the pion remains small over the entire studied area

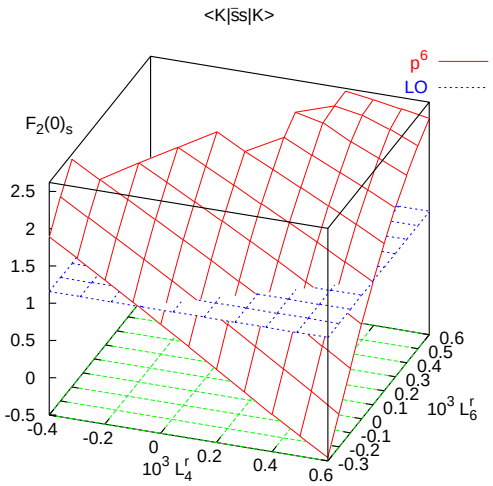

Figure 12: The result for $F_2(0)_s = 2/\sqrt{3}\, F^K_{Ss}(0)/B_0$ as a function of L_4^r and L_6^r. Also shown is the lowest order value $2/\sqrt{3}$.

We have also studied the $\overline{s}s$ current. The results for the pion case are shown in Fig. 11. The result for $F_2(0)_s = 2/\sqrt{3}\, F^K_{Ss}(0)/B_0$ is shown in Fig. 12. Notice that the strange quark content in the pion remains small for the entire studied region.

At this level we cannot determine the values of L_4^r and L_6^r but there is a tendency for the corrections compared to the lowest order to be small for positive L_4^r and a correlated value for L_6^r which is similar for all four cases.

5.4 Slopes and the value of L_4^r and L_6^r

We can use our results to get at the form factors away from zero in two different ways. First we can of course simply use our full ChPT calculation to calculate them to order p^6. Second, from Sect. 5.1 we can calculate the form factors at $t \neq 0$ given the values at zero. We can then require that both should give the same results and obtain in this way constraints on the values of L_4^r and L_6^r.

The results for the pion scalar radius,

$$< r^2 >^\pi_S = \frac{6}{F^\pi_S(0)} \frac{d}{dt} F^\pi_S(t)|_{t=0}\,, \tag{37}$$

are shown in Fig. 13. The dispersive prediction is quite stable except in the region where $F^\pi_S(0)$ vanishes and is away this region entirely predicted by the values of the $\pi\pi$ phase

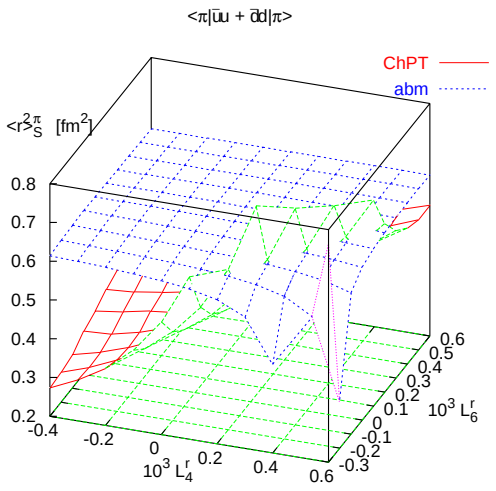

Figure 13: The result for the scalar radius $< r^2 >^S_\pi$ as a function of L^r_4 and L^r_6, with the dispersive estimate (abm) and the ChPT calculation.

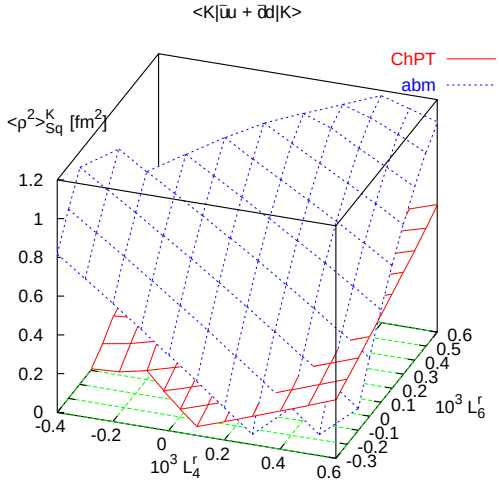

Figure 14: The result for the kaon light quark scalar radius $< \rho^2 >^S_{Kq}$ as a function of L^r_4 and L^r_6, with the dispersive estimate (abm) and the ChPT calculation.

shifts. We see agreement in the same regions as where the corrections to the lowest order results for the form factors at $t = 0$ are fairly small.

The results for the kaon light quark scalar radius,

$$< r^2 >^K_{Sq} = \frac{6}{F^K_{Sq}(0)} \frac{d}{dt} F^K_{Sq}(t)|_{t=0} \,, \tag{38}$$

are not as easily shown, the calculated value of $F^K_{Sq}(0)$ runs through zero in the relevant region, producing strong effects in both the dispersive and ChPT results. We plot instead the scalar radius normalized to the lowest order result

$$< \rho^2 >^K_{Sq} = \frac{6}{B_0} \frac{d}{dt} F^K_{Sq}(t)|_{t=0} \,. \tag{39}$$

This is shown in Fig. 14. The dispersive prediction is less stable since the relative strength of the two solutions varies much more than in the pion case. We see agreement again in the regions where the corrections to the lowest order results for the form factors at $t = 0$ are fairly small, but the results are less reliable than for the pion case.

The strange quark scalar current can be studied in exactly the same way. The pion strange scalar form factor has a very small value at zero, we normalize therefore to the value B_0 and plot

$$< \rho^2 >^\pi_{Ss} = \frac{6}{B_0} \frac{d}{dt} F^\pi_{Ss}(t)|_{t=0} \,. \tag{40}$$

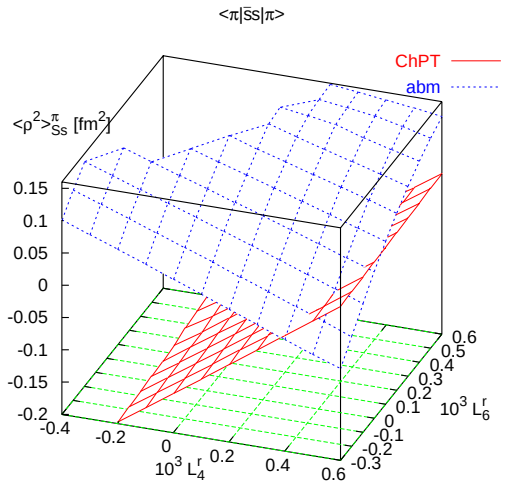

Figure 15: The result for the pion strange quark scalar radius $< \rho^2 >^S_{\pi s}$ as a function of L^r_4 and L^r_6, with the dispersive estimate (abm) and the ChPT calculation.

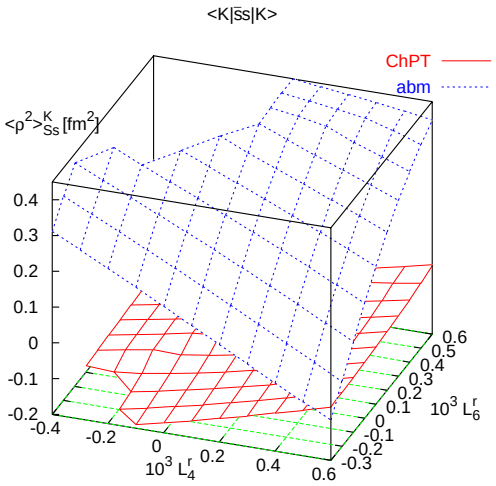

Figure 16: The result for the kaon strange quark scalar radius $< \rho^2 >^S_{Ks}$ as a function of L^r_4 and L^r_6, with the dispersive estimate (abm) and the ChPT calculation.

The results are shown in Fig. 15 both for the dispersive and the ChPT result.

The prediction for the kaon strange quark radius

$$< r^2 >^K_{Ss} = \frac{6}{F^K_{Ss}(0)} \frac{d}{dt} F^K_{Ss}(t)|_{t=0} \,, \tag{41}$$

is very stable for most values of the L^r_4, L^r_6 except for a region near the top of the L^r_4 and the bottom of the L^r_6 range where $F^K_{Ss}(0)$ is near zero. The two predictions never agree there, the dispersive estimate is positive and the ChPT one is negative. If we plot

$$< \rho^2 >^K_{Ss} = \frac{6}{B_0} \frac{d}{dt} F^K_{Ss}(t)|_{t=0} \,, \tag{42}$$

there is a small region of overlap between the dispersive and the ChPT estimates. The latter quantity is plotted in Fig. 16.

It is somewhat difficult to get a final conclusion about L^r_4 and L^r_6 since the effect of the p^6 constants has to be evaluated. This requires a general study of effects in the scalar sector which goes beyond the scope of this paper.[2] The main constraint from the pion scalar radius is

$$L^r_6 \approx L^r_4 - 0.00035 \,. \tag{43}$$

To be precise, this is the curve where the two surfaces shown in Fig. 13 intersect. If in addition we require that the values of the scalar form factors at zero do not deviate too

[2] The analysis might be doable by extending some of the lines of work of [37].

much from their lowest order we obtain that L_4^r should be in the range from $0.0003 to 0.0006$. We have shown several numerical results in Table 2 for grid points in this region.

5.5 Curvatures and the value of C_{12} and C_{13}

Since the values at zero determine the form factors away from zero, we can also compare higher order terms in the expansion in t. Because of the large corrections to the values at zero coming from order p^6, there is a rather large uncertainty in the resulting values. We have therefore not done a full error analysis.

We define the pion scalar curvature via

$$c_S^\pi = \frac{1}{F_S^\pi(0)} \frac{1}{2} \frac{d^2}{dt^2} F_S^\pi(t)|_{t=0} \,. \tag{44}$$

The results both from the dispersive and the ChPT analysis are shown in Fig. 17. Notice that, precisely as in the two flavour case analyzed in [19] a large part of the curvature is due to the loop effects and and the contribution from the p^6 constants is not the dominant one.

The same analysis can be performed for the other form factors. We again run into the problem of normalizing the curvature, just as we had for the slopes. For the rest we use thus quantities normalized to B_0. We show the pion strange scalar curvature

$$\gamma_{Ss}^\pi = \frac{1}{B_0} \frac{1}{2} \frac{d^2}{dt^2} F_{Ss}^\pi(t)|_{t=0} \,. \tag{45}$$

in Fig. 18. Analogously we define

$$\begin{aligned}
\gamma_S^\pi &= \frac{1}{2B_0} \frac{1}{2} \frac{d^2}{dt^2} F_S^\pi(t)|_{t=0} \,, \\
\gamma_{Sq}^K &= \frac{1}{B_0} \frac{1}{2} \frac{d^2}{dt^2} F_{Sq}^K(t)|_{t=0} \,, \\
\gamma_S^K &= \frac{1}{B_0} \frac{1}{2} \frac{d^2}{dt^2} F_{Ss}^K(t)|_{t=0} \,.
\end{aligned} \tag{46}$$

but we have not included plots of these quantities.

The curvatures can now be used to estimate combinations of the C_i^r. We get

$$\begin{aligned}
C_{12}^r + 2C_{13}^r &= -\frac{F_\pi^4}{8} \left(c_S^{\pi\, disp} - c_S^{\pi\, ChPT, C_i^r = 0} \right) = -\frac{F_\pi^4}{8} \left(\gamma_{Ss}^{K\, disp} - \gamma_{Ss}^{K\, ChPT, C_i^r = 0} \right) \,, \\
C_{13}^r &= -\frac{F_\pi^4}{16} \left(\gamma_{Ss}^{\pi disp} - \gamma_{Ss}^{\pi ChPT, C_i^r = 0} \right) \,, \\
C_{12}^r + 4C_{13}^r &= -\frac{F_\pi^4}{8} \left(\gamma_{Sq}^{K\, disp} - \gamma_{Sq}^{K\, ChPT, C_i^r = 0} \right)
\end{aligned} \tag{47}$$

From the figures it is obvious that the extracted values will depend strongly on the input values used for L_4^r and L_6^r. On the other hand, we see that there are several ways to extract

Set	Fit 10	A	B	C
$10^3 \cdot L_4^r$	0.0	0.4	0.5	0.5
$10^3 \cdot L_6^r$	0.0	0.1	0.1	0.2
$F_S^\pi(0)/B_0$ (ChPT, $C_i^r = 0$)	2.54	1.99	1.75	2.12
$F_{Ss}^\pi(0)/B_0$ (ChPT, $C_i^r = 0$)	0.020	0.004	-0.002	0.008
$F_{Sq}^K(0)/B_0$ (ChPT, $C_i^r = 0$)	1.94	1.36	1.12	1.51
$F_{Ss}^K(0)/B_0$ (ChPT, $C_i^r = 0$)	1.77	1.35	1.17	1.45
$< r^2 >_S^{\pi,disp}$ (fm^2)	0.617	0.612	0.610	0.614
$< r^2 >_S^{\pi,ChPT,C_i^r=0}$ (fm^2)	0.384	0.547	0.625	0.563
$10^5\,(C_{12}^r + 2C_{13}^r)\,[c_S^\pi]$	-2.6	-0.56	0.55	-0.71
$10^5\,(C_{12}^r + 2C_{13}^r)\,[\gamma_S^\pi]$	-3.3	-0.55	0.48	-0.75
$10^5\,(C_{12}^r + 2C_{13}^r)\,[\gamma_{Ss}^K]$	-0.55	0.11	0.33	0.15
$10^5\,C_{13}^r[\gamma_{Ss}^\pi]$	-0.56	-0.02	0.15	0.03
$10^5\,(C_{12}^r + 4C_{13}^r)\,[\gamma_{Sq}^K]$	-4.1	-0.27	0.99	-0.08
$10^5\,C_{12}^r[c_S^\pi,\,\gamma_{Ss}^\pi]$	-1.5	-0.52	0.26	-0.78
$10^5\,C_{12}^r[c_S^\pi,\,\gamma_{Sq}^K]$	-1.1	-0.84	0.12	-1.3
F_0 (MeV)	87.7	63.5	70.4	71.0
$F_{\pi^+}^{(4)}$	0.136	0.230	0.253	0.254
$F_{\pi^+}^{(6)}$	-0.083	0.226	0.059	0.048
$F_{K\pi}^{(4)}$	0.169	0.157	0.153	0.159
$F_{K\pi}^{(6)}$	0.051	0.063	0.067	0.061
$\mu_\pi^{(2)}$	0.736	1.005	1.129	0.936
$\mu_\pi^{(4)}$	0.006	-0.090	-0.138	-0.043
$\mu_\pi^{(6)}$	0.258	0.085	0.009	0.107
$\mu_K^{(2)}$	0.687	0.938	1.055	0.874
$\mu_K^{(4)}$	0.007	-0.100	-0.149	-0.057
$\mu_K^{(6)}$	0.306	0.162	0.094	0.183
$\mu_\eta^{(2)}$	0.734	1.001	1.124	0.933
$\mu_\eta^{(4)}$	-0.052	-0.151	-0.197	-0.104
$\mu_\eta^{(6)}$	0.318	0.150	0.073	0.171

Table 2: The results for various quantities for the L_i^r of fit 10 and sets A, B and C. These are described in the text.

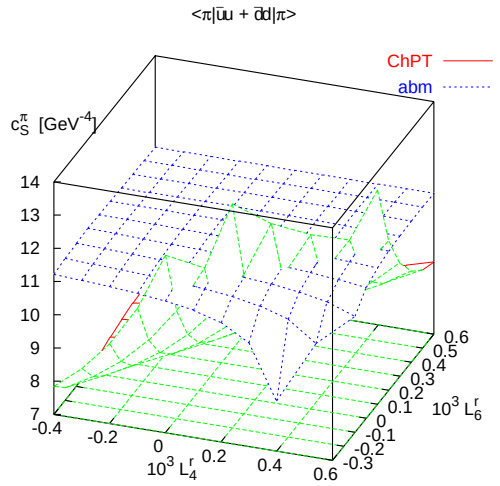

Figure 17: The result for the pion scalar curvature c_S^π as a function of L_4^r and L_6^r, with the dispersive estimate (abm) and the ChPT calculation.

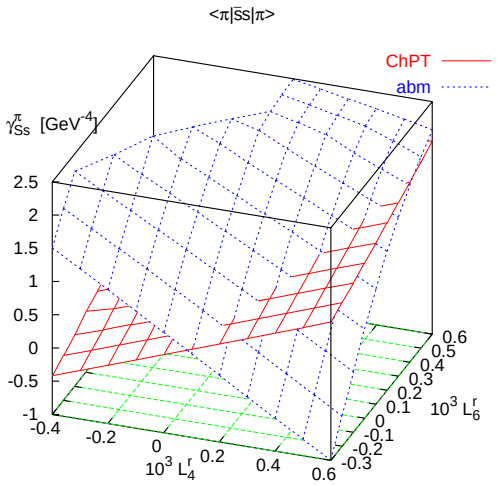

Figure 18: The result for the pion strange scalar curvature γ_{Ss}^π as a function of L_4^r and L_6^r, with the dispersive estimate (abm) and the ChPT calculation.

the same quantities. It turns out that these tend to give similar values in the region of L_i^r preferred in the previous section.

We have collected in Table 2 various results for fours sets of L_i^r. The first one is exactly the same as fit 10 of Ref. [25] and all quantities coincide with the one reported there. The notation $C_{12}^r + 2C_{13}^r[c_S^\pi]$ used in Table 2 means that the quantity $C_{12}^r + 2C_{13}^r$ is determined from c_S^π using the L_i^r as determined from a fit to the same experimental input as used in fit 10 in Ref. [25] but with the values of L_4^r and L_6^r as indicated. The difference between the first two determinations of $C_{12}^r + 2C_{13}^r$ is due to the difference from $F_S^\pi(0)$ from $2B_0$.

A final conclusion on the value of C_{12}^r which is needed for the measurement of V_{us} [26] is not very easy since it depends so strongly on the values of L_4^r and L_6^r as well as on which dispersive quantities are used to evaluate it.

5.6 Masses and Decay Constants

For completeness we present here results for the masses and decay constants for the sets of input parameters we use. These can also be found in Table 2. We present results for

$$\begin{aligned} F_{\pi^+}/F_0 &= 1 + F_{\pi^+}^{(4)} + F_{\pi^+}^{(6)}\,, \\ F_{K^+}/F_{\pi^+} &= 1 + F_{K\pi}^{(4)} + F_{K\pi}^{(6)}\,, \\ m_{\pi^\pm}^2/(m_{\pi^\pm}^2)_{\rm QCD} &= \mu_\pi^{(2)} + \mu_\pi^{(4)} + \mu_\pi^{(6)} \\ m_{K^\pm}^2/(m_{K^\pm}^2)_{\rm QCD} &= \mu_K^{(2)} + \mu_K^{(4)} + \mu_K^{(6)}\,, \end{aligned}$$

$$m_\eta^2/(m_\eta^2)_{\rm phys} \;=\; \mu_\eta^{(2)} + \mu_\eta^{(4)} + \mu_\eta^{(6)}\,. \tag{48}$$

Notice that in [25] the corresponding numbers were quoted for the "Main Fit" which used the old K_{e4} data.

A glance at Table 2 shows that the pion decay constant in the chiral limit F_0 can be substantially different from the value of about 87 MeV. for the case of fit 10.

6 Conclusions

In this paper we have computed the full order p^6 expressions in the isospin limit for all pion and kaon scalar form factors. We also found a new relation between the scalar form factors, analogous to the Sirlin relation for vector form factors. We have used these results to obtain constraints on the values of L_4^r, L_6^r and C_{12}^r, C_{13}^r.

Acknowledgements

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A The proof of the symmetry relation

The relation for the scalar form factors, Eq. (18) can be proven using the same method as used in the second paper of Ref. [30] when the different behaviour of scalar and vector currents under charge conjugation is taken in to account.

We write the Hamiltonian in the form

$$H = H_0 + \lambda_3 v_3\,. \tag{A.1}$$

H_0 is the Hamiltonian in the limit $m_s = m_u$. We now do oldfashioned perturbation theory in λ_3, and look at the variation of

$$F_{su}^{K^0\pi^-}(t) \tag{A.2}$$

w.r.t. λ_3 at $\lambda_3 = 0$. This can be rewritten as an expectation value involving the K^0 and π^- field, v_3 and $\overline{s}u$. Applying the operator $G_V = Cexp(i\pi V_2)$ with V_2 the second generator of V-spin and C charge conjugation we obtain that

$$\frac{\partial}{\partial\lambda_3} F_{su}^{K^0\pi^-}(t)\Big|_{\lambda_3=0} = -\,\frac{\partial}{\partial\lambda_3} F_{su}^{\overline{K^0}\pi^+}(t)\Big|_{\lambda_3=0} \tag{A.3}$$

while charge conjugation and the the fact that it is a scalar current require the full form factors in (A.3) to be equal. The first variation w.r.t. λ_3 of $F_{su}^{K^0\pi^-}(t)$ thus vanishes.

The same argument can be applied to the combination

$$F_{uu}^{\pi^+\pi^+}(t) - F_{ss}^{\pi^+\pi^+}(t) - F_{uu}^{\overline{K^0}\,\overline{K^0}}(t) + F_{ss}^{\overline{K^0}\,\overline{K^0}}(t)\,. \tag{A.4}$$

and in the V-spin limit both combinations are related. This relation can be brought into the form of Eq. (18) using isospin.

B Analytical expressions for $F_S^\pi(t)$

For brevity we only write the pion scalar form factor with the light quark densities. The expressions for the others can be obtained from the authors. The integrals can be found in several places. The functions $\overline{A}$, $\overline{B}$,... can be found in [21] and [24]. The H^F,... are defined in [21] and the V_i can be found in [31]. The method to evaluate the H_i^F was developed in [21] and the V_i are evaluated as described in [31] using the methods of [39].

The result given is not the shortest possible analytical one. There are various relations between the integrals that we have not implemented. The reason is that these relations involve inverse powers of t and we have used simplifying relations to rewrite all the masses in the numerators in terms of m_π^2 and m_K^2. This has as a consequence that, if we had used those relations, the numerators of the $1/t$ poles would not cancel exactly numerically and produce possible instabilities for small t.

We write the pion scalar form factor as

$$F_S^\pi(t) = 2B_0\left(F_S^{\pi(2)} + \frac{1}{F_\pi^2}F_S^{\pi(4)}(t) + \frac{1}{F_\pi^4}F_S^{\pi(6)}(t)\right)\,. \tag{B.5}$$

The superscript indicates the chiral order. The lowest order is simply

$$F_S^{\pi(2)} = 1\,. \tag{B.6}$$

The next order has been calculated in [5] and we agree with their result, it is

$$\begin{aligned} F_S^{\pi(4)}(t) \;=\; & -16L_4^r m_K^2 - 24L_4^r m_\pi^2 + 8L_4^r t - 16L_5^r m_\pi^2 + 4L_5^r t + 32L_6^r m_K^2 \\ & +48L_6^r m_\pi^2 + 32L_8^r m_\pi^2 + 1/6\overline{A}(m_\eta^2) - 1/2\overline{A}(m_\pi^2) + \overline{B}(m_\eta^2, m_\eta^2, t)(1/18m_\pi^2) \\ & +\overline{B}(m_K^2, m_K^2, t)(1/4t) + \overline{B}(m_\pi^2, m_\pi^2, t)(-1/2m_\pi^2 + t)\,. \end{aligned} \tag{B.7}$$

The next order we split in several different parts

$$F_S^{\pi(6)}(t) = F_{S\mathbf{B}}^\pi(t) + F_{S\mathbf{H}}^\pi(t) + F_{S\mathbf{V}}^\pi(t) + F_{S\mathbf{L}}^\pi(t) + F_{S\mathbf{C}}^\pi(t)\,. \tag{B.8}$$

The result for $F_{S\mathbf{C}}^\pi(t)$ can also be found in the main text we simply repeat it here for completeness.

$$\begin{aligned} F_{S\mathbf{C}}^\pi(t) \;=\; & -128m_\pi^2 m_K^2 C_{13}^r - 64m_\pi^2 m_K^2 C_{15}^r + 64m_\pi^2 m_K^2 C_{16}^r + 64m_\pi^2 m_K^2 C_{20}^r \\ & +576m_\pi^2 m_K^2 C_{21}^r + 128m_\pi^2 m_K^2 C_{32}^r + 32m_\pi^2 C_{12}^r t + 64m_\pi^2 C_{13}^r t + 16m_\pi^2 C_{14}^r t \\ & +24m_\pi^2 C_{15}^r t + 32m_\pi^2 C_{16}^r t + 16m_\pi^2 C_{17}^r t + 16m_\pi^2 C_{34}^r t + 16m_\pi^2 C_{36}^r t - 96m_\pi^4 C_{12}^r \\ & -128m_\pi^4 C_{13}^r - 48m_\pi^4 C_{14}^r - 64m_\pi^4 C_{15}^r - 112m_\pi^4 C_{16}^r - 48m_\pi^4 C_{17}^r + 144m_\pi^4 C_{19}^r \\ & +240m_\pi^4 C_{20}^r + 240m_\pi^4 C_{21}^r + 96m_\pi^4 C_{31}^r + 128m_\pi^4 C_{32}^r + 16m_K^2 C_{15}^r t - 64m_K^4 C_{16}^r \\ & +64m_K^4 C_{20}^r + 192m_K^4 C_{21}^r - 8C_{12}^r t^2 - 16C_{13}^r t^2\,. \end{aligned} \tag{B.9}$$

$$
\begin{aligned}
F^{\pi}_{S\mathbf{H}}(t) &= H^{F\prime}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2)(5/6m_\pi^4) + H^{F\prime}(m_\pi^2,m_K^2,m_K^2,m_\pi^2)(-5/8m_\pi^4) \\
&+H^{F\prime}(m_\pi^2,m_\eta^2,m_\eta^2,m_\pi^2)(1/18m_\pi^4) + H^{F\prime}(m_K^2,m_\pi^2,m_K^2,m_\pi^2)(m_\pi^2m_K^2) \\
&+H^{F\prime}(m_K^2,m_K^2,m_\eta^2,m_\pi^2)(-5/6m_\pi^4) \\
&+H^{F\prime}(m_\eta^2,m_K^2,m_K^2,m_\pi^2)(1/2m_\pi^2m_K^2+7/8m_\pi^4) \\
&+H_1^{F\prime}(m_\pi^2,m_K^2,m_K^2,m_\pi^2)(m_\pi^4) + H_1^{F\prime}(m_\eta^2,m_K^2,m_K^2,m_\pi^2)(-m_\pi^4) \\
&+H_{21}^{F\prime}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2)(3m_\pi^4) + H_{21}^{F\prime}(m_\pi^2,m_K^2,m_K^2,m_\pi^2)(-3/8m_\pi^4) \\
&+H_{21}^{F\prime}(m_K^2,m_\pi^2,m_K^2,m_\pi^2)(3m_\pi^4) + H_{21}^{F\prime}(m_\eta^2,m_K^2,m_K^2,m_\pi^2)(9/8m_\pi^4) \\
&+H^{F}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2)(2/3m_\pi^2-2/3t) \\
&+H^{F}(m_\pi^2,m_K^2,m_K^2,m_\pi^2)(-47/24m_\pi^2+1/6t) + H^{F}(m_\pi^2,m_\eta^2,m_\eta^2,m_\pi^2)(1/9m_\pi^2) \\
&+H^{F}(m_K^2,m_\pi^2,m_K^2,m_\pi^2)(5/6m_\pi^2+m_K^2-1/2t) \\
&+H^{F}(m_K^2,m_K^2,m_\eta^2,m_\pi^2)(-13/9m_\pi^2-1/8t) \\
&+H^{F}(m_\eta^2,m_K^2,m_K^2,m_\pi^2)(119/72m_\pi^2+1/2m_K^2+1/24t) \\
&+H_1^{F}(m_\pi^2,m_K^2,m_K^2,m_\pi^2)(2m_\pi^2+1/24t) \\
&+H_1^{F}(m_\eta^2,m_K^2,m_K^2,m_\pi^2)(-2m_\pi^2-1/24t) \\
&+H_{21}^{F}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2)(6m_\pi^2) + H_{21}^{F}(m_\pi^2,m_K^2,m_K^2,m_\pi^2)(-3/4m_\pi^2) \\
&+H_{21}^{F}(m_K^2,m_\pi^2,m_K^2,m_\pi^2)(6m_\pi^2) + H_{21}^{F}(m_\eta^2,m_K^2,m_K^2,m_\pi^2)(9/4m_\pi^2)\,. \qquad \text{(B.10)}
\end{aligned}
$$

$$
\begin{aligned}
F^{\pi}_{S\mathbf{V}}(t) &= +V_0(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(-m_\pi^2t+7/2m_\pi^4+2/3t^2) \\
&+V_0(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-11/12m_\pi^2t+3/2m_\pi^4+1/12t^2) \\
&+V_0(m_\pi^2,m_\pi^2,m_\eta^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(1/18m_\pi^4) \\
&+V_0(m_\pi^2,m_\eta^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(1/4m_\pi^2t-1/6m_\pi^4) \\
&+V_0(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(-17/12m_\pi^2t+3/2m_\pi^4+1/2t^2) \\
&+V_0(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(-2/3m_\pi^2t+2/3m_\pi^4+1/8t^2) \\
&+V_0(m_\eta^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(2/9m_\pi^2m_K^2+13/36m_\pi^2t \\
&\qquad -7/18m_\pi^4-1/9m_K^2t-1/12t^2) \\
&+V_0(m_\eta^2,m_\eta^2,m_\pi^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(1/27m_\pi^4) \\
&+V_0(m_\eta^2,m_\eta^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-5/36m_\pi^2t+1/18m_\pi^4+1/12t^2) \\
&+V_{11}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(13/3m_\pi^2t-4m_\pi^4-4/3t^2) \\
&+V_{11}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(11/6m_\pi^2t-3m_\pi^4-1/6t^2) \\
&+V_{11}(m_\pi^2,m_\eta^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-1/2m_\pi^2t+2/3m_\pi^4) \\
&+V_{11}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(53/12m_\pi^2t-4m_\pi^4-t^2) \\
&+V_{11}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(5/3m_\pi^2t-2m_\pi^4-1/4t^2)
\end{aligned}
$$

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\begin{aligned}
&+V_{11}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-2/9m_\pi^2 m_K^2 - 7/9m_\pi^2 t \\
&\qquad + 8/9m_\pi^4 + 1/9m_K^2 t + 1/6t^2) \\
&+V_{11}(m_\eta^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(5/9m_\pi^2 t - 1/3m_\pi^4 - 1/6t^2) \\
&+V_{12}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(-t^2) \\
&+V_{12}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/3m_\pi^2 t + 1/8t^2) \\
&+V_{12}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/6m_\pi^2 t - 1/8t^2) \\
&+V_{12}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(5/6m_\pi^2 t - 1/2t^2) \\
&+V_{12}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(1/3m_\pi^2 t) \\
&+V_{12}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-5/18m_\pi^2 t + 1/9m_K^2 t + 1/8t^2) \\
&+V_{12}(m_\eta^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/9m_\pi^2 t - 1/8t^2) \\
&+V_{13}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(2/3m_\pi^2 t - 8m_\pi^4 + 2/3t^2) \\
&+V_{13}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(2m_\pi^2 t - 4m_\pi^4) \\
&+V_{13}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/2m_\pi^2 t + 1/3m_\pi^4) \\
&+V_{13}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(13/6m_\pi^2 t - 3m_\pi^4 - 5/24t^2) \\
&+V_{13}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(3/2m_\pi^2 t - 2m_\pi^4 - 1/8t^2) \\
&+V_{13}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-4/9m_\pi^2 m_K^2 - 13/18m_\pi^2 t + 7/9m_\pi^4 \\
&\qquad + 2/9m_K^2 t + 1/6t^2) \\
&+V_{14}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(2/3t^2) \\
&+V_{14}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/2t^2) \\
&+V_{14}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/6m_\pi^2 t - 1/4t^2) \\
&+V_{14}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(7/24t^2) \\
&+V_{14}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(1/4t^2) \\
&+V_{14}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-7/18m_\pi^2 t + 2/9m_K^2 t + 1/6t^2) \\
&+V_{21}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(4m_\pi^2 - 2/3t) \\
&+V_{21}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(3/2m_\pi^2 - 7/12t) \\
&+V_{21}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/2m_\pi^2 + 1/4t) \\
&+V_{21}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(4m_\pi^2 - t) \\
&+V_{21}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(3/2m_\pi^2 - 1/2t) \\
&+V_{21}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/2m_\pi^2 + 1/12t) \\
&+V_{21}(m_\eta^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/2m_\pi^2 - 1/12t) \\
&+V_{22}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(-10/3m_\pi^2 t + 4m_\pi^4 + 2/3t^2) \\
&+V_{22}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-11/12m_\pi^2 t + 3/2m_\pi^4 + 1/12t^2) \\
&+V_{22}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/4m_\pi^2 t - 1/2m_\pi^4) \\
&+V_{22}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(-3m_\pi^2 t + 4m_\pi^4 + 1/2t^2)
\end{aligned}
$$

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\begin{aligned}
&+V_{22}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(-m_\pi^2 t + 3/2 m_\pi^4 + 1/8 t^2) \\
&+V_{22}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(5/12 m_\pi^2 t - 1/2 m_\pi^4 - 1/12 t^2) \\
&+V_{22}(m_\eta^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-5/12 m_\pi^2 t + 1/2 m_\pi^4 + 1/12 t^2) \\
&+V_{23}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(1/3 t^2) \\
&+V_{23}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-5/24 t^2) \\
&+V_{23}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/8 t^2) \\
&+V_{23}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(-1/8 t^2) \\
&+V_{23}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/24 t^2) \\
&+V_{23}(m_\eta^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/24 t^2) \\
&+V_{24}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(-8/3 m_\pi^2 t + t^2) \\
&+V_{24}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/3 m_\pi^2 t - 1/8 t^2) \\
&+V_{24}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/8 t^2) \\
&+V_{24}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(-2 m_\pi^2 t + 1/2 t^2) \\
&+V_{24}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(-1/2 m_\pi^2 t) \\
&+V_{24}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/3 m_\pi^2 t - 1/8 t^2) \\
&+V_{24}(m_\eta^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/3 m_\pi^2 t + 1/8 t^2) \\
&+V_{25}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(-4/3 t) \\
&+V_{25}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(4 m_\pi^2 - 2t) \\
&+V_{25}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-m_\pi^2 + 1/2 t) \\
&+V_{25}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(4 m_\pi^2 - 19/12 t) \\
&+V_{25}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(3 m_\pi^2 - 5/4 t) \\
&+V_{25}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-m_\pi^2 + 1/6 t) \\
&+V_{26}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(4/3 m_\pi^2 t - 2/3 t^2) \\
&+V_{26}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-2 m_\pi^2 t + 4 m_\pi^4) \\
&+V_{26}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/2 m_\pi^2 t - m_\pi^4) \\
&+V_{26}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(-29/12 m_\pi^2 t + 4 m_\pi^4 + 5/24 t^2) \\
&+V_{26}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(-7/4 m_\pi^2 t + 3 m_\pi^4 + 1/8 t^2) \\
&+V_{26}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(5/6 m_\pi^2 t - m_\pi^4 - 1/6 t^2) \\
&+V_{27}(m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2, t, m_\pi^2)(-4/3 t^2) \\
&+V_{27}(m_\pi^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-t^2) \\
&+V_{27}(m_\pi^2, m_\eta^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(1/4 t^2) \\
&+V_{27}(m_K^2, m_K^2, m_\pi^2, m_K^2, m_\pi^2, t, m_\pi^2)(-7/12 t^2) \\
&+V_{27}(m_K^2, m_K^2, m_K^2, m_\eta^2, m_\pi^2, t, m_\pi^2)(-1/2 t^2) \\
&+V_{27}(m_\eta^2, m_\pi^2, m_K^2, m_K^2, m_\pi^2, t, m_\pi^2)(-1/12 t^2)
\end{aligned}
$$

$$
\begin{aligned}
&+V_{28}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(8/3m_\pi^2t-4/3t^2)\\
&+V_{28}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-1/2t^2)\\
&+V_{28}(m_\pi^2,m_\eta^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(1/2m_\pi^2t)\\
&+V_{28}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(-5/6m_\pi^2t-1/12t^2)\\
&+V_{28}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(-1/2m_\pi^2t-1/8t^2)\\
&+V_{28}(m_\eta^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(1/6m_\pi^2t-1/12t^2)\\
&+V_{29}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(-2/3t^2)\\
&+V_{29}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-1/2t^2)\\
&+V_{29}(m_\pi^2,m_\eta^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-1/2m_\pi^2t+1/4t^2)\\
&+V_{29}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(-7/24t^2)\\
&+V_{29}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(-1/4t^2)\\
&+V_{29}(m_\eta^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(1/2m_\pi^2t-1/6t^2)\\
&+V_{210}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(8m_\pi^2-4t)\\
&+V_{210}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(4m_\pi^2-2t)\\
&+V_{210}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(3/2m_\pi^2-3/4t)\\
&+V_{210}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(3/2m_\pi^2-3/4t)\\
&+V_{211}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(-4m_\pi^2t+8m_\pi^4)\\
&+V_{211}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-2m_\pi^2t+4m_\pi^4)\\
&+V_{211}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(-3/4m_\pi^2t+3/2m_\pi^4)\\
&+V_{211}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(-3/4m_\pi^2t+3/2m_\pi^4)\\
&+V_{212}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(-2t^2)\\
&+V_{212}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-t^2)\\
&+V_{212}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(-3/8t^2)\\
&+V_{212}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(-3/8t^2)\\
&+V_{213}(m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2,t,m_\pi^2)(-2t^2)\\
&+V_{213}(m_\pi^2,m_\pi^2,m_K^2,m_K^2,m_\pi^2,t,m_\pi^2)(-t^2)\\
&+V_{213}(m_K^2,m_K^2,m_\pi^2,m_K^2,m_\pi^2,t,m_\pi^2)(-3/8t^2)\\
&+V_{213}(m_K^2,m_K^2,m_K^2,m_\eta^2,m_\pi^2,t,m_\pi^2)(-3/8t^2)\,. \qquad \text{(B.11)}
\end{aligned}
$$

$$
\begin{aligned}
F_{S\mathbf{B}}^{\pi}(t) \;=\;& +\frac{1}{16\pi^2}(-11/81m_\pi^2m_K^2/m_\eta^2\overline{A}(m_\eta^2)-5/108m_\pi^2\overline{A}(m_\eta^2)+5/6m_\pi^2\overline{A}(m_\pi^2)\\
&-1/4m_\pi^2\overline{A}(m_K^2)+7/324m_\pi^4/m_\eta^2\overline{A}(m_\eta^2)-5/54m_K^2\overline{A}(m_\eta^2)+m_K^2\overline{A}(m_\pi^2)\\
&+m_K^2\overline{A}(m_K^2)+16/81m_K^4/m_\eta^2\overline{A}(m_\eta^2))\\
&+\left(\frac{1}{16\pi^2}\right)^2(-55/162m_\pi^2m_K^2\pi^2-1015/432m_\pi^2m_K^2+23/108m_\pi^2\pi^2t
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 23/18m_\pi^2 t - 1171/1296m_\pi^4\pi^2 - 8783/1728m_\pi^4 + 17/216m_K^2\pi^2 t \\
&\quad + 17/36m_K^2 t - 11/36m_K^4\pi^2 - 15/16m_K^4) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)^2(-1/2m_\pi^2 t + 1/4m_\pi^4 + 2/9t^2) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}(m_K^2, m_K^2, t)(-1/24m_\pi^2 t + 7/72t^2) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}(m_\eta^2, m_\eta^2, t)(1/27m_\pi^2 t + 1/18m_\pi^4) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}_1((m_\pi^2, m_\pi^2, t)(-m_\pi^2 t + 10/9t^2) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}_1((m_K^2, m_K^2, t)(-1/12m_\pi^2 t + 17/72t^2) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}_{21}(m_\pi^2, m_\pi^2, t)(4/9t^2) + \overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}_{21}(m_K^2, m_K^2, t)(1/12t^2) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}_{22}(m_\pi^2, m_\pi^2, t)(4/9t) + \overline{B}(m_\pi^2, m_\pi^2, t)\overline{B}_{22}(m_K^2, m_K^2, t)(1/12t) \\
&+\overline{B}(m_\pi^2, m_\pi^2, t)(-1/12m_\pi^2\overline{A}(m_\eta^2) - 15/4m_\pi^2\overline{A}(m_\pi^2) - 2m_\pi^2\overline{A}(m_K^2) \\
&\quad + 1/9t\overline{A}(m_\eta^2) + t\overline{A}(m_\pi^2) + 35/72t\overline{A}(m_K^2)) \\
&+\overline{B}(m_\pi^2, m_\pi^2, 0)(-1/9m_\pi^2\overline{A}(m_\eta^2) + 1/3m_\pi^2\overline{A}(m_\pi^2)) \\
&+\overline{B}(m_K^2, m_K^2, t)^2(1/24t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}(m_\eta^2, m_\eta^2, t)(-1/9m_\pi^2 m_K^2 + 1/24m_\pi^2 t - 1/27m_K^2 t + 1/72t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_1((m_\pi^2, m_\pi^2, t)(-1/12m_\pi^2 t + 1/4t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_1((m_K^2, m_K^2, t)(5/24t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_1((m_\eta^2, m_\eta^2, t)(1/12m_\pi^2 t + 1/36t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_{21}(m_\pi^2, m_\pi^2, t)(1/9t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_{21}(m_K^2, m_K^2, t)(1/12t^2) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_{22}(m_\pi^2, m_\pi^2, t)(1/9t) \\
&+\overline{B}(m_K^2, m_K^2, t)\overline{B}_{22}(m_K^2, m_K^2, t)(1/12t) \\
&+\overline{B}(m_K^2, m_K^2, t)(-1/4m_\pi^2\overline{A}(m_\eta^2) - 3/4m_\pi^2\overline{A}(m_\pi^2) - 1/2m_\pi^2\overline{A}(m_K^2) \\
&\quad - 53/360t\overline{A}(m_\eta^2) + 35/72t\overline{A}(m_\pi^2) + 17/120t\overline{A}(m_K^2)) \\
&+\overline{B}(m_K^2, m_K^2, 0)(1/18m_K^2\overline{A}(m_\eta^2)) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)^2(4/81m_\pi^2 m_K^2 - 7/324m_\pi^4) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)\overline{B}_1((m_\pi^2, m_\pi^2, t)(1/27m_\pi^2 t) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)\overline{B}_1((m_K^2, m_K^2, t)(1/12m_\pi^2 t - 1/27m_K^2 t + 1/24t^2) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)\overline{B}_{21}(m_K^2, m_K^2, t)(1/36t^2) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)\overline{B}_{22}(m_K^2, m_K^2, t)(1/36t) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)(1/108m_\pi^2\overline{A}(m_\eta^2) + 7/36m_\pi^2\overline{A}(m_\pi^2) - 2/9m_\pi^2\overline{A}(m_K^2) \\
&\quad + 13/360t\overline{A}(m_K^2)) \\
&+\overline{B}^\epsilon(m_\pi^2, m_\pi^2, t)\frac{1}{16\pi^2}(-2m_\pi^2 m_K^2 + 14/9m_\pi^2 t - 91/18m_\pi^4 + 19/36m_K^2 t \\
&\quad + 97/72t^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{B}^{\epsilon}(m_K^2,m_K^2,t)\frac{1}{16\pi^2}(-m_\pi^2m_K^2+103/180m_\pi^2t-2/3m_\pi^4+14/45m_K^2t\\
&\qquad+35/72t^2)\\
&+\overline{B}^{\epsilon}(m_\eta^2,m_\eta^2,t)\frac{1}{16\pi^2}(1/6m_\pi^2t-5/54m_\pi^4+1/20m_K^2t+1/24t^2)\\
&+\overline{B}_1^{\epsilon}(m_\pi^2,m_\pi^2,t)\frac{1}{16\pi^2}(25/18m_\pi^2t+17/18m_K^2t-253/72t^2)\\
&+\overline{B}_1^{\epsilon}(m_K^2,m_K^2,t)\frac{1}{16\pi^2}(-71/180m_\pi^2t+17/45m_K^2t-101/72t^2)\\
&+\overline{B}_1^{\epsilon}(m_\eta^2,m_\eta^2,t)\frac{1}{16\pi^2}(-1/6m_\pi^2t-1/10m_K^2t-5/24t^2)\\
&+\overline{B}_{21}^{\epsilon}(m_\pi^2,m_\pi^2,t)\frac{1}{16\pi^2}(55/72t^2)+\overline{B}_{21}^{\epsilon}(m_K^2,m_K^2,t)\frac{1}{16\pi^2}(13/36t^2)\\
&+\overline{B}_{21}^{\epsilon}(m_\eta^2,m_\eta^2,t)\frac{1}{16\pi^2}(1/8t^2)+\overline{B}_{22}^{\epsilon}(m_\pi^2,m_\pi^2,t)\frac{1}{16\pi^2}(-15/2m_\pi^2+95/36t)\\
&+\overline{B}_{22}^{\epsilon}(m_K^2,m_K^2,t)\frac{1}{16\pi^2}(-3m_\pi^2+10/9t)\\
&+\overline{B}_{22}^{\epsilon}(m_\eta^2,m_\eta^2,t)\frac{1}{16\pi^2}(-1/2m_\pi^2+1/4t)+\overline{B}_1((m_\pi^2,m_\pi^2,t)^2(4/9t^2)\\
&+\overline{B}_1((m_\pi^2,m_\pi^2,t)\overline{B}_1((m_K^2,m_K^2,t)(7/36t^2)\\
&+\overline{B}_1((m_\pi^2,m_\pi^2,t)(1/9t\overline{A}(m_\eta^2)+5/9t\overline{A}(m_\pi^2)+17/18t\overline{A}(m_K^2))\\
&+\overline{B}_1((m_K^2,m_K^2,t)^2(1/12t^2)+\overline{B}_1((m_K^2,m_K^2,t)\overline{B}_1((m_\eta^2,m_\eta^2,t)(1/36t^2)\\
&+\overline{B}_1((m_K^2,m_K^2,t)(23/180t\overline{A}(m_\eta^2)-1/12t\overline{A}(m_\pi^2)+2/15t\overline{A}(m_K^2))\\
&+\overline{B}_1((m_\eta^2,m_\eta^2,t)(-1/10t\overline{A}(m_K^2))-1/2m_\pi^{-2}m_K^2\left(\overline{A}(m_\pi^2)\right)^2\\
&+23/18m_\pi^{-2}t\left(\overline{A}(m_\pi^2)\right)^2-4/3m_\pi^2m_K^{-2}\left(\overline{A}(m_K^2)\right)^2\\
&+11/162m_\pi^2m_K^2/m_\eta^4\left(\overline{A}(m_\eta^2)\right)^2-35/144m_\pi^2/m_\eta^2\left(\overline{A}(m_\eta^2)\right)^2\\
&-7/648m_\pi^4/m_\eta^4\left(\overline{A}(m_\eta^2)\right)^2+17/36m_K^{-2}t\left(\overline{A}(m_K^2)\right)^2\\
&-7/36m_K^2/m_\eta^2\left(\overline{A}(m_\eta^2)\right)^2-8/81m_K^4/m_\eta^4\left(\overline{A}(m_\eta^2)\right)^2\\
&+13/36\overline{A}(m_\eta^2)\overline{A}(m_\pi^2)-7/18\overline{A}(m_\eta^2)\overline{A}(m_K^2)+1/36\left(\overline{A}(m_\eta^2)\right)^2-\overline{A}(m_\pi^2)\overline{A}(m_K^2)\\
&-1037/144\left(\overline{A}(m_\pi^2)\right)^2-7/4\left(\overline{A}(m_K^2)\right)^2\ . \qquad \text{(B.12)}
\end{aligned}
$$

$$
\begin{aligned}
F_{S\mathbf{L}}^{\pi}(t) \;=\;& +\frac{1}{16\pi^2}(-16/9m_\pi^2m_K^2L_2^r-16/27m_\pi^2m_K^2L_3^r+12m_\pi^2m_K^2L_4^r-24m_\pi^2m_K^2L_6^r\\
&+4m_\pi^4L_1^r+74/9m_\pi^4L_2^r+56/27m_\pi^4L_3^r+16/3m_\pi^4L_4^r+16/3m_\pi^4L_5^r\\
&-32/3m_\pi^4L_6^r-32/3m_\pi^4L_8^r+104/9m_K^4L_2^r+86/27m_K^4L_3^r+8/3m_K^4L_4^r\\
&+4/3m_K^4L_5^r-16/3m_K^4L_6^r-8/3m_K^4L_8^r)\\
&+\overline{B}(m_\pi^2,m_\pi^2,t)(8m_\pi^2m_K^2L_4^r-16m_\pi^2m_K^2L_6^r-24m_\pi^2L_1^rt-8m_\pi^2L_2^rt-12m_\pi^2L_3^rt\\
&-12m_\pi^2L_4^rt-26/3m_\pi^2L_5^rt+32m_\pi^2L_6^rt+64/3m_\pi^2L_8^rt+48m_\pi^4L_1^r+16m_\pi^4L_2^r
\end{aligned}
$$

$$
\begin{aligned}
&+24m_\pi^4 L_3^r - 36m_\pi^4 L_4^r - 16m_\pi^4 L_5^r + 56m_\pi^4 L_6^r + 24m_\pi^4 L_8^r - 32/3m_K^2 L_4^r t \\
&+64/3m_K^2 L_6^r t) \\
&+\overline{B}(m_\pi^2, m_\pi^2, 0)(32/3m_\pi^2 m_K^2 L_4^r - 64/3m_\pi^2 m_K^2 L_6^r + 16/3m_\pi^4 L_4^r + 16/3m_\pi^4 L_5^r \\
&-32/3m_\pi^4 L_6^r - 32/3m_\pi^4 L_8^r) \\
&+\overline{B}(m_K^2, m_K^2, t)(32m_\pi^2 m_K^2 L_1^r + 8m_\pi^2 m_K^2 L_3^r - 32m_\pi^2 m_K^2 L_4^r - 8m_\pi^2 m_K^2 L_5^r \\
&+32m_\pi^2 m_K^2 L_6^r + 16m_\pi^2 m_K^2 L_8^r - 4/3m_\pi^2 L_4^r t + 8/3m_\pi^2 L_6^r t - 16m_K^2 L_1^r t \\
&-4m_K^2 L_3^r t - 8/3m_K^2 L_5^r t + 16m_K^2 L_6^r t + 16/3m_K^2 L_8^r t) \\
&+\overline{B}(m_K^2, m_K^2, 0)(4/3m_\pi^2 m_K^2 L_4^r - 8/3m_\pi^2 m_K^2 L_6^r + 8/3m_K^4 L_4^r + 4/3m_K^4 L_5^r \\
&-16/3m_K^4 L_6^r - 8/3m_K^4 L_8^r) \\
&+\overline{B}(m_\eta^2, m_\eta^2, t)(64/9m_\pi^2 m_K^2 L_1^r + 32/27m_\pi^2 m_K^2 L_3^r - 104/9m_\pi^2 m_K^2 L_4^r \\
&-64/27m_\pi^2 m_K^2 L_5^r + 16m_\pi^2 m_K^2 L_6^r - 128/9m_\pi^2 m_K^2 L_7^r + 8/9m_\pi^2 L_1^r t \\
&+4/27m_\pi^2 L_3^r t - 4/9m_\pi^2 L_4^r t - 2/9m_\pi^2 L_5^r t - 16/9m_\pi^4 L_1^r \\
&-8/27m_\pi^4 L_3^r + 20/9m_\pi^4 L_4^r + 16/27m_\pi^4 L_5^r - 8/3m_\pi^4 L_6^r + 128/9m_\pi^4 L_7^r \\
&+40/9m_\pi^4 L_8^r - 32/9m_K^2 L_1^r t - 16/27m_K^2 L_3^r t + 16/9m_K^2 L_4^r t) \\
&+\overline{B}_1((m_\pi^2, m_\pi^2, t)(-48m_\pi^2 L_1^r t - 16m_\pi^2 L_2^r t - 24m_\pi^2 L_3^r t - 8/3m_\pi^2 L_5^r t + 32m_\pi^2 L_6^r t \\
&+64/3m_\pi^2 L_8^r t - 32/3m_K^2 L_4^r t + 64/3m_K^2 L_6^r t + 32L_1^r t^2 + 24L_2^r t^2 \\
&+16L_3^r t^2 + 32/3L_4^r t^2 + 16/3L_5^r t^2) \\
&+\overline{B}_1((m_K^2, m_K^2, t)(-32m_\pi^2 L_1^r t - 8m_\pi^2 L_3^r t + 44/3m_\pi^2 L_4^r t + 4m_\pi^2 L_5^r t \\
&+8/3m_\pi^2 L_6^r t - 8m_K^2 L_4^r t - 8/3m_K^2 L_5^r t + 16m_K^2 L_6^r t + 16/3m_K^2 L_8^r t + 16L_1^r t^2 \\
&+8L_2^r t^2 + 6L_3^r t^2 + 16/3L_4^r t^2 + 4/3L_5^r t^2) \\
&+\overline{B}_1((m_\eta^2, m_\eta^2, t)(-16/3m_\pi^2 L_1^r t - 8/9m_\pi^2 L_3^r t + 16/3m_\pi^2 L_4^r t + 8/9m_\pi^2 L_5^r t \\
&+8/3L_1^r t^2 + 4/3L_2^r t^2 + 8/9L_3^r t^2) \\
&+\overline{B}_{21}(m_\pi^2, m_\pi^2, t)(-8L_1^r t^2 - 16L_2^r t^2 - 4L_3^r t^2 + 32/3L_4^r t^2 + 16/3L_5^r t^2) \\
&+\overline{B}_{21}(m_K^2, m_K^2, t)(-8L_2^r t^2 - 2L_3^r t^2 + 16/3L_4^r t^2 + 4/3L_5^r t^2) \\
&+\overline{B}_{21}(m_\eta^2, m_\eta^2, t)(-4/3L_2^r t^2 - 4/9L_3^r t^2) \\
&+\overline{B}_{22}(m_\pi^2, m_\pi^2, t)(32m_\pi^2 L_1^r + 64m_\pi^2 L_2^r + 16m_\pi^2 L_3^r - 16L_1^r t - 32L_2^r t - 8L_3^r t \\
&+32/3L_4^r t + 16/3L_5^r t) \\
&+\overline{B}_{22}(m_K^2, m_K^2, t)(32m_\pi^2 L_2^r + 8m_\pi^2 L_3^r - 16L_2^r t - 4L_3^r t + 16/3L_4^r t + 4/3L_5^r t) \\
&+\overline{B}_{22}(m_\eta^2, m_\eta^2, t)(16/3m_\pi^2 L_2^r + 16/9m_\pi^2 L_3^r - 8/3L_2^r t - 8/9L_3^r t) \\
&-128m_\pi^2 m_K^2 L_4^r L_5^r + 1536m_\pi^2 m_K^2 L_4^r L_6^r + 512m_\pi^2 m_K^2 L_4^r L_8^r - 256m_\pi^2 m_K^2 (L_4^r)^2 \\
&+256m_\pi^2 m_K^2 L_5^r L_6^r - 1024m_\pi^2 m_K^2 L_6^r L_8^r - 2048m_\pi^2 m_K^2 (L_6^r)^2 + 104m_\pi^2 L_1^r \overline{A}(m_\pi^2) \\
&+32m_\pi^2 L_1^r \overline{A}(m_K^2) - 4/3m_\pi^2 L_2^r \overline{A}(m_\eta^2) + 48m_\pi^2 L_2^r \overline{A}(m_\pi^2) - 4/9m_\pi^2 L_3^r \overline{A}(m_\eta^2) \\
&+52m_\pi^2 L_3^r \overline{A}(m_\pi^2) + 8m_\pi^2 L_3^r \overline{A}(m_K^2) - 8/3m_\pi^2 L_4^r \overline{A}(m_\eta^2) - 424/3m_\pi^2 L_4^r \overline{A}(m_\pi^2) \\
&-172/3m_\pi^2 L_4^r \overline{A}(m_K^2) - 4/3m_\pi^2 L_5^r \overline{A}(m_\eta^2) - 244/3m_\pi^2 L_5^r \overline{A}(m_\pi^2) - 24m_\pi^2 L_5^r \overline{A}(m_K^2) \\
&+800/3m_\pi^2 L_6^r \overline{A}(m_\pi^2) + 248/3m_\pi^2 L_6^r \overline{A}(m_K^2) + 32m_\pi^2 L_7^r \overline{A}(m_\eta^2) + 16m_\pi^2 L_8^r \overline{A}(m_\eta^2)
\end{aligned}
$$

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\begin{aligned}
&+464/3m_\pi^2 L_8^r \overline{A}(m_\pi^2) + 48m_\pi^2 L_8^r \overline{A}(m_K^2) - 128m_\pi^4 L_4^r L_5^r + 512m_\pi^4 L_4^r L_6^r \\
&+384m_\pi^4 L_4^r L_8^r - 64m_\pi^4 (L_4^r)^2 + 512m_\pi^4 L_5^r L_6^r + 384m_\pi^4 L_5^r L_8^r - 64m_\pi^4 (L_5^r)^2 \\
&-1280m_\pi^4 L_6^r L_8^r - 768m_\pi^4 (L_6^r)^2 - 512m_\pi^4 (L_8^r)^2 + 64/3m_K^2 L_1^r \overline{A}(m_\eta^2) \\
&+64m_K^2 L_1^r \overline{A}(m_K^2) + 16/3m_K^2 L_2^r \overline{A}(m_\eta^2) + 16m_K^2 L_2^r \overline{A}(m_K^2) + 16/3m_K^2 L_3^r \overline{A}(m_\eta^2) \\
&+20m_K^2 L_3^r \overline{A}(m_K^2) - 88/3m_K^2 L_4^r \overline{A}(m_\eta^2) - 104/3m_K^2 L_4^r \overline{A}(m_\pi^2) \\
&-248/3m_K^2 L_4^r \overline{A}(m_K^2) - 32/9m_K^2 L_5^r \overline{A}(m_\eta^2) - 52/3m_K^2 L_5^r \overline{A}(m_K^2) \\
&+112/3m_K^2 L_6^r \overline{A}(m_\eta^2) + 208/3m_K^2 L_6^r \overline{A}(m_\pi^2) + 304/3m_K^2 L_6^r \overline{A}(m_K^2) \\
&-64/3m_K^2 L_7^r \overline{A}(m_\eta^2) + 104/3m_K^2 L_8^r \overline{A}(m_K^2) - 128m_K^4 L_4^r L_5^r + 1024m_K^4 L_4^r L_6^r \\
&+256m_K^4 L_4^r L_8^r - 256m_K^4 (L_4^r)^2 + 256m_K^4 L_5^r L_6^r - 512m_K^4 L_6^r L_8^r - 1024m_K^4 (L_6^r)^2 \\
&-8/3L_1^r t\overline{A}(m_\eta^2) - 24L_1^r t\overline{A}(m_\pi^2) - 16L_1^r t\overline{A}(m_K^2) - 8L_2^r t\overline{A}(m_\pi^2) - 4/9L_3^r t\overline{A}(m_\eta^2) \\
&-12L_3^r t\overline{A}(m_\pi^2) - 4L_3^r t\overline{A}(m_K^2) + 4/3L_4^r t\overline{A}(m_\eta^2) + 20/3L_4^r t\overline{A}(m_\pi^2) \\
&+16/3L_4^r t\overline{A}(m_K^2) + 2/3L_5^r t\overline{A}(m_\eta^2) + 10/3L_5^r t\overline{A}(m_\pi^2) + 10/3L_5^r t\overline{A}(m_K^2)\,. \quad \text{(B.13)}
\end{aligned}
$$

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